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## Two questions concerning vector-valued holomorphic functions

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# TWO QUESTIONS CONCERNING VECTOR-VALUED HOLOMORPHIC FUNCTIONS

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Let  $\Omega \subset \mathbb{C}$  be an open and connected subset and  $E$  a complex Hausdorff locally convex space which is locally complete, i.e.: every bounded set is contained in a bounded Banach-disc.  $H(\Omega, E)$  denotes the set of all holomorphic (= weakly holomorphic) functions  $\Omega \rightarrow E$ . For a Banach-disc  $B$  and its Minkowski-gauge-functional  $m_B$  write  $[B] := (\text{span } B, m_B)$  for the Banach-space which is spanned up by  $B$ .

(P1) GLOBAL FACTORIZATION: *Characterize those pairs  $(\Omega, E)$  such that for every  $f \in H(\Omega, E)$  there is a bounded Banach-disc  $B$  in  $E$  such that  $f : \Omega \rightarrow [B] \hookrightarrow E$  holomorphically.*

(P2) INTERPOLATION: *Given a sequence  $(z_n)$  in  $\Omega$  which is discrete in  $\Omega$  and a sequence  $(x_n)$  in  $E$ . Under which circumstances does  $f \in H(\Omega, E)$  exist with  $f(z_n) = x_n$  for all  $n \in \mathbb{N}$ ?*

It is well-known that every  $f \in H(\Omega, E)$  factors locally through a compact Banach-disc. So it is easy to see

(1) *If  $E'_{co}$  has the countable neighbourhood property (i.e. for every sequence  $(U_n)$  of neighbourhood of zero there are  $\lambda_n > 0$  such that  $\bigcap_n \lambda_n U_n$  is a neighbourhood of zero) then  $H(\Omega, E)$  admits global factorization for all  $\Omega$ . The Banach-disc for the factorization can be chosen even to be compact.*

This is true e.g. for Fréchet-spaces  $E$ . As a consequence

(2) *If  $f \in H(\Omega, E)$  factors globally it factors globally through a Banach-space  $[B]$  where  $B$  is a compact Banach-disc.*

If  $E = \hat{H}(\{0\})$ , the nuclear (LS)-space of germs of holomorphic functions in  $\{0\} \subset \mathbb{C}$ , then the holomorphic function  $f : \mathbb{C} \setminus \{0\} \rightarrow \hat{H}(\{0\})$  defined by

$$f(z) = \frac{1}{z-}$$

(the "moving-pole function") admits no global factorization.

(3) If  $A \subset \Omega$  is a set of uniqueness for holomorphic functions in  $H(\Omega)$  and  $F \subset E$  is a closed subspace, then every  $f \in H(\Omega, E)$  with  $f(A) \subset F$  satisfies  $f(\Omega) \subset F$ .

This follows from a simple Hahn-Banach-argument. In particular, there is for every  $f \in H(\Omega, E)$  a compact set  $K \subset E$  (namely  $f(\overline{K(z_0, \varepsilon)})$ ) such that  $f(\Omega) \subset \overline{\text{span } K}$ . A Baire-argument (in  $\Omega$ ), together with (3), implies

(4) If  $E = \bigcup_{n=1}^{\infty} E_n$  where  $E_n$  are closed subspaces of  $E$ , then there is for every  $f \in H(\Omega, E)$  an  $n_0 \in \mathbb{N}$  such that  $f(\Omega) \subset E_{n_0}$ .

This answers the interpolation question to the negative in the case of strict inductive limits  $E$ , e.g. there is no entire function  $f$  with values in  $\varphi := \varinjlim_{m \rightarrow} \mathbb{C}^m$  such that  $f(n) = e_n$  (the  $n$ -th unit vector) for all  $n \in \mathbb{N}$ . The same trick as for scalar-valued functions gives

(5) If  $\frac{\partial}{\partial \bar{z}} : \mathcal{C}^{\infty}(\Omega, E) \rightarrow \mathcal{C}^{\infty}(\Omega, E)$  is surjective, then there is always an interpolating function  $f$  (as in (P2)).

For Fréchet-spaces  $E$  the  $\frac{\partial}{\partial \bar{z}}$ -operator is always surjective: since  $\mathcal{C}^{\infty}(\Omega) \otimes_{\pi} E = \mathcal{C}^{\infty}(\Omega, E)$  and the scalar  $\frac{\partial}{\partial \bar{z}}$ -operator on  $\mathcal{C}^{\infty}(\Omega)$  is onto this is a consequence of the fact that the  $\pi$ -tensorproduct of surjective operators between Fréchet-spaces is surjective. However, in general,  $\frac{\partial}{\partial \bar{z}}$  is not onto, e.g. not on  $\mathcal{C}^{\infty}(\mathbb{C}, H(K(0, 1)))$ ; but it is onto on  $\mathcal{C}^{\infty}(\mathbb{C}, H(\{0\}))$  (see [3], p. 23). In the class of (DF)-spaces  $E$  (more general: of those spaces with a countable basis of bounded sets) D. Vogt [3] characterized the surjectivity of  $\frac{\partial}{\partial \bar{z}}$  on  $\mathcal{C}^{\infty}(\mathbb{C}, E)$  in terms of the existence of a somehow dominating bounded set in  $E$ .

R. M. Aron, J. Globevnik, and M. Schottenloher [1] investigated the

*bounded* interpolation problem on  $\Omega := K(0,1)$  : which are the sequences  $(z_n)$  in  $K(0,1)$  such that for every *bounded* sequence  $(x_n)$  in  $E$  there is a *bounded*  $f \in H(K(0,1), E)$  which interpolates. Their result is: the sequences  $(z_n)$  are the same for all Banach-spaces  $E$  (different from zero), i.e. the same as in the scalar case.

## REFERENCES

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