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In: Zdeněk Frolík (ed.): Abstracta. 9th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1981. pp. 56–63.

Persistent URL: <http://dml.cz/dmlcz/701225>

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NINTH WINTER SCHOOL ON ABSTRACT ANALYSIS (1981)

DENSITY THEOREMS FOR MEASURABLE TRANSFORMATIONS

Ryszard Grzaślewicz

1. Introduction

Let m denote Lebesgue measure on Borel σ -algebra of the unit interval $[0,1]$. A function $\tau: [0,1] \rightarrow [0,1]$ which is Borel-measurable and nonsingular (i.e. $m(A)=0 \Rightarrow m\tau^{-1}(A)=0$) is called a transformation. We identify transformations which differ only on a set of measure zero. A transformation τ is called invertible if τ^{-1} exists and is also a transformation. τ is called measure-preserving if $m\tau^{-1}(A)=m(A)$ for all Borel A . The group of all invertible transformations is denoted by G . By G_m we denote the group of all invertible measure-preserving transformations.

Every invertible transformation τ induces a positive invertible isometry $T_\tau^{(p)}$ of $L^p(m)$, $1 \leq p < \infty$, defined by

$$T_\tau^{(p)}(f)(t) = \omega_\tau^{1/p}(t) f(\tau^{-1}(t)) ,$$

where $f \in L^p(m)$, $\omega_\tau = dm\tau^{-1} / dm$. If $\tau \in G_m$, then $\omega_\tau = 1$.

By a classical result (see e.g. Ionescu Tulcea [2], footnote 3), for every $1 \leq p < \infty$ we can identify G with

the group $G^{(1)}$ of all positive invertible isometries of $L^p(m)$ (i.e. with the set of all Banach lattice automorphisms of $L^p(m)$). Therefore we can define a topology in G as the strong operator topology inherited from $L(L^p(m))$. For all $1 \leq p < \infty$ these topologies coincide (Choksi, Kakutani [1], Theorem 8). Moreover, L^p - strong and weak operator topologies in G_m coincide, since the strong and weak topologies on the unitary group in $L(L^2(m))$ are the same and all L^p - weak operator topologies coincide on the compact set of doubly stochastic operators. It is not hard to see that the family of sets of the form

$$\{\tau \in G: m(\tau(A_i) \Delta B(A_i)) < \varepsilon \text{ for } i=1, \dots, n \text{ and } \|\omega_\tau - \omega_5\|_1 < \varepsilon\}$$

where $\varepsilon > 0$, $B \in G$ and A_1, \dots, A_n is a partition of $[0, 1]$ into subintervals is a neighborhood base for the strong operator topology in G .

In this paper we prove that the group G_m and G are topologically finitely generated (Theorem 1 and 2).

2. Invertible measure-preserving transformations.

We will use following property of permutations

Lemma1. Let n be a natural number. The group of all permutations of $\{1, \dots, 2n\}$ is generated by the following two elements

$$\alpha = \begin{pmatrix} 1, 2, \dots, 2n \\ 2, 3, \dots, 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1, 2, \dots, n-1, n, n+1, \dots, 2n \\ 2, 3, \dots, n, 1, n+1, \dots, 2n \end{pmatrix}$$

Proof. Since every permutation can be decomposed into transpositions, it suffices to show that α and β generate every transposition. Moreover, because of the nature of α and β it is enough to prove that some transposition, e.g.

$$\gamma = \begin{pmatrix} 1, 2, \dots, n, n+1, \dots, 2n \\ 1, 2, \dots, 2n, n+1, \dots, n \end{pmatrix}$$

can be expressed as a composition of α and β . In fact it is not hard to see that $\gamma = \alpha^{n-1} \beta \alpha^n \beta$.

For $a \in [0, 1]$ we write $\alpha_a(t) = t + a \pmod{1}$.

Moreover, we define

$$\beta_a(t) = \begin{cases} t + a \pmod{1/2} & \text{for } 0 \leq t < 1/2 \\ t & \text{for } 1/2 \leq t \leq 1 \end{cases}.$$

Obviously $\alpha_a, \beta_a \in G_m$.

Theorem 1. Let a and b be irrational numbers. Then the group generated by α_a and β_b is dense in G_m .

Proof. It is easy to see that for every real number c , the transformations α_c and β_c belong to the closure \mathcal{H}_m in G_m of the

group generated by α_a and β_b .

Now given $n \in \mathbb{N}$, we partition $[0,1]$ into $2n$ subintervals of equal length. It is sufficient to show that \mathcal{H}_m contains every piecewise linear transformation ξ which permutes these subintervals. From Lemma 1 we can express ξ as a certain composition of transformations α_c and β_c for $c=1/2n$.

3. Invertible transformations.

Let I_1, \dots, I_n and J_1, \dots, J_n be partitions of the interval $[0,1]$ into subintervals. The notation $\varphi: I_1 \rightarrow J_1, \dots, I_n \rightarrow J_n$ will mean that φ is the piecewise linear transformation that maps I_i linearly (with positive slope) onto J_i for all $i \leq n$.

Now for $a \in [0,1]$ and $b > 0$ we define $\varphi_{a,b}: [0, a/(b+1)] \rightarrow [0, ab/(b+1)], (a/(b+1), a] \rightarrow (ab/(b+1), a], (a,1] \rightarrow (a,1]$. Note that the first interval is stretched and the second is shrunk by the factor of b .

Let \mathcal{H} denote the group generated by G_m and Ψ where $\Psi = \varphi_{1/4, 2} \circ \alpha_{1/2} \circ \varphi_{1/4, 3} \circ \alpha_{1/2}$:
 $[0, 1/12] \rightarrow [0, 1/6], [1/12, 1/4] \rightarrow [1/6, 1/4],$
 $[1/4, 1/2] \rightarrow [1/4, 1/2], [1/2, 9/16] \rightarrow [1/2, 11/16],$
 $[9/16, 3/4] \rightarrow [11/16, 3/4], [3/4, 1] \rightarrow [3/4, 1].$

Lemma 2. Let $a \in [0, 1]$. Then $\varphi_{a,2}$, $\varphi_{a,3}$ belong to \mathcal{H} .

Proof. We may assume that $a \leq 1/12$ (since $G_m \subset \mathcal{H}$, several conjugates of φ can be composed together, if necessary). Let $\xi \in G_m$ be defined by ξ :

$$[0, 1/6] \rightarrow [1/12 + 2a], [2a + 1/4, 1/6, 2a + 1/4] \rightarrow [0, 2a + 1/12],$$

$$[2a + 1/4, 1/2] \rightarrow [2a + 1/4, 1/2], [1/2, 11/16] \rightarrow [9/16, 3/4],$$

$$[11/16, 3/4] \rightarrow [1/2, 9/16], [3/4, 1] \rightarrow [3/4, 1].$$

The transformation $\varphi = \psi \xi \psi$ transforms linear intervals $I_1 = (1/12 - a, 1/12)$ and $I_2 = (1/4, 1/4 + 2a)$ onto $(1/4, 2a + 1/4)$ and $(1/6, a + 1/6)$, respectively. It is easy to check that for all intervals I with $I \cap (I_1 \cup I_2) = \emptyset$ we have $m(\varphi(I)) = m(I)$. This implies the existence of two transformations $\xi_1, \xi_2 \in G_m$ such that $\varphi_{a,2} = \xi_1 \varphi \xi_2$. Therefore we obtain $\varphi_{a,2} \in \mathcal{H}$.

By analogous arguments, $\varphi_{a,3} \in \mathcal{H}$.

Lemma 3. If $\varphi_{a,b}$, $\varphi_{a,c} \in \mathcal{H}$ for all $a \in [0, 1]$ and some $b, c > 0$, then $\varphi_{a,bc} \in \mathcal{H}$ for all $a \in [0, 1]$.

Proof. Let $\delta > 0$ be such that $\delta(b+1)(c+1) \leq 1/2$. We put $\eta = \varphi_{b+1,b} \circ \alpha_{1/2} \circ \varphi_{b,c+1,c} \circ \alpha_{1/2}$:

$$[0, \delta] \rightarrow [0, \delta b], [\delta, \delta(b+1)] \rightarrow [\delta b, \delta(b+1)],$$

$$[\delta(b+1), 1/2] \rightarrow [\delta(b+1), 1/2], (1/2, \delta b + 1/2] \rightarrow (1/2, \delta b c + 1/2].$$

$$(\delta b + 1/2, b(c+1) + 1/2] \rightarrow (\delta bc + 1/2, b(c+1) + 1/2],$$

$$(\delta b(c+1) + 1/2, 1] \rightarrow (\delta b(c+1) + 1/2, 1]. \text{ We have } .$$

Let now $\xi \in G_m$ be defined by $\xi : [0, \delta b) \rightarrow [1/2, \delta b + 1/2)$,

$$[\delta b, \delta(b+1)) \rightarrow [0, \delta], [\delta(b+1), 1/2) \rightarrow [\delta(b+1), 1/2],$$

$$[1/2, \delta bc + 1/2) \rightarrow [\delta b + 1/2, \delta b(c+1) + 1/2),$$

$$[\delta bc + 1/2, \delta b(c+1)) \rightarrow [\delta, \delta(b+1)),$$

$$[\delta b(c+1) + 1/2, 1] \rightarrow [\delta b(c+1) + 1/2, 1]. \text{ The transforma-}$$

tion $\varphi = \eta \circ \xi \circ \eta$ transforms $I_1 = (0, \delta]$ and $I_2 = (\delta b + 1/2, \delta b(c+1) + 1/2]$ onto $(1/2, \delta bc + 1/2]$ and $(\delta b, \delta(b+1)]$ respectively, and

for intervals I with $I \cap (I_1 \cup I_2) = \emptyset$ we have $m(\eta \circ \xi \circ \eta(I)) = m(I)$.

Hence $\varphi_{a,bc} = \delta \eta \tau$ for some $\delta, \tau \in G_m$ and we obtain $\varphi_{a,bc} \in \mathcal{H}$.

Corollary. The closure of \mathcal{H} contains all $\varphi_{a,b}$ for $0 \leq a \leq 1$ and $b > 0$.

Proof. The transformation $\varphi_{a,b}$ belongs to \mathcal{H} if and only if $\varphi_{a, 1/b}$ belongs to \mathcal{H} since G_m . Therefore using Lemma 2 and Lemma 3 we obtain that $\varphi_{a,b} \in \mathcal{H}$ for $b = 2^k / 3^m$ with $k, m \in \mathbb{N}$. Because the set $\{2^k / 3^m : k, m \in \mathbb{N}\}$ is dense in \mathbb{R}_+ the proof is complete.

The following Proposition is implicitly contained in [3]; we omit the proof.

Proposition. Let D be a dense subset of $[0, 1]$. Then the family of all invertible transformations τ of the form $\tau : I_1 \rightarrow J_1, \dots, I_n \rightarrow J_n$, where (I_k) and (J_k) are partitions of $[0, 1]$ into subintervals with endpoints in $D \cup \{0, 1\}$, is dense in G .

Theorem 2. Let α, β be irrational numbers. Then the group generated by α, β and ψ is dense in G .

Proof. In view of Theorem 1 and Proposition , it is sufficient to show that for every partitions $0=a_0 < a_1 < \dots < a_{n+1}=1$ and $0=b_0 < b_1 < \dots < b_{n+1}=1$ with a_i, b_i of the form $2^k/3^m$ for $1 \leq i \leq n$ there exists a transformation in \mathcal{H} which takes $[a_i, a_{i+1})$ linearly onto $[b_i, b_{i+1})$ for $i=1, \dots, n$.

By the last corollary, we may assume that $a_n \leq 1/4$ and $b_n \leq 1/4$.

Now $\xi_1 = \varphi_{a_1+b_1, b_1/a_1}$ maps $[0, a_1)$ onto $[0, b_1)$. The function $\varphi: [0, b_1) \rightarrow [0, b_1)$, $[b_1, \xi_1(a_2)) \rightarrow [b_1, b_2)$, $[\xi_1(a_2), \xi_1(a_2)+b_2-b_1) \rightarrow [b_2, \xi_1(a_2)+b_2-b_1)$, $[\xi_1(a_2)+b_2-b_1, 1] \rightarrow [\xi_1(a_2)+b_2-b_1, 1]$ clearly satisfies $\varphi = \sigma \varphi_{x,y} \tau$ for some $\sigma, \tau \in \mathcal{H}_m$ and $x, y \in \mathbb{R}_+$ and so $\varphi \in \mathcal{H}$. Therefore $\xi_2 = \varphi \xi_1 \in \mathcal{H}$ and it is easy to see that ξ_2 takes $[a_1, a_{i+1})$ onto $[b_1, b_{i+1})$ for $i=0, 1$. Continuing this process by induction, we can construct a transformation $\xi_n \in \mathcal{H}$ such that ξ_n takes $[a_i, a_{i+1})$ onto $[b_i, b_{i+1})$ for $i=0, 1, \dots, n$.

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