WSAA 9

Michel Talagrand

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In: Zdeněk Frolík (ed.): Abstracta. 9th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1981. pp. 173–175.

NINTH WINTER SCHOOL ON ABSTRACT ANALYSIS (1981)

Separation of orthogonal sets of measures

Michel Talagrand (Results of G.Mokobodzki)

Let K be a compact space. Let X be the set of positive measures on X of mass ≤ 1. A subset of X is said to be <u>measure-convex</u> if for each compact set $L \subset A$ and each Radon measure μ on L we have $\int x \, d\mu(x) \in A$.

A function $\varphi: [0,1]^{\mathbb{N}} \longrightarrow \mathbb{R}$ is called a <u>medial limit</u> if it is strongly affine (i.e. universally measurable, and $\varphi(\int x \, d\mu(x)) = \int \varphi(x) \, d\mu(x)$. for each Radon measure μ on $[0,1]^{\mathbb{N}}$) and if for each $x = (x_n) \in [0,1]^{\mathbb{N}}$, $\liminf x_n \leqslant \varphi(x) \leqslant \limsup x_n$. Mokobodzki proved that continuum hypothesis implies the existence of a medial limit [1]. (It is known now that Martin's axiom is enough to imply the result).

<u>Theorem</u> (Mokobodzki) . Assume there exists a medial limit. Then given two K-analytic measure convex sets $A, B \subset X$ which are orthogonal (i.e. $\mu \in A$, $\nu \in B \Rightarrow \mu \perp \nu$) there exists a universally measurable set $V \subset K$ with $\mu \in A \Rightarrow \mu(V) = 1$, $\mu \in B \Rightarrow \mu(V) = 0$.

Note that, as the result of D .Preiss shows, it is impossible in general to take V Borel.

<u>Proof</u>. We first think to X as a convex compact set of its own, forgetting about its special structure. For any set $A \subset X$, $x \in X$, let

$$\hat{1}_A(x) = \sup\{\mu^*(A), \delta_x < \mu\}$$

where δ_x is the Dirac measure in x, $\mu \in M^+(X)$, and $\prec$ is the Choquet order, i.e. $\delta_x \prec \mu \Leftrightarrow f(x) \le \mu(f)$ for each convex continuous f. It is classical that if A is compact

$$\hat{1}_A(x) = \operatorname{Inf}\{f(x) \,;\, f \text{ affine continuous, } 1_A \le f\} \quad (1)$$

Hence for each decreasing sequance (A_n) of compact sets,

$$\hat{1}_{\cap A_n}(x) = \operatorname*{Inf}_n \hat{1}_{A_n}(x) .$$

Now for $A,B \subset X$, let

$$\mathcal{C}'(A,B) = \operatorname*{Sup}_n \hat{1}_A(x) + \hat{1}_B(x) - 1 .$$

If $(A_n), (B_n)$ are two decreasing sequences of compact sets, $\mathcal{C}'(\cap A_n, \cap B_n) = \inf \mathcal{C}'_n(A_n,B_n)$.

For $A,B \subset X$, let

$$\mathcal{C}(A,B) = \operatorname{Inf}\{\varepsilon \,;\, \exists f \text{ strongly affine on } X \text{ with } 1_A \le f \quad 1 - 1_B + \varepsilon\} .$$

Let A_n, B_n be two increasing sequances of sets in X. For each n let $\varepsilon_n \le 2^{-n} + \mathcal{C}(A_n,B_n)$ such that

$1_{A_n} \le f_n < 1 - 1_{B_n} + \varepsilon_n$, where f_n is strongly affine.

Let $f(x) = \varphi((f_n(x)))$ where φ is a medial limit (note that $f_n(x) \in [0,1]$, $\forall_n, \forall x$). Then f is strongly affine, and $1_A \le f \le 1 - 1_B + \varepsilon$, where $\varepsilon = \limsup \varepsilon_n$. This proves that $\mathcal{C}(\cup A_n, \cup B_n) = \sup_n \mathcal{C}(A_n,B_n)$.

Now, suppose A,B compact. If $1_A \le f < 1 - 1_B + \varepsilon$ where f is strongly affine, from (1) we get $\hat{1}_A \le f < 1 - \hat{1}_B + \varepsilon$ so $\hat{1}_A + \hat{1}_B - 1 < \varepsilon$. Moreover, if $\hat{1}_A + \hat{1}_B - 1 < \varepsilon$, then $\hat{1}_A \le 1 - \hat{1}_B + \varepsilon$, and since $\hat{1}_A$ is concave u.s.c., $1 - \hat{1}_B + \varepsilon$ concave ℓ.s.c., the Hahn-Banach theorem shows that there exists an affine continuous f with , $1_A \leq f < 1 - \hat{1}_B + \varepsilon$. We have shown that $\mathcal{C}(A,B) = \mathcal{C}'(A,B)$.

We have shown that $\mathcal{C}(A,B)$ is a capacity. Let $A,B \subset X$ as in the statement. The capacitability theorem shows that

$$\mathcal{C}(A,B) = \operatorname{Sup}\{\mathcal{C}(A_1,B_1) \ , \ A_1 \subset A,\ B_1 \subset B,\ A_1,B_1 \text{ compact}\}.$$

Since A,B are measure convex, we can assume A_1,B_1 convex. Let $\varkappa \in X$. It is easy to see.that $\varkappa = ay + (1-a)y = by' + (1-b)y'$ where $a = 1_{\hat{A}_1}(\varkappa)$, $b = 1_{\hat{B}_1}(\varkappa)$, $y \in A_1$, $y' \in B_1$.

Now by hypothesis there exists a Borel set $V \subset K$ with $y(V) = 1$, $y'(V) = 0$. Hence we get $1-b \geqslant \varkappa(V) \geqslant a$ and so $a + b - 1 \leqslant 0$, that is $\mathcal{C}'(A,B) = \mathcal{C}(A_1,B_1) = 0$. Hence $\mathcal{C}(A,B) = 0$. Using again medial limits, we get a strongly affine f on X with $1_A \leq f \leq 1 - 1_B$. Let g on K given by $g(t) = f(\delta_t)$ for $t \in K$. Since f is strongly affine, for each measure μ on K , $f(\mu) = \int g(t)\, d\mu(t)$. It is clear now the universally measurable set $V = \{t \in K;\ g(t) = 1\}$ works.

[1] P.A.Meyer, Limites mediales, d'après Mokobodski Seminaire de Probabilités de Strasbourg,1971/72,Springer, Lecture Notes