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ON THE REPRESENTATION OF ORLICZ LATTICES

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1. Definition. A function $\varrho : L \to [0,\infty)$, where L is a vector lattice, is called a modular, if it satisfies the following conditions :

(ϱ1) $\varrho(x) = 0$ if and only if $x = 0$,

(ϱ2) $|x| \le |y|$ implies $\varrho(x) \le \varrho(y)$,

(ϱ3) $x \perp y$ implies $\varrho(x+y) = \varrho(x) + \varrho(y)$,

(ϱ4) $\varrho(\alpha x + \beta y) \le \varrho(x) + \varrho(y)$ whenever $\alpha, \beta \ge 0$, $\alpha + \beta = 1$,

(ϱ5) $0 \le x_n \uparrow x$ implies $\varrho(x_n) \uparrow \varrho(x)$,

(ϱ6) $\varrho(\lambda x) \to 0$ whenever $\lambda \to 0$ ($\lambda \in [0,\infty)$) .

A modular ϱ defines a monotone norm on L (the so-called Musielak -Orlicz norm) as follows :

$$|x|_\varrho = \inf \{ \varepsilon > 0 : \varrho(\frac{x}{\varepsilon}) \le \varepsilon \}$$

It is a well known fact that $|x_n - x|_\varrho \to 0$ if and only if $\forall \lambda > 0 \ \varrho(\lambda(x_n - x)) \to 0$.

2. Definition. If ϱ is a modular on L and L is complete with respect to the F - norm $|\cdot|_\varrho$, then the pair $(L, |\cdot|_\varrho)$, which will be denoted by L^ϱ is called an Orlicz lattice.

Let (Ω, Σ, ν) be a finite positive measure space. A function $\Psi : R_+ \times \Omega \to R_+$ is called a (convex) Musielak - Orlicz function if :

1) $\Psi(r,s)$ is Σ - measurable for each r ,

and there exists a set A of Σ - measure zero such that for each $s \in \Omega \setminus A$

2) $\Psi(r,s) = 0$ iff $r = 0$,

3) $\Psi(r,s)$ is monotone and left - continuous with respect to the first variable ,

(3') $\Psi(r,s)$ is a convex function with respect to the first variable)

4) $\Psi(r,s) = 0$ for each $s \in A$ and each $r \in R_+$.

Let $\mathfrak{F} = \{ x :$ x is a real Σ - measurable function $\}$, and let Ψ be a (convex) Musielak - Orlicz function. Then the function $M : L^\Psi_a \to [0,\infty)$ ($L^\Psi \to [0,\infty)$) given by $M(x) = \int_\Omega \Psi(|x(s)|,s)\, d\nu$ is a(convex) modular where $L^\Psi = \{ x \in \mathfrak{F} : \exists \alpha > 0 \int_\Omega \Psi(\alpha|x(s)|,s) d\nu < \infty \}$ and $L_a = \{ x \in \mathfrak{F} : \forall \alpha > 0 \int_\Omega \Psi(\alpha|x(s)|,s) \ldots$

The space $(L^\Psi, \|\cdot\|_M)$, where $\|\cdot\|_M$ is an Orlicz norm, is an Orlicz lattice for the ordering $x \le y$ if $x(s) \le y(s)$ ν - a.e.

An element e of a vector lattice L is called a weak unit of L if $e \in L$ and $e \wedge |x| = 0$ imply $x = 0$.
Two F - lattices $(L_1, \|\cdot\|_1)$ and $(L_2, \|\cdot\|_2)$ are said to be isometrically lattice isomorphic (denoted $L_1 \cong L_2$) if there exists a linear isometry (onto) $T : L_1 \to L_2$ such that T is a lattice isomorphism.

Theorem 1.
Every Orlicz lattice L^{ϱ} with a weak unit is isometrically lattice isomorphic to some space $L_a^{\Psi}(S, \Lambda, \mu)$ where S is a compact space and μ is finite.

The above theorem applies evidently to the case when the Musielak - Orlicz function in question is is u^p for $0 < p < \infty$, that is , gives the representation of the standard lattices L^p for any finite measure (atomic or not) and for any p between zero and infinity , thus giving a generalisation of some classical results of Kakutani, Bohenblust, Claas - Zaanen , not necessarily for the Banach space case.

Indeed, it is not even necessary to request that the measure space be finite, i.e. it is superfluous to insist that the lattice have a weak unit. It is summarized in the following

Theorem 2.
Let L^{ϱ} be an Orlicz lattice . Then there exist a Musielak - Orlicz function Ψ and a measure space (S, Λ, μ) such that

$$L^{\varrho} \cong L_a(S, \Lambda, \mu) .$$

Further investigations have showed that the topological completion of a lattice with an F - norm generated by a modular is some Musielak - Orlicz space (i.e. the completeness assumption in Def. 2 is not necessary).
The second author has applied these theorems on the representation of Orlicz lattices in the investigations of the form of ultraproducts of some families of Orlicz spaces (cf. S.Heinrich, Ultraproducts in Banach space theory, J.Reine Angew. Math.) .

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