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## On infinite-dimensional manifolds

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Let  $I$  denote the segment  $[-1,1]$ ,  $Q$  the Hilbert cube  $I^\infty$  and  $\ell_2$  the separable Hilbert space of square-summable sequences. In [4] and [5], the author has proved the following characterizations of spaces locally homeomorphic to  $Q$  or  $\ell_2$ , respectively:

1. A locally compact ANR-space,  $X$ , is a  $Q$ -manifold iff the following condition is satisfied for each  $n$ :

(\*)<sub>n</sub> any map  $f: I^n \times \{1,2\} \rightarrow X$  can arbitrarily closely be approximated by maps  $g$  with  $g(I^n \times 1) \cap g(I^n \times 2) = \emptyset$ .

2. A separable complete-metrizable ANR-space,  $X$ , is an  $\ell_2$ -manifold iff

(\*\*) given an open covering  $\mathcal{U}$  of  $X$  and a map  $f: D \rightarrow X$ , where  $D$  is the disjoint union  $I^0 \cup I^1 \cup \dots$ , there is a map  $g: D \rightarrow X$  such that  $\{g(I^n)\}_n$  is a discrete collection in  $X$  and  $g$  is  $\mathcal{U}$ -close to  $f$  (i.e.  $\{f(d), g(d)\}$  refines  $\mathcal{U}$ ,  $\forall d \in D$ ).

Here, we present an application of 2 to showing that certain topological groups are actually  $\ell_2$ -manifolds, and we report on some recent results of R.J.Daverman and J.Walsh related to result 1.

### § 1. Topological groups which are Hilbert manifolds.

T.Dobrowolski and the autor have jointly proved the following result:

Theorem ([3]). Let  $G$  be a metrizable topological group and  $X$  its separable complete-metrizable subspace which is multiplicative (i.e.  $1 \in X$  and  $xy \in X$  for  $x, y \in X$ ). In order that  $X$  be an  $\ell_2$ -manifold it suffices that  $X \in \text{ANR}$  and no neighbourhood of  $1$  in  $X$

be totally bounded in the right structure of  $G$ .

Combined with earlier known facts this shows the following:

Corollary 1. Let  $X$  be a complete-metrizable separable ANR.

If  $X$  admits a topological group structure then either this is a Lie group structure, and  $X$  is a finite-dimensional manifold, or  $X$  is an  $l_2$ -manifold.

Corollary 2. Let  $X$  be a separable closed convex subset of a Banach space (or of a  $B_0$ -space). Then,  $X$  is either homeomorphic to  $l_2$  or is locally compact (and then homeomorphic to one of the sets  $I^k \times R^l \times [0,1]^m$  where  $k \leq \infty$ ,  $l < \infty$ ,  $m \leq 1$  and  $\min(m,1)=0$ ; see [1]).

Question: Do the analogues of the above corollaries hold true for non-separable spaces  $X$ ? (C.f. the characterization of non-separable Hilbert manifolds in [5]).

Outline of the proof of the Theorem. Let  $d$  be a right-invariant metric for  $G$ . We fix  $\mathcal{U}$  and  $f: D \rightarrow X$  in (\*\*\*) and let

$$\alpha(x) = \sup \{ \text{dist}_d(x, X \setminus U) : U \in \mathcal{U} \} / 2, \quad x \in X,$$

$$D_k = \{ d \in D : \alpha f(d) \geq 1/k \}, \quad k = 1, 2, \dots$$

Using the fact that no neighbourhood of 1 in  $X$  is totally bounded in the metric  $d$  we construct sequences  $\{g_k: D \rightarrow X\}_{k \geq 0}$  and  $\{\varepsilon_k\}_{k \geq 0} \subset (0, \infty)$  so that the following conditions hold for  $k \geq 1$

$$(1)_k \quad g_k = f \text{ on } D \setminus D_{k+1} \text{ and } g_k = g_{k-1} \text{ on } D_{k-1};$$

$$(2)_k \quad d(g_k(I^n \cap D_k), g_k(I^m)) > \varepsilon_k \text{ for } m < n;$$

$$(3)_k \quad d(g_k, g_{k-1}) < \varepsilon_{k-1}/4$$

$(4)_k \quad \varepsilon_k < \min \{1/k, \varepsilon_{k-1}/4\} \quad (\text{Here, } \varepsilon_0 = 1).$

It is not difficult to see that  $g = \lim g_k$  is the map desired in  $(\#)$ . (Hint: First check that  $d(g(x), f(x)) < \alpha f(x)$  for  $x \in D$ . To show that  $\{g(I^n)\}_{n \geq 0}$  is discrete assume that  $(g(x_i))_1$  converges to a point  $y \in X$  and distinct  $x_i$ 's belong to distinct cells in  $D$ . Then  $\inf \alpha f(x_i) > 0$  - for otherwise  $(f(x_i))$  would contain a sub-sequence converging to  $y$ , yielding  $\alpha(y) = 0$ . Thus there is a  $k \in \mathbb{N}$  with  $\{x_i\}_{i=0}^\infty \subset D_k$  and  $(1)$  and  $(2)_k$  yield  $d(g(x_i), g(x_j)) \geq \varepsilon_k$  for  $i \neq j$ . This contradicts the assumed convergence of  $(g(x_i))$ .

The construction of the  $g_k$ 's and  $\varepsilon_k$ 's (outline). Assume for simplicity that  $D_1 = D$ . In this case the sequences will terminate after the first step, which is as follows, Let  $\varepsilon_1$  be so small that no compact set in  $G$  is an  $\varepsilon_1$  - net in

$B = \{x \in X : d(x, 1) < 1\}$ . We let  $g_1 \upharpoonright I^0 = f \upharpoonright I^0$  and, if

$g_1 \upharpoonright I^0 \cup \dots \cup I^n$  is already defined, we select  $p \in B$  with

$$d(p, x) > \varepsilon_1 \text{ for } x \in \{ab^{-1} : a, b, \in g_1(I^0 \cup \dots \cup I^n) \cup f(I^{n+1})\}$$

and we put  $g_1(x) = pf(x)$  for  $x \in I^{n+1}$ . In this way we inductively define  $g_1 \upharpoonright I^n$  so that the resulting map  $g_1$  satisfies  $(2)_1$ .

The general case (where no  $D_1$  equals  $D$ ) is technically more involved but follows the same idea. See [3] for details.

§ 2. Homology characterizations of  $\mathcal{Q}$ -manifolds. R.J.Daverman and J.Walsh have recently showed that, in the result 1, the "disjoint  $n$ -cube property" of  $(\#)_n$  can for  $n > 2$  be replaced by a disjointness property in homologies. To be more specific let us

say that, whenever  $(U, V)$  is an open pair in  $X$  and  $\alpha \in H_{\mathbb{Z}}(U, V)$  (integer coefficients), a compact pair  $(A, B) \subset (U, V)$  is said to be a Čech carrier for  $\alpha$  iff any neighbourhood of  $(A, B)$  in  $(U, V)$  contains a cycle homologous to  $\alpha$  in  $(U, V)$ .

Theorem ([2]). Let  $X$  be a locally compact ANR. Then,  $X$  is a  $Q$ -manifold iff it satisfies  $(\mathfrak{K})_2$  and the following condition:

$(\mathfrak{K})'$  given open pairs  $(U_i, V_i)$  in  $X$  and relative cycles

$\alpha_i \in H_{\mathbb{Z}}(U_i, V_i)$ ,  $i = 1, 2$ , there are Čech carriers  $(A_i, B_i)$

for  $\alpha_i$  with  $A_1 \cap A_2 = \emptyset$ .

It is unknown whether  $(\mathfrak{K})'$  is satisfied by any infinite-dimensional homology manifold  $X$  (i.e., by any locally compact ANR such that

$H_{\mathbb{Z}}(X, X \setminus \{x\}) = 0$  for each  $x \in X$ ). It is easy to show that  $(\mathfrak{K})'$

is satisfied if the homology manifold has the property that any relative cycle in it admits a finite-dimensional Čech carrier; see

[2]. Also, property  $(\mathfrak{K})'$  is relatively easy to prove for certain CE-images of  $Q$ -manifolds, and thereby can be used to prove that these images are manifolds themselves. A sample application is:

Corollary ([2]). If  $X$  is a space such that  $X \times I^n \cong Q$  for some finite  $n$  then  $X \times I^2 \cong Q$ .

It is unknown whether, in the statement above,  $I^2$  can be replaced by  $I$ . This is related to the open problem whether  $X \times I$  satisfies  $(\mathfrak{K})_2$  for any infinite-dimensional homology manifold; this problem is of great interest also for <sup>homology</sup> manifolds of finite dimension  $n \geq 4$ . The author has recently observed that, at least,  $X \times D$  satisfies  $(\mathfrak{K})_2$  for any  $X$  as above and  $D$  a dendron with a dense set of separating points. Denoting by  $p : D \rightarrow I$  the natural retraction having  $\infty$  non-trivial point inverses, all of which are

dendra, one thus faces the following

Question: If  $X \times D \cong Q$ , is  $1_X \times p : X \times D \rightarrow X \times I$  approximable by homeomorphisms?

### References

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Editorial note:

This is an abstract of a talk presented by the author at the 8th Winter School (1980)