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Note on orthocomplemented posets II.

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NOTE ON ORTHOCOMPLEMENTED POSETS II.

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Abstract. Orthocomplemented posets are partially ordered sets with an antitone involution - the orthocomplementation. For some posets is up to an isomorphism only one orthocomplementation. We give an example of a finite lattice with two non-isomorphic orthocomplementations. The ortho-double is introduced and the orthoposets of order ≤ 10 are classified.

0. Boolean algebras are special distributive lattices. There is a natural non-distributive, non-lattice generalization of Boolean algebras: the orthocomplemented poset (orthoposet). We recall the definition.

$\mathcal{P} = (P, \leq, 0, 1, ^{\perp})$ is an order-theoretic object satisfying the following conditions:

1. $(P, \leq)$ is a partially ordered set (poset).

2. 0 (zero) is the smallest and 1 (unit) the greatest element in P with respect to $\leq$.

3. $^{\perp}: P \to P$ is a unary operation with following properties:

(i) $a^{\perp\perp} = a$ for all $a \in P$,

(ii) $a \leq b$ implies $b^{\perp} \leq a^{\perp}$,

(iii) for all $a \in P$, the supremum of a and its orthocomplement $a^{\perp}$ exists and is 1, $\sup(a, a^{\perp}) = a \vee a^{\perp} = 1$.

Dually to (iii) one has always $\inf(a, a^{\perp}) = a \wedge a^{\perp} = 0$.
Furthermore $0^{\perp} = 1 \quad 1^{\perp} = 0$.

$\mathcal{P}$ is called an ortholattice if the underlying poset is a lattice. In ortholattices the other two de Morgan laws are fulfilled:
$(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp} \quad (a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$.

1. The ortho-double of a poset

For any poset $\mathcal{P} = (P, \leq, 0, 1)$ with at least three elements it is possible to construct an orthoposet $d(\mathcal{P})$, which we will call the ortho-double of $\mathcal{P}$, in the following way. The underlying set of $d(\mathcal{P})$ is the quotient set of $P \times \{0,1\}$ with respect to the equivalence relation $\sim$ which only amalgamates the pairs (0,0),(1,1) on the one and (1,0), (0,1) on the other hand. The partial order in $P \times \{0,1\}/_\sim$ is stipulated by

$$\{(0,0),\ (1,1)\} \text{ is the zero } 0,$$
$$\{(1,0),\ (0,1)\} \text{ is the unit } 1,$$
$$(p,0) \leq (q,0) \text{ iff } p \leq q \text{ in } \mathcal{P},$$
$$(p,1) \leq (q,1) \text{ iff } q \leq p \text{ in } \mathcal{P}.$$

The orthocomplementation in $P \times \{0,1\}/_\sim$ is determined by

$$0^{\perp} = 1, \quad 1^{\perp} = 0$$
$$(p,0)^{\perp} = (p,1),\ (p,1)^{\perp} = (p,0).$$

A natural interpretation of the construction $\mathcal{P} \to d(\mathcal{P})$ is possible by taking the poset $\mathcal{P}$ and it inverse ordered exemplar $\mathcal{P}'$ and amalgamate these two exemplars at the bottom and the top. The orthocomplementation is tantamount to the reflection.

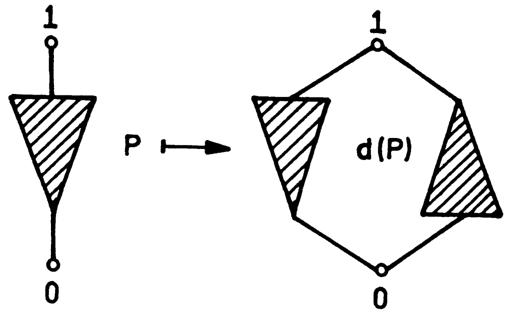


Figure 1

Proposition 1.1.
Let $\mathcal{P} = (P, \leq, 0, 1)$ be any poset with at least 3 elements.

1. The ortho-double $d(\mathcal{P})$ of $\mathcal{P}$ is isomorphic to the orthocomplemented poset of the left and right intervals of $\mathcal{P}$.
2. $d(\mathcal{P})$ is an ortholattice iff $\mathcal{P}$ is a lattice.
3. If $\mathcal{P}$ contains a chain $0 < p < q < 1$, then the ortho-double $d(\mathcal{P})$ fails to be orthomodular.

Proof.

1. The orthoposet of the left and right intervals of $\mathcal{P}$ is defined in [1]. (Clearly an isomorphism between two orthoposets is a bijective map which is an order isomorphism and preserves orthocomplementation). The wanted isomorphism is established by the following correspondence:

$$(p,0) \mapsto \,]\!]\leftarrow, p]\!] \qquad (0 \prec p < 1 \text{ in } \mathcal{P})$$
$$(p,1) \mapsto [\![p, \rightarrow[\![\qquad (0 \prec p < 1 \text{ in } \mathcal{P}).$$

2. The inner parts of $\mathcal{P}$ and of it inverse ordered counterpart $\mathcal{P}'$ are incomparable. This implies the lattice statement.

3. $q = p \vee (q \wedge p^{\perp})$ for every pair $p \leq q$ in some orthomodular poset. Now let $0 < p < q < 1$ in $\mathcal{P}$. Then $0 < (p,0) < (q,0) < 1$ in $d(\mathcal{P})$ and $(q,0) \wedge (p,0)^{\perp} = (q,0) \wedge (p,1) = 0$ in $d(\mathcal{P})$, consequently, $(p,0) \vee ((q,0) \wedge (p,0)^{\perp}) = (p,0) < (q,0)$ in $d(\mathcal{P})$, and $d(\mathcal{P})$ is not orthomodular. ◻

With the process of the ortho-double we are able to decide the following question:
Is there a finite poset with two non-isomorphic orthocomplementations? The positive answer is given in the following

Example 1.1.

We start with the ortho-double of the 8-element Boolean algebra. Fig. 2b presents the Hasse diagram of this ortho-double, which is not orthomodular. But Fig. 2a is the amalgamation of two Boolean algebras with unchanged complementation, which is an orthomodular ortholattice.
(Is 14 the smallest possible cardinality of such poset or is there an example of order 12?)

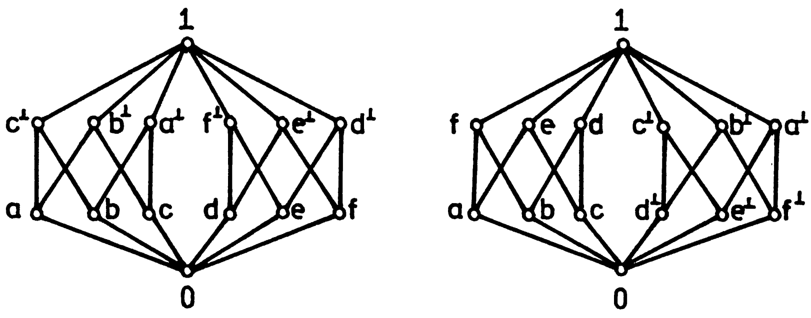


Figure 2a 2b

2. The finite orthocomplemented posets up to order 10

Proposition 2.1.

1. The order of every finite orthocomplemented poset is even, i.e. card $\mathcal{P} = 2n$, $n \geq 1$.

2. For every natural number n there exists orthocomplemented posets of order 2n.

Proof.

1. $p \neq p^{\perp}$ must hold for every $p \in \mathcal{P}$ because otherwise

$$1 = p \vee p^{\perp} = p$$
$$0 = p \wedge p^{\perp} = p.$$

$P = \bigcup \{p,p^{\perp}\} \vdots (p \in P)$ is a decomposition into disjoint pairs, since $\{p,p^{\perp}\} \cap \{q,q^{\perp}\} \neq \emptyset$ implies $p=q$ or $p=q^{\perp}$ and consequently $\{p,p^{\perp}\} = \{q,q^{\perp}\}$.

2. The Hasse diagram of figure 3a demonstrates the validity of the assertion. □

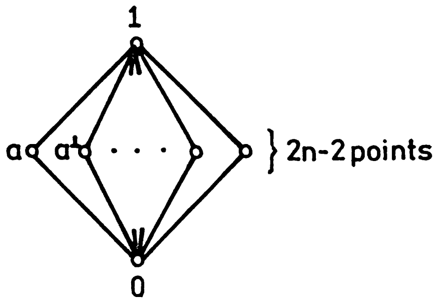
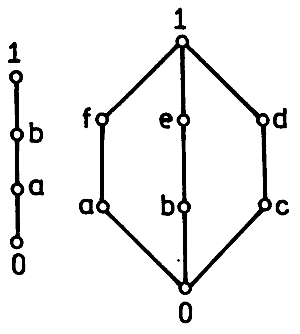


Figure 3a



3b

Example 2.1.

There exist finite posets with even order for which no orthocomplementation is possible. (vid. Figure 3b).

Now we go to classify the orthoposets up to order 10. The procedure makes use of the discussion of immediately successors and the symmetry with respect to the $\perp$-map. The ortho-double is also helpful. Furthermore some inductive steps will be applied, e.g.:

1. The amalgamation with the 4-element Boolean algebra.
2. The dismemberment of an edge placing a new vertex x and the dismemberment of the orthocomplemented edge by a new vertex $x^{\perp}$
3. Strengthening the order by taking new joining edges.

4. Joining non-consecutive edges a,b and $b^{\perp}$,$a^{\perp}$ by a chain a,x,b respectively $b^{\perp}$,$x^{\perp}$,$a^{\perp}$.

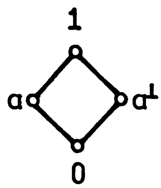
The result is the following

<u>List of Hasse diagrams of the non-isomorphic types of the orthoposets of order ≤ 10</u>

<u>Order 2</u>

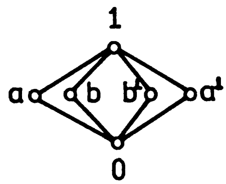
1.

<u>Order 4</u>



1.

<u>Order 6</u>



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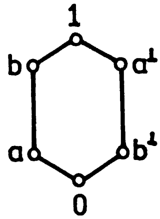
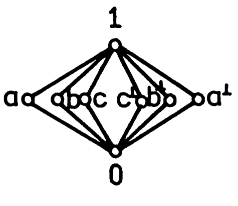


2.

<u>Order 8</u>



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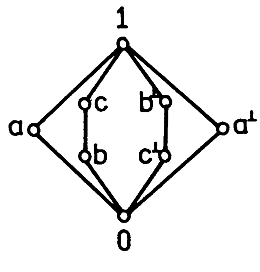

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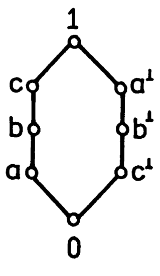

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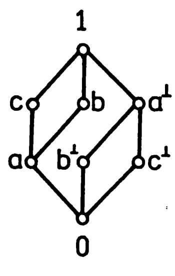

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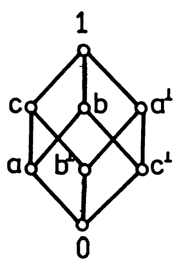
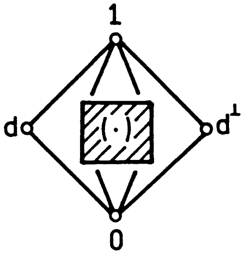
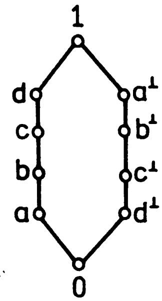


5.

Order 10



1.-5.



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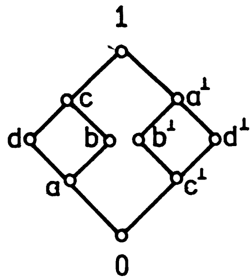

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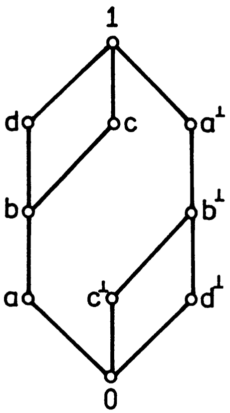

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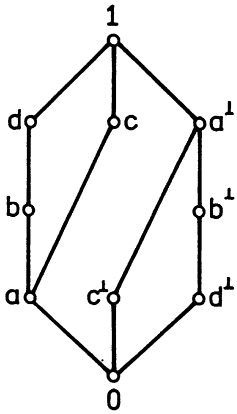

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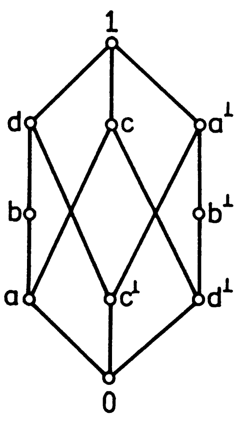

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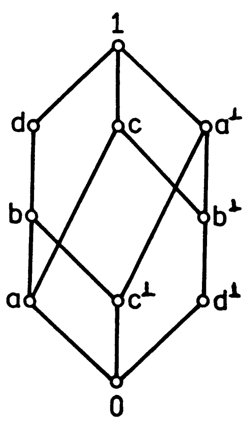

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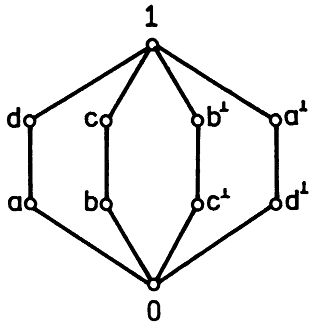

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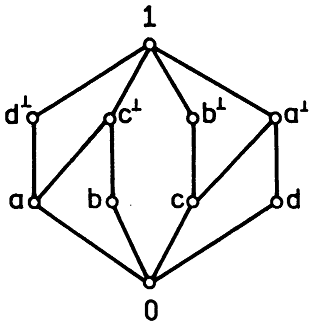

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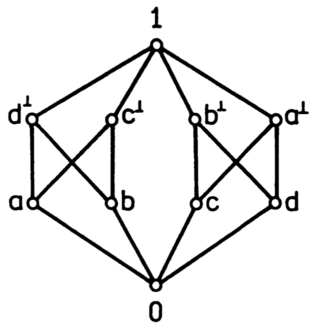

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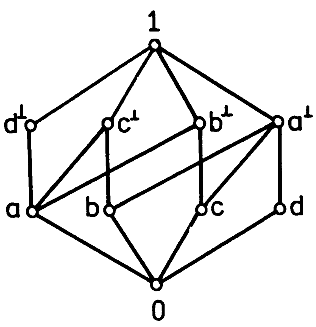

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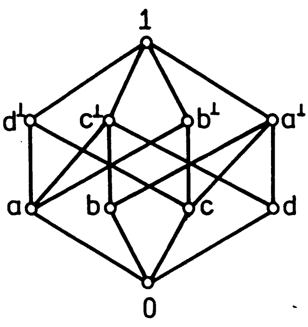

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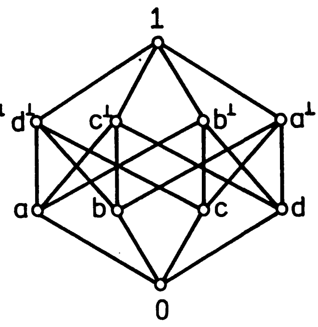

17.

Comments:

Order 2,4: These are Boolean algebras. The last is the ortho-double of a 3-element chain.

Order 6: The first is the ortho-double of the 4-element Boolean algebra.
The second is the ortho-double of the 4-element chain.

Order 8: 1)-4) are ortho-doubles. 4) is also to get from order 6 number 2) by building brigdes a,c,1 and 0,$c^{\perp}$,$a^{\perp}$.
We get 5) by strengthening the order of 4).

Order 10: We yield the casses 1)-5) from order 8 by amalgamation with the 4-element Boolean algebra. 6)-9) are ortho-doubles. 6) can also be derived by an edge dismemberment form order case 8 number 3). Also 8) and 9) are to get in this way from order case 8 number 4).
7) is obtained by brigdes a,d,c resp. $c^{\perp}$,$d^{\perp}$,$a^{\perp}$ from order case 8 number 3).

10) has stronger order than 9). But 10) and also 11) are constructed from order 8 number 5) by an edge dismemberment. 12) we also get from order 8 number 2) by an edge dismemberment. It is also an ortho-double. 13)-17) have stronger order than 12).
The cases 1) and 5) are orthomodular ortholattices. The cases 14), 16) and 17) are non-ortholattices.

3. Remark to the non-isomorphic orthocomplementations

M.D. MacLaren discusses in his paper [4] as a byproduct the question about the existence of a poset with two different orthocomplementations. He considered a left vector space V over a division ring equipped with a semi-inner product. As usually an orthogonality relation in V can be defined by $x \perp y$ whenever the semi-inner product for the pair x,y is zero. Then the set of all closed subspaces X of V, i.e. $X^{\perp\perp} = X$, forms with respect to inclusion and the orthocomplementation $X \mapsto X^{\perp}$ an orthocomplemented lattice L(V). Now for some spaces V_1, V_2 it may be that the lattices $L(V_1)$ and $L(V_2)$ are isomorphic without being ortho-isomorphic.

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