

Jiří Vinárek

Remarks on dimensions of graphs

In: Zdeněk Frolík (ed.): Abstracta. 8th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1980. pp. 180–193.

Persistent URL: <http://dml.cz/dmlcz/701206>

Terms of use:

© Institute of Mathematics of the Academy of Sciences of the Czech Republic, 1980

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

EIGHTH WINTER SCHOOL ON ABSTRACT ANALYSIS (1980)

REMARKS ON DIMENSIONS OF GRAPHS

• Jiří Vinárek

1. Preliminaries

The well-known Dushnik - Miller dimension of partly ordered sets (see [DM]) was shown by Ore ([O]) to coincide with the necessary number of linearly ordered factors in a product $\prod L_i$ into which the given poset can be fully embedded. It is a particular case of a characteristic of objects based on representations of products of subdirectly irreducibles.

Recall a definition of a subdirectly irreducible (SI) object for a productive hereditary class \underline{C} of digraphs (i.e. a class closed to categorical products and full subgraphs): A \underline{C} -graph (i.e. a digraph $A \in \underline{C}$) is SI iff for every full subgraph $m : A \rightarrow \prod_{i=1}^n A_i$ such that all $p_i m$ are onto (p_i are projections) at least one $p_i m$ is an isomorphism. (This is a special case of the general categorical definition of a SI object - see e.g. [PV].)

One can see easily that under the assumption of pro-

ductivity and hereditariness of a class \underline{C} , every

$$\underline{CX} = (\{R \subset X^2 ; (X,R) \in \underline{C}\}, \cap)$$

is a complete meet semilattice. A digraph $A = (X,R)$ is

called meet irreducible (MI) iff for $R = \bigcap_{i=1}^n R_i$ at

least one $R_i = R$. One can see easily (cf. [PV]) that every

SI is MI.

Now, three types of dimensions based on MI and SI can be defined : Let $A = (X,R)$ be an object of \underline{C} . Then

a meet dimension $m\text{-dim}_{\underline{C}}(X,R) = \min \{n ; \exists R_1, \dots, R_n,$

(X_i, R_i) are MI for $i=1, \dots, n$ and $R = \bigcap_{i=1}^n R_i\}$, a pro-

duct dimension $p\text{-dim}_{\underline{C}} A = \min \{n ; A$ is a full subgraph

of $\prod_{i=1}^n A_i$ with A_i SI $\}$, and a subdirect dimension

$s\text{-dim}_{\underline{C}} A = \min \{n ; A$ is a full subgraph of $\prod_{i=1}^n A_i$

with A_i SI and $p_i m$ onto (p_i are projections, m is an

embedding) $\}$, i.e. $s\text{-dim}$ is the smallest number of

factors in a subdirect representation of A .

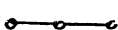
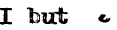
Remark. The original Dushnik - Miller dimension was $m\text{-dim}$

of posets. The product dimension of graphs was studied

L. Lovász, J. Nešetřil, A. Pultr etc. (see e.g. [LNF], [N₂,

[Tr₁], [Tr₂]).

As we mentioned, for \underline{C} a class of reflexive

posets, there is $p\text{-dim}_{\underline{C}} \equiv m\text{-dim}_{\underline{C}}$ (and also $\equiv s\text{-dim}_{\underline{C}}$). Another example is the class of all the antireflexive antisymmetric digraphs (where $m\text{-dim}_{\underline{C}} \equiv p\text{-dim}_{\underline{C}} \equiv s\text{-dim}_{\underline{C}} \leq 2$). But in the general case, these three dimensions can be different. One can see easily that $p\text{-dim}_{\underline{C}} \equiv s\text{-dim}_{\underline{C}}$ iff the subdirect irreducibility is hereditary in \underline{C} . (An example of non-validity of this equality are bipartite graphs where  is SI but  is not.)

Notation. Denote \underline{P} a class of all the antireflexive posets, \underline{Q} a class of all the digraphs (X, R) such that $\text{card}(R \cap R^{-1}) \leq 1$ and if (x, y) and $(y, x) \in R$ then $\{(x, z), (z, x)\} \cap R \neq \emptyset$ implies $z = x$. (Actually, \underline{Q} contains antireflexive antisymmetric digraphs with possible one isolated loop added.)

2. Digraphs

Definition. A class \underline{C} of digraphs is called trivial if every \underline{C} -graph has at most one vertex.


The aim of this chapter is to prove the following


Theorem 1. Let \underline{C} be a productive hereditary class of digraphs. If $p\text{-dim}_{\underline{C}} \equiv m\text{-dim}_{\underline{C}} \equiv s\text{-dim}_{\underline{C}}$ then either \underline{C} is trivial or $\underline{P} \subset \underline{C} \subset \underline{Q}$.

Lemma 1. If $s\text{-dim}_{\underline{C}} A \leq m\text{-dim}_{\underline{C}} A$ for any $A \in \underline{C}$ then \underline{C} contains no digraph with two loops.

Proof. Let G be a maximal \underline{C} -graph with two vertices and two loops.

Consider three cases :

1. $G = \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array}$. Then  is in \underline{C} , it is MI but it is not SI.

2. $G = \begin{array}{c} \rightarrow \\ \rightleftarrows \\ \rightarrow \end{array}$. Then  is in \underline{C} , it is MI but it is not SI.

3. $G = \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array}$. Then  is in \underline{C} , it is MI but it is not SI.

In all these cases an existence of an object which is MI but not SI contradicts the assumption $s\text{-dim}_{\underline{C}} \leq m\text{-dim}_{\underline{C}}$.

Lemma 2. If $s\text{-dim}_{\underline{C}} A \equiv m\text{-dim}_{\underline{C}} A$ for any $A \in \underline{C}$ then $\begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} \notin \underline{C}$.


Proof. Suppose $G = \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} \in \underline{C}$. Then

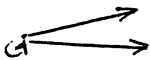
$$H = \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \in \underline{C}.$$

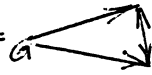

By Lemma 1, H is maximal hence MI. But on the other hand, H is a full subgraph of G^2 and therefore it is not SI which is a contradiction.

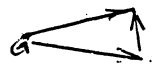

Lemma 3. If $s\text{-dim}_{\underline{C}} \equiv p\text{-dim}_{\underline{C}} \equiv m\text{-dim}_{\underline{C}}$ then $\begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \notin \underline{C}$, $\begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \notin \underline{C}$.


Proof. a/ Suppose $G = \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array}$, $H = \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \in \underline{C}$. Then

$K =$  is a full subgraph of $G \times H$ hence $K \in \underline{C}$ and it is not SI. But according to Lemma 1 and Lemma 2, K is MI which is a contradiction.

b/ Suppose $G \in \underline{C}$, $H \notin \underline{C}$. Let K be a maximal \underline{C} -graph containing G as a full subgraph. (Such a graph exists because e.g.  is a \underline{C} -graph containing G as a full subgraph.)

/i/ Suppose $K =$ . Then $L =$  is a full subgraph of $K \times G$ hence L is a subdirectly reducible \underline{C} -graph. But one can see easily (according to previous lemmas) that L is MI which is a contradiction.

/ii/ Suppose $K =$ . Then $M =$  is a full subgraph of $K \times G$ hence M is a subdirectly reducible \underline{C} -graph. According to the maximality of K and the assumption $H \notin \underline{C}$, M is MI.

/iii/ Suppose $K =$ . Then K is MI but it is not SI which is a contradiction.

c/ Using the same technique as in b/ one can prove that also under the assumption $G \notin \underline{C}$, $H \in \underline{C}$ one obtains a contradiction.

Proposition 1. Let \underline{C} be a productive hereditary class of digraphs. If $s\text{-dim}_{\underline{C}} \equiv p\text{-dim}_{\underline{C}} \equiv m\text{-dim}_{\underline{C}}$, $G = (X, R) \in \underline{C}$

then for every $Y \subset X$ such that $\text{card } Y = 3$ there is
 $\text{card}(R \cap Y \times Y) \leq 3$.

Proof.

1. If Y contains a loop vertex of G then the assertion follows from Lemmas 2 and 3.

2. Suppose there exists an antireflexive \underline{C} -graph with 3 vertices and more than 3 edges. Let G be a maximal \underline{C} -graph with these properties.

a/ $G = \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$. Then $H = \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$ is a subgraph of $G \times \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$ hence it is meet reducible and

$K = \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$ is a \underline{C} -graph. Therefore, $L = \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array} =$

$= \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array} \wedge \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array} \wedge \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array} \wedge \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$
 is a \underline{C} -graph, $m\text{-dim}_{\underline{C}} L = 4$ but L is a full subgraph of $G \times \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$ and hence $s\text{-dim}_{\underline{C}} L = 2$.

b/ $G = \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$. Then again $L = \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$ is a \underline{C} -graph and $m\text{-dim}_{\underline{C}} L = 4$, $s\text{-dim}_{\underline{C}} L = 2$.


c/ $G = \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$. Then $M = \begin{array}{c} \updownarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$ is a full subgraph of G^2 hence it is a subdirectly reducible \underline{C} -graph.

But according to the maximality of G , M is also maximal (and hence MI) which is a contradiction.

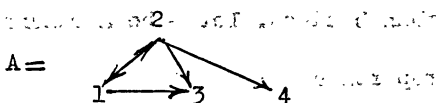
d/ $G = \begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$. Then $\begin{array}{c} \leftarrow \quad \rightarrow \\ \leftarrow \quad \rightarrow \end{array}$ is a full subgraph of

$G \times \longleftrightarrow$ hence it is a subdirectly reducible C-graph, it must be also meet reducible and therefore also



is a C-graph. By a similar technique, one can prove that  is a C-graph. Hence, all the tournaments with 3 vertices are meet reducible in C.

Denote $A = G \wedge B \wedge C$.





Since A is a full subgraph of G^2 it is a C-graph and $s\text{-dim}_{\underline{C}} A = 2$. Therefore, $m\text{-dim}_{\underline{C}} A = 2$ and $A = B \wedge C$.

According to a/, b/, c/ neither B nor C have edges $(3,1)$, $(3,2)$, $(4,2)$.

/i/ Suppose B has both edges $(3,4)$ and $(4,3)$. Then C contains none of edges $(3,4)$, $(4,3)$ and a subdirectly reducible two-point discrete graph is a full subgraph of C which is a contradiction.

/ii/ Suppose B has only one of edges $(3,4)$ and $(4,3)$. Then B contains a tournament with three vertices as a full subgraph which is a contradiction with the meet reducibility of all the tournaments with 3 vertices.

e/ For the case $G =$  or $G =$  one can use a similar technique as in d/.

Q.E.D.

Proposition 2. Let \underline{C} be a productive hereditary class of digraphs. If $s\text{-dim}_{\underline{C}} \equiv p\text{-dim}_{\underline{C}} \equiv m\text{-dim}_{\underline{C}}$ then every \underline{C} -graph is antisymmetric.

Proof. Suppose the contrary. Then \underline{C} contains a symmetric graph G with two vertices. By Lemma 1 and Lemma 2,

$G = \longleftrightarrow$. Take a maximal \underline{C} -graph H with three vertices containing G as a full subgraph.

Consider two cases :

a/ H has a loop. According to Proposition 1, $H = \updownarrow \circ$.

Take a maximal \underline{C} -graph K with 4 vertices containing H as a full subgraph. According to Proposition 1, the fourth vertex of K cannot be connected with both vertices of the symmetric edge by an edge. Hence, H contains a discrete graph D_2 with two vertices as a full subgraph which is a contradiction with the assumption $p\text{-dim}_{\underline{C}} \equiv s\text{-dim}_{\underline{C}}$ (because $\cdot \cdot = \longleftrightarrow \times \cdot$).

b/ H has no loop. Then H contains D_2 as a full subgraph and it is a contradiction with the assumption $p\text{-dim}_{\underline{C}} \equiv s\text{-dim}_{\underline{C}}$. Q.E.D.

Proposition 3. Let \underline{C} be a productive hereditary class of digraphs. If $s\text{-dim}_{\underline{C}} \equiv p\text{-dim}_{\underline{C}} \equiv m\text{-dim}_{\underline{C}}$ then either \underline{C} is trivial, or $\underline{C} \rightarrow \underline{P}$.

Proof. Suppose that \underline{C} is not trivial. Since \underline{C} is productive,

and hereditary, it suffices to prove that \underline{C} contains all the antireflexive linear orderings L_1, L_2, \dots

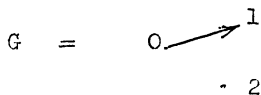
Suppose that there exists an n such that $L_n \in \underline{C}$, $L_{n+1} \notin \underline{C}$. Consider three cases :

1/ $n=0$. Then according to Proposition 2, \underline{C} contains no digraphs with proper edges. Since \underline{C} is not trivial, there are the following possibilities :

a/ $\underline{C} = \underline{SET}$ (the system of all the discrete graphs). But for D_3 (a discrete graph with three vertices) there is $s\text{-dim}_{\underline{C}} D_3 = 2$, $m\text{-dim}_{\underline{C}} D_3 = 1$ which is a contradiction.

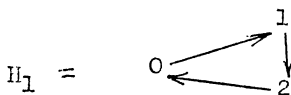
b/ $\underline{C} = \underline{SET}_2$ (the system of all the digraphs with at most one loop and with no proper edge). But then $s\text{-dim}_{\underline{C}} 2 = 2$, $m\text{-dim}_{\underline{C}} 2 = 1$ which is a contradiction.

2/ $n=1$. Take

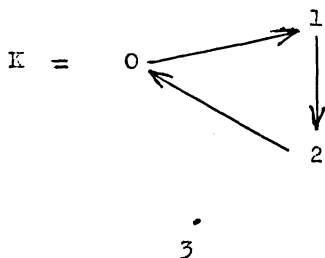


Then $s\text{-dim}_{\underline{C}} G = 2 \Rightarrow m\text{-dim}_{\underline{C}} G = 2$. Therefore, there exist

III \underline{C} -graphs H_1, H_2 such that $G = H_1 \wedge H_2$. We can suppose that H_1 has no loop. Then



Take



Then $s\text{-dim}_{\underline{C}} K = 2 \Rightarrow m\text{-dim}_{\underline{C}} K = 2$ and there exist MI \underline{C} -graphs K_1, K_2 such that $K = K_1 \wedge K_2$. We can assume that K_1 has no loop. Since $s\text{-dim}_{\underline{C}} \equiv p\text{-dim}_{\underline{C}}$ is supposed and D_2 is not SI, K_1 has to be a tournament. But there is no tournament with 4 vertices which does not contain L_2 as a full subgraph; it is a contradiction.

/3/ $n \geq 2$. Then take $G = L_{n+1} \setminus \{(n-1, n), (n, n+1)\}$. G is a full subgraph of L_n^2 ; hence, G is a \underline{C} -graph and $s\text{-dim}_{\underline{C}} G = 2$. Suppose $m\text{-dim}_{\underline{C}} G = 2$, $G = G_1 \wedge G_2$ where G_1 and G_2 are MI \underline{C} -graphs. Since $p\text{-dim}_{\underline{C}} \equiv s\text{-dim}_{\underline{C}}$, neither G_1 nor G_2 contains D_2 as a full subgraph. Every vertex of G is an initial or an end vertex of some edge; thus, neither G_1 nor G_2 contains a loop. Hence, G_1 and G_2 are tournaments. Since $G_1 \neq L_{n+1}$, $(n-1, n)$ and $(n, n+1)$ are edges of G_1 and $(n-1, n), (n, n+1)$ are edges of G_2 . Thus, $G_2 = L_{n+1}$ which is a contradiction. Q.E.D.

This finishes also the proof of Theorem 1.

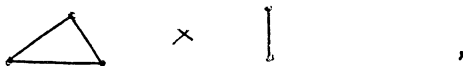
3. Undirected graphs

In this part, we are going to study dimensions in subclasses of a class \underline{G} of all the undirected graphs without loops.

Denote \underline{G}_1 the system of all the graphs of a degree less or equal to 1.

Proposition 4. Let \underline{C} be a productive hereditary subclass of \underline{G} . Then $s\text{-dim}_{\underline{C}} A \geq m\text{-dim}_{\underline{C}} A$ for every $A \in \underline{C}$ iff either \underline{C} is trivial, or $\underline{C} = \underline{SET}$, or $\underline{C} = \underline{G}_1$.

Proof. 1. Suppose that $\triangle \in \underline{C}$ or $\begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \in \underline{C}$.
Since $\begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array}$ is a full subgraph of



in both these cases $\begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \in \underline{C}$ and $m\text{-dim}_{\underline{C}} D_3 = 3$ while $s\text{-dim}_{\underline{C}} D_3 = 2$ which is a contradiction.

Hence, either \underline{C} is trivial, or $\underline{C} = \underline{SET}$, or $\underline{C} = \underline{G}_1$.

2. a/ If $\underline{C} = \underline{SET}$ then $m\text{-dim}_{\underline{C}} \equiv 1$.

b/ \underline{G}_1 has only two SI graphs : \bullet and $\begin{array}{c} | \\ \bullet \end{array}$.

III \underline{G}_1 -graphs are just graphs with $2n$ vertices and $n-1$ or n edges and graphs with $2n+1$ vertices and n edges.

One can see easily that $m\text{-dim}_{\underline{C}} A \leq s\text{-dim}_{\underline{C}} A$ for every $A \in \underline{C}$.

Theorem 2. Let \underline{C} be a productive hereditary subclass of \underline{G} . Then $s\text{-dim}_{\underline{C}} \equiv m\text{-dim}_{\underline{C}}$ iff \underline{C} is trivial.

Proof follows directly from Proposition 4 because SET and \underline{G}_1 does not satisfy the condition $s\text{-dim}_{\underline{C}} \equiv m\text{-dim}_{\underline{C}}$.

Theorem 3. Let \underline{C} be a productive hereditary subclass of \underline{G} . Then $p\text{-dim}_{\underline{C}} \equiv m\text{-dim}_{\underline{C}}$ iff \underline{C} is trivial.

Proof. Suppose $p\text{-dim}_{\underline{C}} \equiv m\text{-dim}_{\underline{C}}$. Then $m\text{-dim}_{\underline{C}} A \leq s\text{-dim}_{\underline{C}} A$ for every $A \in \underline{C}$. According to Proposition 4, there are three possibilities :

/i/ \underline{C} is trivial - the assertion holds trivially.

/ii/ $\underline{C} = \text{SET}$. Then $m\text{-dim}_{\underline{C}} D_3 = 1$ while $p\text{-dim}_{\underline{C}} D_3 = 2$ which is a contradiction.

/iii/ $\underline{C} = \underline{G}_1$. Then $m\text{-dim}_{\underline{C}} \int \epsilon = 1$ while

$p\text{-dim}_{\underline{C}} \int \epsilon = 2$ which is a contradiction. Q.E.D.

For a graph G denote (similarly as in [NP₁]) $SP(G)$ the system of all the full subgraphs of G^k where k is a non-negative integer. Denote by K_n the complete (anti-reflexive) graph with n vertices.

Theorem 4. Let \underline{C} be a productive hereditary subclass of \underline{G} .

Then $s\text{-dim}_{\underline{C}} \equiv p\text{-dim}_{\underline{C}}$ iff either $\underline{C} = \underline{\text{SET}}$, or $\underline{C} = \text{SP}(K_n)$ for some n .

Proof. If $\underline{C} = \underline{\text{SET}}$ or $\underline{C} = \text{SP}(K_n)$ then the assertion holds. If $\underline{C} \neq \text{SP}(K_n)$ then there exists a SI \underline{C} -graph which contains D_2 as a full subgraph. Since $s\text{-dim}_{\underline{C}} \equiv p\text{-dim}_{\underline{C}}$, D_2 is SI. Hence, \underline{C} does not contain --- and $\underline{C} = \underline{\text{SET}}$. Q.E.D.

R e f e r e n c e s

- [DM] Ben Dushnik and E.W. Miller : Partially ordered sets, Amer.J. of Math. 63 (1941) , 600-610
- [LNP] L.Lovász, J.Nešetřil and A.Pultr : On a product dimension of graphs, to appear in J. of Comb. Theory B
- [NP₁] J.Nešetřil and A.Pultr : On classes of relations and graphs determined by subobjects and factorobjects, Discr. Math. 22 (1978), 287-300
- [NP₂] J.Nešetřil and A.Pultr : A Dushnik-Miller type dimension of graphs and its complexity, Proc. of the Conf. on Fundamentals of Computation Theory, Lect. Notes in Comp.Sci. 56, Springer 1977, 482-493

- [O] O.Ore : Theory of graphs, AMS Colloq.Publ. Vol.
XXXVIII, Providence, Rhode Island, 1962
- [P] A.Pultr : On productive classes of graphs determined by prohibiting given subgraphs,
Colloq.Math.Soc.J.Bolyai 18(1976),
805-819
- [PV] A.Pultr and J.Vinárek : Productive classes and subdirect irreducibility, in particular for graphs, Discr. Math. 20(1977),
159-176
- [Tr₁] W.T.Trotter, Jr. : A note on Dilworth's embedding theorem, Proc. of the AMS 52 (1975),
33-39
- [Tr₂] W.T.Trotter, Jr. : Embedding finite posets in cubes, Discr.Math. 12 (1975) ,165-172