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0. Introduction:

During the last decade measurable selection results have been proved by numerous mathematicians and the field has become so extensive that it seems almost impossible to give a complete survey. It was a great achievement of Wagner ([57] and [58]) that he, nevertheless, gave an almost complete account of the results proved before 1979. For guidance to the literature, historical remarks, and statements of many of the known selection results we refer the reader to Wagner's articles. In the present paper we will limit ourself to explaining very few basic principles concerning the existence proofs for measurable selections and try to point out that a large class of selection results can already be obtained from these principles. In a first preliminary section we introduce the basic notions and facts about selections. In the second section we prove the fundamental selection theorem (due to Rokhlin [50] (in a special case), Kuratowski-Ryll-Nardzewski [37], and Castaing [6]) and show that many classical selection results (as, for instance, the Jankov-von Neumann measurable choice theorem and the Fillipov lemma of control theory) can be deduced from this theorem. Although all results in this section are well-known it seems worthwile to include them here because there seems to be no easy accessible account of the classical measurable selection results together with their proofs. Moreover, the method of proof we have chosen for the fundamental selection theorem, in our opinion, quite clearly reveals the conditions for the existence of measurable selections for correspondences with metric range, and, therefore, immediately leads over to the generalizations discussed in section 3. While for the classical selection theorems the range spaces of the correspondences in question are always separable the results in section 3 deal with correspondences taking closed values in a non-separable metric space. To obtain these results we have to assume that the measurable spaces, which are considered, satisfy some additivity condition stronger than σ-additivity. Inspired by results of Hansell [26] and Maitra-Rao [45] we show that there is a common principle behind many of the known selection results in the non-separable metric case (for instance, behind the results of Kaniewski-Pol [35], Frolík-Holický [19], and Fremlin [16]) and the fundamental selection theorem.

Section 4 deals with a rather unusual method for constructing measurable selections using an existence theorem for dominated Boolean homomorphisms. The idea for the main theorem in this section originated in the technique of dualization for Boolean correspondences as developed in [20]. The theorem mentioned above enables us to prove measurable selection results for correspondences whose range space is neither metrizable nor separable. Kupka [36] pointed out that these results are very useful in the framework of topological group theory. We shall summarize some of Kupka's results along with results of Sion [54] and Hasumi [28] which are all consequences of the main theorem in section 4.

The paper contains only very few hints to applications of measurable selection theorems. It also completely leaves out the topic of parametrizations of correspondences. Inspite of these disadvantages we hope that the paper will be useful to those who want to apply measurable selection results.

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1. Preliminaries.

Let X and Y be sets.

1.1 Definition:

A map $F: X \to \mathfrak{P}^*(Y) := \mathfrak{P}(Y) \setminus \{\emptyset\}$ is called a correspondence (or multi-function or non-empty set-valued map). A map $f: X \to Y$ is a selection (choice function) for F iff $f(x) \in F(x)$ for all $x \in X$.

A basic selection statement in mathematics is the

1.2 Axiom of choice:

Every correspondence has a selection.

The following example introduces an important special case of a correspondence.

1.3 Example and definition:

Let $p: Y \to X$ be surjective. Then $F: X \to \mathfrak{P}^*(Y)$, $x \to p^{-1}(x)$ is a correspondence and f is a selection for F if and only if $p \circ f = id_X$. The selections for this special F are called sections for p.

1.4 Definition:

Let $F: X \to \mathfrak{P}^*(Y)$ be given. Then

$Gr(F) := \{(x,y) \in X \times Y \mid y \in F(x)\}$ is called the graph of F.

By $\Pi_X: Gr(F) \to X$ and $\Pi_Y: Gr(F) \to Y$ we denote the canonical projections.

1.5 Remark:

$g: X \to Gr(F)$ is a section for Π_X if and only if $\Pi_Y \circ g$ is a selection for F.

1.6 Definition:

Let $\mathfrak{A} \subset \mathfrak{P}(X)$ and $\mathfrak{B} \subset \mathfrak{P}(Y)$ be given. A map $f: X \to Y$ is called $\mathfrak{A}$-$\mathfrak{B}$-measurable iff $f^{-1}(B) \in \mathfrak{A}$ for all $B \in \mathfrak{B}$.

1.7 Remark:

If $\mathfrak{A}$ and $\mathfrak{B}$ are topologies then $\mathfrak{A}$-$\mathfrak{B}$-measurability is just continuity.

The most general form of the problem we are going to consider is the following.

1.8 Measurable selection problem:

Let $\mathfrak{A} \subset \mathfrak{P}(X)$ and $\mathfrak{B} \subset \mathfrak{P}(Y)$ be given. Under what conditions does a correspondence $F: X \to \mathfrak{P}^*(Y)$ have an $\mathfrak{A}$-$\mathfrak{B}$-measurable selection ?

In what follows we almost always know that $\mathfrak{A}$ and $\mathfrak{B}$ are σ-fields, in which case we are looking for selections which are measurable in the usual sense.
Simple examples show that, without imposing some kind of "measurability" assumption on the correspondence F, there is no hope for the existence of a measurable selection for F.

The following definitions provide the basic notions needed for the formulation of a first positive result.

1.9 Definitions:
Let $\mathfrak{A} \subset \mathfrak{P}(X)$ be given.
By $\mathfrak{A}_\sigma$ we denote the collection $\{\bigcup_{n\in\mathbb{N}} A_n \mid A_n \in \mathfrak{A}\}$, by $\mathfrak{A}_\delta$ the collection $\{\bigcap_{n\in\mathbb{N}} A_n \mid A_n \in \mathfrak{A}\}$ and by $\mathfrak{A}^*$ the collection $\mathfrak{A}\setminus\{\emptyset\}$.
For $A \subset X$ let A^c stand for the complement $X\setminus A$ of A in X.
$\mathfrak{A}$ is called a field iff $\emptyset, X \in \mathfrak{A}$, $A, A' \in \mathfrak{A}$ implies $A \cup A' \in \mathfrak{A}$, and $A \in \mathfrak{A}$ implies $A^c \in \mathfrak{A}$ for all A, A'.
$\mathfrak{A}$ is called a σ-field iff $\mathfrak{A}$ is a field with $\mathfrak{A} = \mathfrak{A}_\sigma$. By $\sigma(\mathfrak{A})$ we denote the σ-field generated by $\mathfrak{A}$. If $\mathfrak{A}$ is a σ-field then $(X,\mathfrak{A})$ is said to be a measurable space.
By a measure on a measurable space $(X,\mathfrak{A})$ we always mean a countably additive map $\mu: \mathfrak{A} \to \overline{\mathbb{R}}_+ = [0,+\infty]$.
For a σ-field $\mathfrak{A}$ and a measure $\mu: \mathfrak{A} \to \overline{\mathbb{R}}_+$ we denote by μ^* the outer measure induced by μ on $\mathfrak{P}(X)$ and by $\mathfrak{A}_\mu$ the completion $\{A \,\Delta\, N \mid A \in \mathfrak{A} \text{ and } \mu^*(N)=0\}$. A measure space $(X,\mathfrak{A},\mu)$ is said to be complete iff $\mathfrak{A} = \mathfrak{A}_\mu$.

1.10 Notation:
Let Y be a topological space. We denote by

- $\mathfrak{G}(Y)$ the topology of Y,
- $\mathfrak{F}(Y)$ the collection of closed subsets of Y,
- $\mathfrak{K}(Y)$ the collection of compact subsets of Y,
- $\mathfrak{B}(Y)$ the Borel σ-field $\sigma(\mathfrak{G}(Y))$ of Y.

By abuse of notation the sets in $\mathfrak{G}(Y)_\delta$ are called G_δ-sets.

1.11 Definitions:
Let X be a set and Y a topological space, $F: X \to \mathfrak{P}^*(Y)$ a correspondence, and $\mathfrak{A} \subset \mathfrak{P}(X)$. For $B \subset Y$ the set $F^{-1}(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}$ is called the inverse image of B w.r.t. F.

F is said to be <u>weakly-$\mathfrak{A}$-measurable</u> (resp. <u>$\mathfrak{A}$-measurable</u>) iff $F^{-1}(\mathfrak{B}) \in \mathfrak{A}$ for all $\mathfrak{B} \in \mathfrak{G}(Y)$ (resp. $\mathfrak{B} \in \mathfrak{F}(Y)$).
If X is also a topological space then F is called <u>upper semi-continuous</u> (u.s.c) iff, F is $\mathfrak{F}(X)$-measurable.

<u>1.12 Examples</u>:
Let Y be a topological space.

(i) If X is a topological space and $p: Y \to X$ is surjective then p is closed if and only if $F:=p^{-1}$ is u.s.c. In particular $F=p^{-1}$ is u.s.c. if X is a Hausdorff, Y a compact Hausdorff space, and p is continuous.

(ii) If X is a set and $\mathfrak{A} \subset \mathfrak{P}(X)$ and if $\mathfrak{G}(Y) \subset \mathfrak{F}_\sigma(Y)$ then every $\mathfrak{A}$-measurable correspondence $F: X \to \mathfrak{P}^*(Y)$ is weakly $\mathfrak{A}_\sigma$-measurable.

(iii) If X is a set and $\mathfrak{A} \subset \mathfrak{P}(Y)$ and if Y is metrizable then every weakly $\mathfrak{A}$-measurable correspondence $F: X \to \mathfrak{K}^*(Y)$ is $\mathfrak{A}_\delta$-measurable.

<u>2. Some classical measurable selection theorems.</u>

The aim of this section is to show that many of the classical selection results are direct consequences of a fundamental selection theorem due to Rokhlin [49] (in a special case), Kuratowski-Ryll-Nardzewski [37], and, independently, Castaing [6]. This seems to be a well-known fact and has, for instance, been worked out by Mägerl [43].

<u>2.1 Theorem</u>: (Rokhlin [50], Kuratowski-Ryll-Nardzewski [37], Castaing [6])

Let $(X,\mathfrak{A})$ be a measurable space, Y a Polish space, and $F:X \to \mathfrak{F}^*(Y)$ weakly $\mathfrak{A}$-measurable. Then there exists an $\mathfrak{A}$-$\mathfrak{B}(Y)$-measurable selection for F.

<u>Proof</u>: For a weakly $\mathfrak{A}$-measurable correspondence $G: X \to \mathfrak{F}^*(Y)$ and an open cover $\mathfrak{U}=(U_n)_{n\in N}$ of Y we define

$$G_{\mathfrak{U}} : X \to \mathfrak{F}^*(Y) \quad \text{by}$$
$$G_{\mathfrak{U}}(x)= \overline{G(x) \cap U_n} \quad \text{if } x \in G^{-1}(U_n)\setminus \bigcup_{\nu=1}^{n-1} G^{-1}(U_\nu)$$

Then $G_{\mathfrak{U}}$ has the following properties:

(i) $\sup_{x\in X} \operatorname{diam} G_{\mathfrak{U}}(x) \leq \sup_{n\in N} \operatorname{diam} (U_n)$

(ii) $\forall x \in X$: $G_{\mathfrak{U}}(x) \subset G(x)$

(iii) $\forall V \in \mathfrak{G}(Y): G_{\mathfrak{U}}^{-1}(V) = \bigcup_{n\in N} [G^{-1}(V\cap U_n) \setminus \bigcup_{\nu=1}^{n-1} G^{-1}(U_\nu)] \in \mathfrak{A}$.

Now, for every $n \in \mathbb{N}$, we choose a countable open cover $\mathfrak{U}_n$ of Y with diam $(U) \leq \frac{1}{n}$ for all $U \in \mathfrak{U}_n$ and define

$$F_1 := F_{\mathfrak{U}_1} \quad , \quad F_{n+1} := (F_n)_{\mathfrak{U}_{n+1}}$$

Since Y is complete and, for each $x \in X$, $(F_n(x))_{n\in\mathbb{N}}$ is a decreasing sequence of closed sets whose diameter tends to zero we know that $\bigcap_{n\in\mathbb{N}} F_n(x)$ is a singleton. The map $f: X \to Y$ defined by

$f(x) \in \bigcap_{n\in\mathbb{N}} F_n(x)$ is obviously a selection for F. To prove that f is

$\mathfrak{A}$-$\mathfrak{B}(Y)$-measurable let $V \in \mathfrak{G}(Y)$ be given. We set $V_n := \{y \in Y \mid \text{dist}(y,V^c) > \frac{1}{n}\}$. Then V_n is open and we will show

$$f^{-1}(V) = \bigcup_{n\in\mathbb{N}} F_n^{-1}(V_n),$$

hence $f^{-1}(V) \in \mathfrak{A}$.

"$\subset$": Let $x \in F^{-1}(V)$ be arbitrary. Then there exists an $n \in \mathbb{N}$ with $\text{dist}(f(x),V^c) > \frac{1}{n}$, i.e. $f(x) \in V_n$. Since $f(x) \in F_n(x)$ we deduce $F_n(x) \cap V_n \neq \emptyset$, i.e. $x \in F_n^{-1}(V_n)$. Thus we have

$$f^{-1}(V) \subset \bigcup_{n\in\mathbb{N}} F_n^{-1}(V_n).$$

"$\supset$": For $x \in F_n^{-1}(V_n)$ we have $F_n(x) \cap V_n \neq \emptyset$.
Since diam $(F_n(x)) \leq \frac{1}{n}$ the definition of V_n implies $F_n(x) \cap V^c = \emptyset$, hence $f(x) \in F_n(x) \subset V$, which proves $\bigcup_{n\in\mathbb{N}} F_n(x) \subset f^{-1}(V)$.

Our first corollary is due to Castaing [6](see also Castaing-Valadier [7], p.67), but has a predecessor by Novikov[46a].
It characterizes weak measurability of a correspondence in terms of the existence of "many" measurable selections.

<u>2.2 Corollary</u>: (Castaing [6])
Let $(X,\mathfrak{A})$ be a measurable space, Y a Polish space, and $F: X \to \mathfrak{F}^*(Y)$.
Then the following statements are equivalent:

(i) F is weakly $\mathfrak{A}$-measurable

(ii) There exists a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathfrak{A}$-$\mathfrak{B}(Y)$-measurable maps from X to Y such that

$$F(x) = \overline{\{f_n(x) \mid n \in \mathbb{N}\}}$$

for all $x \in X$.

Proof:

(i) $\Rightarrow$ (ii): Let $\mathfrak{B}$ be a countable base for the topology of Y. For $B \in \mathfrak{B}$ define $F_B: X \to \mathfrak{F}^*(Y)$ by

$$F_B(x) = \begin{cases} \overline{F(x) \cap B}, & x \in F^{-1}(B) \\ F(x), & x \notin F^{-1}(B). \end{cases}$$

For $U \in \mathfrak{G}(Y)$ we have

$$F_B^{-1}(U) = F^{-1}(B \cap U) \cup (F^{-1}(U) \setminus F^{-1}(B)) \in \mathfrak{A}$$

so that F_B is weakly $\mathfrak{A}$-measurable. According to Theorem 2.1 there is an $\mathfrak{A}$-$\mathfrak{B}(Y)$-measurable selection f_B for F_B. Obviously $F(x) = \overline{\{f_B(x) \mid B \in \mathfrak{B}\}}$ holds for all $x \in X$.

(ii) $\Rightarrow$ (i): For $U \in \mathfrak{G}(Y)$ we obtain

$$F^{-1}(U) = \{x \in X \mid \exists n \in \mathbb{N}: f_n(x) \in U\} = \bigcup_{n \in \mathbb{N}} f_n^{-1}(U) \in \mathfrak{A},$$

so that F is weakly $\mathfrak{A}$-measurable.

The following special case of Theorem 2.1 can, essentially, already be found in Saks [51], p.282, Lemma 7.1.

2.3 Collary:

Let X and Y be compact metrizable spaces and $p: Y \to X$ a continuous surjective map. Then there exists a $\mathfrak{B}(X)$-$\mathfrak{B}(Y)$-measurable section for p.

Proof: Apply Theorem 2.1 to $F = p^{-1}$.

The compactness condition on Y in the above corollary is essential. This was, for instance, shown by Christensen [9], p. 82 who proved that there is a Polish space Y, a compact metrizable space X, and a continuous surjective map $p: Y \to X$ without any $\mathfrak{B}(X)$-$\mathfrak{B}(Y)$-measurable section. Nevertheless any such p admits a section with slightly weaker measurability properties. For the statement of the corresponding result we need the notion of analyticity for topological spaces.

2.4 Definition:

a) A Hausdorff space is called analytic (or Suslin) iff it is the continuous image of a Polish space.
   For a Hausdorff space X we denote by $\mathfrak{A}(X)$ the collection of all analytic subsets of X.
b) Let $\mathfrak{B}_u(X)$ be the σ-field of all universally measurable subsets

of the Hausdorff space X, i.e. $\mathcal{B}_u(X)$ is the intersection of all completions $\mathcal{B}(X)_\mu$ where μ runs through all finite measures on $\mathcal{B}(X)$.

Some basic facts about analytic spaces are collected in the next lemma, whose proofs can, for instance, be found in Hoffmann-Jørgensen [30] or Schwarz [53].

2.5 Lemma:

(i) If X is a Hausdorff space then all analytic subsets of X are universally measurable and $\mathcal{A}(X)$ is closed w.r.t. countable unions and intersections.

(ii) If X is analytic then all Borel subsets of X are analytic.

(iii) A continuous map from a Hausdorff space X to a Hausdorff space Y maps analytic sets onto analytic sets.

(iv) A map p from an analytic space Y to an analytic space X is $\mathcal{B}(X)$-$\mathcal{B}(Y)$-measurable (i.e. Borel measurable) iff $Gr(p)$ is analytic.

Our next result is the celebrated measurable choice theorem due to Jankov [34] and, independently, von Neumann [56].

2.6 Corollary: (Jankov [34], v. Neumann [56])
Let X and Y be analytic spaces and $p: Y \to X$ continuous and surjective. Then there exists a $\mathcal{B}_u(X)$-$\mathcal{B}(Y)$-measurable section for p.

Proof: Since Y is analytic there is a Polish space Z and a continuous map q from Z onto Y. The correspondence $F = q^{-1} \circ p^{-1}$ is obviously closed-valued and for $U \in \mathcal{O}(Z)$ we have

$$F^{-1}(U) = p \circ q(U)$$

which, according to Lemma 2.5 (i) & (iii), belongs to $\mathcal{A}(X) \subset \mathcal{B}_u(X)$. Thus F is weakly $\mathcal{B}_u(X)$-measurable and, due to Theorem 2.1, has a $\mathcal{B}_u(X)$-$\mathcal{B}(Z)$-measurable selection g. Obviously $q \circ g$ is a section for p with the desired properties.

2.7 Remark:
As the proof shows the section in the above corollary can be chosen $\sigma(\mathcal{A}(X))$-$\mathcal{B}(Y)$-measurable.

2.8 Corollary:
Let X and Y be analytic spaces and let $F: X \to \mathcal{P}^*(Y)$ have analytic

graph. Then F has a $\mathcal{B}_u(X)$-$\mathcal{B}(Y)$-measurable selection.

Proof: The canonical projection π_X: Gr(F) $\to$ X is continuous and surjective. By Corollary 2.6 it, therefore, has a $\mathcal{B}_u(X)$-$\mathcal{B}(Gr(F))$-measurable section g. If π_Y: Gr(F) $\to$ Y denotes the canonical projection onto Y then $\pi_Y$og is obviously a $\mathcal{B}_u(X)$-$\mathcal{B}(Y)$-measurable selection for F.

The following corollary is one of the possible variants of the famous Fillipov lemma (see [15]) in control theory. It can also be interpreted as a generalization of a measurable implicit function theorem special cases of which have already been proved by Lusin [42] and Novikov [46].

2.9 Corollary:
Let X,Y and Z be analytic spaces, f: X x Y $\to$ Z a $\mathcal{B}(X \times Y)$-$\mathcal{B}(Z)$-measurable map, and Ω: X $\to$ $\mathcal{P}^*(Y)$ a correspondence with analytic graph. Moreover, let μ: $\mathcal{B}(X) \to \mathbb{R}_+$ be a measure and g: X $\to$ Z $\mathcal{B}(X)_\mu$-$\mathcal{B}(Z)$-measurable. Then $X_o := \{x \in X \mid g(x) \in f(x,\Omega(x))\}$ belongs to $\mathcal{B}(X)_\mu$ and there exists a $\mathcal{B}(X_o)_\mu$-$\mathcal{B}(X)$-measurable selection ω for Ω with $g(x)= f(x,\omega(x))$ for all $x \in X_o$.

Proof: Since $\mathcal{B}(Z)$ is countably generated as a σ-field (cf. Hoffmann-Jørgensen [30], p.111, Thm.3) there exists a set $N \in \mathcal{B}(X)$ of μ-measure 0 such that $g_{|X \setminus N}$ is $\mathcal{B}(X \setminus N)$-$\mathcal{B}(Z)$-measurable (cf. [24], p.70, Lemma). Since what happens inside a μ-nullset does not effect our claim we may, without loss of generality, assume that g is $\mathcal{B}(X)$-$\mathcal{B}(Y)$-measurable. Define F: X $\to$ $\mathcal{P}(Y)$ by $F(x)=\{y \in \Omega(x) \mid g(x)= f(x,y)\}$ Then GrF= Gr(Ω) $\cap$ $\{(x,y) \in X \times Y \mid f(x,y)=g(x)\}$ is analytic, since Gr(Ω) is analytic and $\{(x,y) \in X \times Y \mid f(x,y)= g(x)\}$ is Borel, hence analytic (see Lemma 2.5 (i) & (ii)). Thus $X_o=\Pi_X$(GrF) is also analytic (see Lemma 2.4 (iii)), hence belongs to $\mathcal{B}(X)_\mu$ (see Lemma 2.5 (i)). By Corollary 2.8 there exists a $\mathcal{B}_u(X_o)$-$\mathcal{B}(Z)$-measurable selection ω for $F_{|X_o}$. Obviously ω has the desired properties.

Dubin-Savage [11](p.38, Lemma 6) used optimal selections in game theory. A generalization of their result is stated below and turns out to be a special case of Fillipov's lemma.

2.10 Corollary:
Let X and Y be analytic spaces, Ω: X $\to$ $\mathcal{P}^*(Y)$ a correspondence with analytic graph, and f: X x Y $\to$ $\mathbb{R}$ a $\mathcal{B}(X \times Y)$-measurable map such that

$f(x,\ )$ attains $\sup f(x,\Omega(x))$ for all $x \in X$. Moreover let μ be a finite Borel measure on X. Then $g: X \to \mathbb{R}$ defined by $g(x)=\max f(x,F(x))$ is $\mathcal{B}(X)_\mu$-measurable and there exists a $\mathcal{B}(X)_\mu$-$\mathcal{B}(Y)$-measurable selection ω for Ω with $f(x,\omega(x))=g(x)$ for all $x \in X$.

Proof: If we can show that g is $\mathcal{B}(X)_\mu$-measurable then Corollary 2.9 yields the other conclusions of the corollary. For $\alpha \in \mathbb{R}$ we obtain

$$\begin{aligned}\{x \in X \mid g(x)>\alpha\} &= \{x \in X \mid \exists y \in \Omega(x):\ f(x,y) > \alpha\} \\ &= \Pi_X(Gr(\Omega) \cap \{(x,y) \in X \times Y \mid f(x,y)>\alpha\}) \in \mathcal{A}(X) \subset \mathcal{B}(X)_\mu .\end{aligned}$$

Thus g is $\mathcal{B}(X)_\mu$-measurable.

2.11 Remark: (Examples of applications)
Measurable selection theorems have been applied in many fields. It would take to much space to list them all. We must be satisfied with some examples:

a) von Neumann proved his measurable choice theorem for use in the theory of W^*-algebras. It was the main tool in his proof of the fact that the commutant of the direct integral of W^*-algebras is isomorphic to the direct integral of the commutants of the single factors (see von Neumann [56], p.452, Lemma 7 and p.459, Lemma 13).
b) That measurable selections play an important role in the general representation theory of C^*-algebras can be seen by reading pp.81-101 in Arveson's book [1].
c) Using measurable selections Azoff and Gilfeather [3] have shown that reductive operators are normal if every operator has a non-trivial invariant subspace.
d) Fillipov [15] used his lemma in control theory to show that if a system admits any control then it also has a measurable control. For further use of selection results in control theory see for instance Rockafellar [49].
e) Aumann [2] used a variant of the measurable choice theorem to show the existence of so-called preference orderings in mathematical economics. For further use of measurable selection results in mathematical economics see the book of Hildebrand [29].

2.12 Remark:
The results considered above have been generalized in several aspects. We will not quote any of these generalizations and instead refer the reader to the survey papers of Wagner [57], [58]. Let us only dwell on one aspect of possible generalizations: The conclusion

of the fundamental selection theorem (Theorem 2.1) still holds if one replaces the condition of Y being Polish by other conditions ; for instance, the theorem obviously remains true if Y is a metrizable σ-compact space. Saint-Raymond [52] has shown that a metrizable space Y which is a Lusin space (i.e. a continuous injective image of a Polish space) has the property that for every measurable space $(X,\mathfrak{A})$ and any weakly $\mathfrak{A}$-measurable correspondence $F:X \to \mathfrak{F}^*(Y)$ there exists an $\mathfrak{A}$-$\mathfrak{B}(Y)$-measurable selection for F if and only if Y is the union of a Polish space and a σ-compact space.

This leads to the following problem:

2.13 Problem:
Characterize the (metrizable) topological spaces Y such that for every measurable space $(X,\mathfrak{A})$ and every weakly $\mathfrak{A}$-measurable correspondence $F:X \to \mathfrak{F}^*(Y)$ there exists an $\mathfrak{A}$-$\mathfrak{B}(Y)$-measurable selection for F.

3. Selection theorems for correspondences with non-separable metric range.

The proof of the fundamental selection theorem (Theorem 2.1) uses the separability of the range at a crucial point. Nevertheless an analysis of this proof leads to selection results for correspondences with non-separable metric range: By imposing some "additivity" assumptions on the measurable space $(X,\mathfrak{A})$ the step from a measurable correspondence F to a smaller one, of the same kind, say $F_{\mathfrak{U}}$, whose values are contained in the sets of a given covering $\mathfrak{U}$ of the range space, is made possible again. What we mean by "additivity" is made precise in the following definitions.

3.1 Definitions:
Let X be a set, $\mathfrak{H} \subset \mathfrak{P}(X)$, and $(H_\lambda)_{\lambda\in\Lambda}$ a family in $\mathfrak{H}$.

a) $(H_\lambda)_{\lambda\in\Lambda}$ is called $\mathfrak{H}$-additive if $\bigcup_{\lambda\in\Lambda'} H_\lambda \in \mathfrak{H}$ for all $\Lambda' \subset \Lambda$.

b) $(H_\lambda)_{\lambda\in\Lambda}$ is called weakly $\mathfrak{H}$-heriditarily additive (weakly $\mathfrak{H}$-h.a.) if, for every $\mathfrak{H}$-additive family $(D_\lambda)_{\lambda\in\Lambda}$, the family $(D_\lambda \cap H_\lambda)_{\lambda\in\Lambda}$ is $\mathfrak{H}$-additive.

c) $(H_\lambda)_{\lambda\in}$ is said to be $\mathfrak{H}$-reducible if there exists a weakly $\mathfrak{H}$-h.a. family $(H'_\lambda)_{\lambda\in\Lambda}$ of pairwise disjoint sets with $\bigcup_{\lambda\in\Lambda} H_\lambda = \bigcup_{\lambda\in\Lambda} H'_\lambda$ and $H'_\lambda \subset H_\lambda$.

3.2 Remark:

A family which is $\mathfrak{K}$-heriditarily additive in the sense of Hansell [26] is weakly $\mathfrak{K}$-h.a.

3.3 Examples:

(i) Let X be a set, $\mathfrak{K} \subset \mathfrak{P}(X)$, Y a topological space and $F: X \to \mathfrak{P}^*(Y)$ weakly-$\mathfrak{K}$-measurable. If $(U_\lambda)_{\lambda\in\Lambda}$ is a family of open sets in Y then $(F^{-1}(U_\lambda))_{\lambda\in\Lambda}$ is $\mathfrak{K}$-additive.

(ii) If $(X, \mathfrak{K})$ is a measurable space and $(H_\lambda)_{\lambda\in\Lambda}$ is a countable family in $\mathfrak{K}$ then $(H_\lambda)_{\lambda\in\Lambda}$ is weakly $\mathfrak{K}$-h.a. and $\mathfrak{K}$-reducible.

3.4 Definition: Let X be a set, $\mathfrak{K} \subset \mathfrak{P}(X)$, Y a topological space, and $F: X \to \mathfrak{P}^*(Y)$ a correspondence. F is called $\mathfrak{K}$-reducible if there exists a base $\mathfrak{L}$ for the topology Y such that for every open cover $(U_\lambda)_{\lambda\in\Lambda}$ of Y by sets from $\mathfrak{L}$ every $\mathfrak{K}$-additive cover $(H_\lambda)_{\lambda\in\Lambda}$ of X with $H_\lambda \subset F^{-1}(U_\lambda)$ is $\mathfrak{K}$-reducible.

The following theorem is in the spirit of the results proved by Hansell [26] , and was also influenced by the work of Maitra-Rao [45].

3.5 Theorem:

Let X be a set, $\mathfrak{K} \subset \mathfrak{P}(X)$, Y a complete metric space, $F: X \to \mathfrak{F}^*(Y)$ weakly $\mathfrak{K}$-measurable and $\mathfrak{K}$-reducible. Then there exists an $\mathfrak{K}_\sigma$-$\mathfrak{B}(Y)$-measurable selection for F.

Proof: Let $\mathfrak{L}$ be a base for the topology of Y having the properties described in Definition 3.4. Let $\mathfrak{U} = (U_\lambda)_{\lambda\in\Lambda}$ be a covering of Y by sets from $\mathfrak{L}$, $G: X \to \mathfrak{F}^*(Y)$ weakly $\mathfrak{K}$-measurable with $G(x) \subset F(x)$ for all $x \in X$. Then $(G^{-1}(U_\lambda))_{\lambda\in\Lambda}$ is an $\mathfrak{K}$-additive cover of X with $G^{-1}(U_\lambda) \subset F^{-1}(U_\lambda)$. Since F is $\mathfrak{K}$-reducible we know that $(G^{-1}(U_\lambda))_{\lambda\in\Lambda}$ is $\mathfrak{K}$-reducible. Hence there exists a weakly $\mathfrak{K}$-h.a. family $(H'_\lambda)_{\lambda\in\Lambda}$ of pairwise disjoint sets with $\bigcup_{\lambda\in\Lambda} H'_\lambda = X$ and $H'_\lambda \subset G^{-1}(U_\lambda)$. Define $G_\mathfrak{U}: X \to \mathfrak{F}^*(Y)$ by

$G_\mathfrak{U}(x) = \overline{G(x) \cap U_\lambda}$ for $x \in H'_\lambda$.

For $U \subset Y$ open we obtain

$$G_\mathfrak{U}^{-1}(U) = \bigcup_{\lambda\in\Lambda} G^{-1}(U \cap U_\lambda) \cap H'_\lambda \cdot$$

Since $(G^{-1}(U \cap U_\lambda))_{\lambda\in\Lambda}$ is $\mathfrak{K}$-additive and $(H'_\lambda)_{\lambda\in\Lambda}$ is weakly $\mathfrak{K}$-h.a. the family $(G^{-1}(U \cap U_\lambda) \cap H'_\lambda)_{\lambda\in\Lambda}$ is $\mathfrak{K}$-additive, hence $G_\mathfrak{U}^{-1}(U) \in \mathfrak{K}$. Thus $G_\mathfrak{U}$ is weakly-$\mathfrak{K}$-measurable. Moreover we have $G_\mathfrak{U}(x) \subset G(x)$ and $\operatorname{diam}(G_\mathfrak{U}(x)) \leq \sup_{\lambda\in\Lambda} d(U_\lambda)$. With the obvious modifications the proof can now be finished as the proof of Theorem 2.1.

3.6 Remark:

If $\mathfrak{X} \subset \mathfrak{P}(X)$ is a field, Y a Polish space, and $F:X \to \mathfrak{F}^*(Y)$ is weakly $\mathfrak{X}_\sigma$-measurable then F is $\mathfrak{X}_\sigma$-reducible: Take for $\mathfrak{L}$ any countable base for the topology of Y. Thus Theorem 2.1 is an immediate consequence of Theorem 3.5.

3.7 Definitions:

Let X be a uniform space.

a) A family $(A_\lambda)_{\lambda\in\Lambda}$ in $\mathfrak{P}(X)$ is called discrete iff, for every $\varepsilon > 0$, there is a uniformly continuous pseudo-metric ρ on X such that for $\lambda, \lambda' \in \Lambda$ with $\lambda \neq \lambda'$ one has $\rho(A_\lambda, A_{\lambda'}) > \varepsilon$.

b) A family $(A_\lambda)_{\lambda\in\Lambda}$ in $\mathfrak{P}(X)$ is called σ-discretely-decomposable (σ-dd) iff there exists a family $(A_{\lambda,n})_{(\lambda,n)\in\Lambda\times\mathbb{N}}$ such that, for every $\lambda\in\Lambda$, $A_\lambda = \bigcup_{n\in\mathbb{N}} A_{\lambda n}$ and, for every $n \in \mathbb{N}$, $(A_{\lambda n})_{\lambda\in\Lambda}$ is discrete.

c) A family $(A_\lambda)_{\lambda\in\Lambda}$ in $\mathfrak{P}(X)$ is called σ-discretely base-like refinable (σ-db) iff there exists a $\mathfrak{L} \subset \mathfrak{P}(X)$ which is the union of a countable number of discrete families and satisfies

$$A_\lambda = \bigcup\{B \in \mathfrak{L} \mid B \subset A_\lambda\}$$

for every $\lambda \in \Lambda$.

3.8 Remark:

Obviously σ-dd. implies σ-db.

3.9 Lemma:

Let X be a uniform space and let $\mathfrak{X} \subset \mathfrak{P}(X)$ have the following properties

(i) $\mathfrak{X} = \mathfrak{X}_\sigma$

(ii) $\forall A,B \in \mathfrak{X}$: $A \cap B \in \mathfrak{X}$

(iii) $\forall (A_n)_{n\in\mathbb{N}}$ in $\mathfrak{X}$ $\exists (B_n)_{n\in\mathbb{N}}$ pairwise disjoint in $\mathfrak{X}$:

$$B_n \subset A_n \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} A_n$$

(iv) $\forall (C_i)_{i\in I}$ discrete in $\mathfrak{X}$: $\bigcup_{i\in I} C_i \in \mathfrak{X}$

(v) $\mathfrak{F}(X) \subset \mathfrak{X}$

Then every σ-db family in $\mathfrak{X}$ is $\mathfrak{X}$-reducible.

Proof: Let $(H_\lambda)_{\lambda\in\Lambda}$ be a σ-db family in $\mathcal{H}$. Then there exists a $\mathcal{L}\subset\mathfrak{P}(X)$ and a sequence $(\mathcal{L}^n)_{n\in\mathbb{N}}$ of discrete families with $\bigcup_{n\in\mathbb{N}}\mathcal{L}^n=\mathcal{L}$ such that, for each $\lambda\in\Lambda$,

$$H_\lambda = \bigcup\{B\in\mathcal{L}\mid B\subset H_\lambda\}$$

and , w.l.o.g, each $B\in\mathcal{L}$ is contained in some H_λ.
By the axiom of choice we can find a map $\mathcal{L}\to\Lambda$, $B\to\lambda_B$ such that $B\subset H_{\lambda_B}$.
For $\lambda\in\Lambda$ and $n\in\mathbb{N}$ define

$$H_{\lambda n}=\bigcup\{\overline{B}\cap H_\lambda\mid\lambda_B=\lambda,\ B\in\mathcal{L}^n\}\subset H_\lambda$$

Since -according to (v) and (ii)- $\{\overline{B}\cap H_\lambda\mid\lambda_B=\lambda,\ B\in\mathcal{L}^n\}$ is a discrete family in $\mathcal{H}$ we deduce from (iv) that $H_{\lambda n}\in\mathcal{H}$. For the same reasons we have

$$H_n:=\bigcup_{\lambda\in\Lambda}H_{\lambda n}=\bigcup\{\overline{B}\cap H_\lambda\mid\lambda\in\Lambda, B\in\mathcal{L}^n,\ \lambda_B=\lambda\}\in\mathcal{H}$$

According to (iii) there is a sequence $(B_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{H}$ with $B_n\subset H_n$ and

$$\bigcup_{n\in\mathbb{N}}B_n=\bigcup_{n\in\mathbb{N}}H_n.$$

Now define

$$H'_\lambda:=\bigcup_{n\in\mathbb{N}}(H_{\lambda n}\cap B_n)\subset H_\lambda.$$

According to (i) and (ii) H'_λ belongs to $\mathcal{H}$. It follows immediately from the definition that $(H'_\lambda)_{\lambda\in\Lambda}$ is a family of pairwise disjoint sets with

$$\bigcup_{\lambda\in\Lambda}H'_\lambda=\bigcup_{n\in\mathbb{N}}H_n=\bigcup\{\overline{B}\cap H_\lambda\mid\lambda\in\Lambda, B\in\mathcal{L},\ \lambda_B=\lambda\}$$

$$=\bigcup_{\lambda\in\Lambda}H_\lambda.$$

That $(H'_\lambda)_{\lambda\in\Lambda}$ is weakly $\mathcal{H}$-h.a. can be seen by repeating the arguments used above.

3.10 Lemma:
Let X be a uniform space and $\mathcal{H}\subset\mathfrak{P}(X)$ a σ-field satisfying properties (iv) and (v) of Lemma 3.9. If $(D_\lambda)_{\lambda\in\Lambda}$ is a σ-db family in $\mathcal{H}$ and $(H_\lambda)_{\lambda\in\Lambda}$ is an $\mathcal{H}$-additive family with $H_\lambda\subset D_\lambda$ for every $\lambda\in\Lambda$ then $(H_\lambda)_{\lambda\in\Lambda}$ is $\mathcal{H}$-reducible.

Proof: According to our assumptions there exists a $\mathcal{L}\subset\mathfrak{P}(X)$ and a sequence $(\mathcal{L}^n)_{n\in\mathbb{N}}$ of discrete families with $\mathcal{L}=\bigcup_{n\in\mathbb{N}}\mathcal{L}^n$ such that, for each $\lambda \in \Lambda$, $D_\lambda = \cup\{B \in \mathfrak{B} \mid B \subset D_\lambda\}$. Now let $\leq$ be a well-ordering on Λ and define for $\lambda \in \Lambda$ and $n \in \mathbb{N}$

$$H_{\lambda n} = \cup\{\bar{B} \cap (H_\lambda \setminus \bigcup_{\lambda' < \lambda} H_{\lambda'}) \mid B \in \mathfrak{B}^n\},$$

Since $(H_\lambda)_{\lambda \in \Lambda}$ is $\mathcal{H}$-additive and $\{\bar{B} \cap (H_\lambda \setminus \bigcup_{\lambda' < \lambda} H_{\lambda'}) \mid B \in \mathfrak{B}^n\}$ is discrete it follows from (iv) and (v) that $H_{\lambda n}$ belongs to $\mathcal{H}$. The proof is then finished in almost the same way as that of Lemma 3.9.

3.11 Remark:

It would be interesting to know whether the conclusions of Lemma 3.10 hold under the assumptions of Lemma 3.9.

3.12 Definitions:

Let X and Y be uniform spaces and let $F: X \to \mathfrak{P}^*(Y)$ be a correspondence.

F is said to be inverse σ-dd (σ-db) preserving iff $(F^{-1}(A_\lambda))_{\lambda \in \lambda}$ is σ-dd (σ-db) for every σ-dd (σ-db) family $(A_\lambda)_{\lambda \in \Lambda}$ in Y.

3.13 Remark:

It is easy to see that every inverse σ-dd preserving correspondence is inverse σ-db preserving. The following corollary of Theorem 3.5 is a modification of a result due to Frolík-Holický [19] (p.656, Lemma 1).

3.14 Corollary:

Let X be a uniform space, $\mathcal{H} \subset \mathfrak{P}(X)$ (a σ-field) satisfying conditions (i) to (v) of Lemma 3.9. Moreover, let Y be a complete metric space and $F: X \to \mathfrak{F}^*(Y)$ a weakly $\mathcal{H}$-measurable correspondence which is inverse σ-dd(σ-db) preserving. Then there exists an $\mathcal{H}$-$\mathfrak{G}(Y)$-measurable selection for F.

Proof: Since Y is metrizable there exists a σ-discrete base $\mathfrak{B}$ for the topology of Y. For every cover $(U_\lambda)_{\lambda \in \Lambda}$ of Y by sets from $\mathfrak{B}$ we have -according to our assumptions- that $(F^{-1}(U_\lambda))_{\lambda \in \Lambda}$ is σ-dd (resp. σ-db). Since every family $(A_\lambda)_{\lambda \in \Lambda}$ with $A_\lambda \subset F^{-1}(U_\lambda)$ is again σ-dd it follows from Lemma 3.9 and Remark 3.13 (resp. Lemma 3.10) that F is $\mathcal{H}$-reducible. The conclusion of Corollary 3.13, therefore, follows from Theorem 3.5.

3.15 Remark:

Using the theory of non-separable analytic spaces developed by Frolik and Holicky ([17], [18]) one can use Corollary 3.14 as a tool to derive other selection results in the non-separable case just as the fundamental selection theorem in section 1 was used to deduce selection results for the separable case. The corresponding results have first been obtained by Frolík-Holický [19].
To apply the above results to more concrete situations we shall need the following deep result due to Kaniewski-Pol [35] and Hansell [27], which has recently been generalized by Frolík-Holický [19a]. Its proof can be found in [35], p.1045. Let us recall that a family $(A_\lambda)_{\lambda\in\Lambda}$ in X is said to be point-finite (point-countable) iff, for every $x \in X$, the set $\{\lambda \in \Lambda | x \in A_\lambda\}$ is finite (at most countable).

3.16 Lemma: (Hansell [27], Kaniewski-Pol [35])
If X is a metrizable analytic space then every point-finite $\mathcal{A}(X)$-additive family is σ-dd.

3.17 Definitions and remarks:
Let X be a metrizable space.
a) Define $\mathcal{G}_0 := \mathfrak{G}(X)$ and for an even ordinal $\alpha > 0$,
$\mathcal{G}_\alpha := (\bigcup_{\xi<\alpha}\mathcal{G}_\xi)_\sigma$ and $\mathcal{G}_{\alpha+1} = (\mathcal{G}_\alpha)_\delta$.
Then $\alpha \leq \alpha'$ implies $\mathcal{G}_\alpha \subset \mathcal{G}_{\alpha'}$.
Define, moreover,
$$\mathfrak{L}_\alpha(X) = \begin{cases} \mathcal{G}_\alpha, & \alpha \text{ even} \\ \{A^c \mid A \in \mathcal{G}_\alpha\}, & \alpha \text{ odd} \end{cases}$$
Then, for $\alpha > 0$, we have $\mathfrak{F}(X) \subset \mathfrak{L}_\alpha(X) \subset \mathcal{A}(X)$.
b) Let Y be any topological space. A map $F:X \to \mathfrak{P}(Y)$ (resp. $f: X \to Y$) is said to be of class α iff $F^{-1}(U) \in \mathcal{A}_\alpha$ (resp. $f^{-1}(U) \in \mathcal{A}_\alpha$) for all $U \in \mathfrak{G}(Y)$.

Using these notions we obtain the following consequence of Corollary 3.14.

3.18 Corollary: (Kaniewski-Pol [35])
Let X be a metric analytic space, Y a metric space, and $F:X \to \mathcal{K}^*(Y)$ of class $\alpha > 0$. Then F has a selection of class α.

Proof: It is known that $\mathfrak{L}_\alpha(X)$ satisfies conditions (i) to (v) of Lemma 3.9. Without loss of generality we may assume that Y is complete, otherwise we take the completion. We claim that F is inverse σ-dd preserving. To prove this it is enough to show that for every discrete family $(U_\lambda)_{\lambda\in\Lambda}$ of open sets in Y the family $(F^{-1}(U_\lambda))_{\lambda\in\Lambda}$ is σ-dd. Since $(F^{-1}(U_\lambda))_{\lambda\in\Lambda}$ is $\mathcal{A}(X)$-additive and point-finite (F(x), being compact, intersects only finitely many of the U_λ's) we obtain this conclusion from Lemma 3.16. Thus the corollary follows from Corollary 3.14.

Let us mention the following problem which seems still to be open.

<u>3.19 Problem</u>: Let X and Y be as in Corollary 3.18.
Does every weakly-$\mathcal{B}(X)$-measurable $F:X \to \mathcal{F}^*(Y)$ admit a Borel measurable selection ?

<u>3.20 Remark</u>:
Quite recently Jayne and Rogers [34a] have proved the following astonishing result:
Let X and Y be metrizable spaces and F: $X \to \mathcal{P}^*(Y)$ u.s.c. Then F has a selection of the second class.
Fremlin [16] has investigated selection problems for correspondences with values in a non-separable metric space. We will now show how his results are related to our approach.

<u>3.21 Definition</u>:
Let X be a set and $\mathcal{X} \subset \mathcal{P}(X)$.
$(X,\mathcal{X})$ satisfies the (<u>point-finite</u> ; <u>point-countable</u>) <u>reduction property</u> iff every (point-finite, point-countable) $\mathcal{X}$-additive family is $\mathcal{X}$-reducible.

<u>3.22 Corollary</u>:
Let X be a set, $\mathcal{X} \subset \mathcal{P}(X)$, Y a complete metric space, and $F:X \to \mathcal{F}^*(Y)$ weakly-$\mathcal{X}$-measurable. Then F has an $\mathcal{X}_\sigma$-$\mathcal{B}(Y)$-measurable selection in each of the following cases:
(i) $(X,\mathcal{X})$ has the reduction property
(ii) $(X,\mathcal{X})$ has the point-finite reduction property and $F(x) \in \mathcal{K}^*(Y)$ for all $x \in X$.
(iii) $(X,\mathcal{X})$ has the point-countable reduction property and F(x) is separable for all $x \in X$.

<u>Proof</u>: It is easy to see that F is $\mathcal{X}$-reducible in each of the above cases. Thus Theorem 3.5 yields the statement of the corollary.

3.23 Definition and remark:
Let $(X,\mathfrak{A})$ be a measurable space and $\mathfrak{n}$ a σ-ideal in $\mathfrak{P}(X)$. Define $\mathfrak{A}_{\mathfrak{n}} := \{A \subset X \mid \exists A_1, A_2 \in \mathfrak{A}: A_1 \subset A \subset A_2$ and $A_2 - A_1 \in \mathfrak{n}\}$. Then $\mathfrak{A}_{\mathfrak{n}}$ is a σ-field.

The following theorem collects some of the results of Fremlin [16] without giving the proofs.

3.24 Theorem: (Fremlin [16])

(i) [Martin's axiom] Let $(X,\mathfrak{A})$ be a measurable space. If there is a σ-ideal $\mathfrak{n}$ in $\mathfrak{P}(X)$ with $\mathfrak{n} \subset \mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{n}$ satisfies the countable chain condition (CCC) and there exists a countably generated σ-field $\mathfrak{A}_o$ with $\mathfrak{A} = (\mathfrak{A}_o)_{\mathfrak{n}}$, then $(X,\mathfrak{A})$ has the point-countable reduction property.

(ii) [Martin's axiom] Let X be a Hausdorff space, μ a finite Radon measure on X, and $\mathfrak{A}$ the σ-field of μ-measurable sets. If $\mathfrak{A}/\mu^{-1}(0)$ is of power less or equal to the continuum then $(X,\mathfrak{A})$ has the point-countable reduction property.

(iii) Let X be a Hausdorff space, μ a finite Radon measure on X, and $\mathfrak{A}$ the σ-field of μ-measurable sets. Then $(X,\mathfrak{A})$ has the point-finite reduction property.

(iv) Let X be compact with CCC, $\mathfrak{A} := \{U \Delta K \mid U$ open, K of first category in X$\}$ the σ-field of Baire property sets. Then $(X,\mathfrak{A})$ has the point-finite reduction property.

Our above considerations suggest the following problem.

3.25 Problem:
Give necessary and sufficient conditions (in terms of "additivity" properties) on a measurable space $(X,\mathfrak{A})$ such that every weakly-$\mathfrak{A}$-measurable correspondence from X to the closed (compact, closed separable) subsets of a complete metric space has a Borel-measurable selection.

4. A Boolean homomorphism approach to measurable selections.

The methods used in the proofs of the fundamental selection theorem (Theorem 2.1) and the main result of Chapter 3 (Theorem 3.5) seem to work only the case of correspondences with metrizable range. Here we shall discuss a method for proving measurable selection results in more general situations. This method was inspired by the following observations: A correspondence $F: X \to \mathfrak{P}(Y)$ is uniquely

determined by the inverse map $F^{-1}:\mathfrak{P}(Y) \to \mathfrak{P}(X)$. The same is true for a map $f:X \to Y$ and its inverse $f^{-1}:\mathfrak{P}(Y) \to \mathfrak{P}(X)$ which is a Boolean homomorphism. It is easy to see that f is a selection for F if and only if $f^{-1}(B) \subset F^{-1}(B)$ for all $B \in \mathfrak{P}(Y)$.
The idea of the following considerations is to first find a Boolean homomorphism φ with $\varphi(B) \subset F^{-1}(B)$ and then show that φ is almost equal to f^{-1} for some map $f: X \to Y$. $\mathfrak{A}$-$\mathfrak{B}$-measurability of f amounts -roughly speaking- to $\varphi(\mathfrak{B}) \subset \mathfrak{A}$.

To carry out this program we need some more definitions.

4.1 Definition:
Let X and Y be sets, $\mathfrak{A} \subset \mathfrak{P}(X)$ and $\mathfrak{B} \subset \mathfrak{P}(Y)$ closed under finite unions and containing the empty set and the whole space.
A map $\Phi: \mathfrak{B} \longrightarrow \mathfrak{A}$ is called a U-homomorphism if it satisfies
(i) $\Phi(\emptyset) = \emptyset$, $\Phi(Y) = X$
(ii) $\forall B,B' \in \mathfrak{B}: \Phi(B \cup B') = \Phi(B) \cup \Phi(B')$
If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras then $\varphi: \mathfrak{B} \to \mathfrak{A}$ is called a Boolean homomorphism if it satisfies (i) and (ii) and, in addition,
(iii) $\forall B,B' \in \mathfrak{B}: \Phi(B \cap B') = \Phi(B) \cap \Phi(B')$

4.2 Examples:
a) If $F: X \to \mathfrak{P}(Y)$ is a correspondence then $F^{-1}:\mathfrak{P}(Y) \to \mathfrak{P}(X)$ is a U-homomorphism.
b) For a map $f: X \to Y$ the map $f^{-1}:\mathfrak{P}(Y) \to \mathfrak{P}(X)$ is a Boolean homomorphism.

4.3 Definition:
Let $\mathfrak{H}$ be a field on X and κ a cardinal number. $\mathfrak{H}$ is said to be κ-complete iff each family $(H_i)_{i\in I}$ in $\mathfrak{H}$ with card $I \leq \kappa$ has a supremum $\bigvee\{H_i | i \in I\}$ in $\mathfrak{H}$. $\mathfrak{H}$ is said to be complete iff it is κ-complete for every cardinal κ.

4.4 Remark:
In general one has $\bigvee\{H_i | i \in I\} \supsetneq \bigcup_{i\in I} H_i$.

We are now able to state the following lemma whose proof can, for instance, be found in [23] .

4.5 Lemma:
Let $\mathfrak{A}$ and $\mathfrak{B}$ be fields of subsets of X and Y respectively, $\Phi: \mathfrak{B} \to \mathfrak{A}$

a $\cup$-homorphism, and $\mathfrak{A}$ κ-complete for all $\kappa <$ card $\mathfrak{L}$. Then there exists a Boolean homomorphism $\varphi: \mathfrak{L} \to \mathfrak{A}$ with $\varphi(B) \subset \Phi(B)$ for all $B \in \mathfrak{L}$.

4.6 Definitions:

a) For a set X, $\mathcal{H} \subset \mathfrak{P}(X)$, and κ a cardinal denote by $\mathcal{H}_\kappa = \{ \bigcup_{i \in I} H_i \mid (H_i)_{i \in I}$ family in $\mathcal{H}$ with card $I \leq \kappa\}$

b) For a regular Hausdorff space Y, a base $\mathfrak{B}$ for the topology of Y, and an open set $U \subset Y$ let $\kappa(\mathfrak{B},U)$ be the smallest cardinal κ such that there is a family $(V_i)_{i \in I}$ in $\mathfrak{B}$ with card $I = \kappa$ and $U = \bigcup_{i \in I} V_i = \bigcup_{i \in I} \overline{V_i}$

Now we can formulate the main theorem of this section and sketch its proof.

4.7 Theorem: (Graf [23])

Let X be a set, $\mathcal{H} \subset \mathfrak{P}(X)$ a field, Y a regular Hausdorff space which has a base $\mathfrak{B}$ for its topology such that $\mathcal{H}$ is κ-complete for all $\kappa <$ card $\mathfrak{B}$, $F: X \to \mathcal{K}(Y)$ a correspondence, and $\Phi: \mathfrak{F}(Y) \to \mathcal{H}$ a $\cup$-homomorphism with $\Phi(A) \subset F^{-1}(A)$ for all $A \in \mathfrak{F}(Y)$. Then there exists a selection f for F with $f^{-1}(U) \in \mathcal{H}_{\kappa(\mathfrak{B},U)}$ for all $U \in \mathfrak{G}(Y)$.

Proof: Let $\mathfrak{L}$ be the field generated by $\mathfrak{B}$. According to our assumptions $\mathcal{H}$ is κ-complete for all $\kappa <$ card $\mathfrak{L}$. The map $\bar{\Phi}: \mathfrak{L} \to \mathcal{H}$ defined by $\bar{\Phi}(B) = \Phi(\bar{B})$ is obviously a $\cup$-homomorphism. Thus Lemma 4.5 yields the existence of Boolean homomorphism $\varphi: \mathfrak{L} \to \mathcal{H}$ with

$$\varphi(B) \subset \bar{\Phi}(B) = \Phi(\bar{B}) \subset F^{-1}(\bar{B}) \qquad (*)$$

for all $B \in \mathfrak{L}$. For $x \in X$ define $\mathfrak{F}_x = \{B \in \mathfrak{L} \mid x \in \varphi(B)\}$. Then $\mathfrak{F}_x$ is an ultrafilter in $\mathfrak{L}$ and it follows from $(*)$ that $F(x) \cap \bar{B} \neq \emptyset$ for all $B \in \mathfrak{F}_x$. Since $F(x)$ is compact $\mathfrak{F}_x$ converges in $F(x)$. If we define $f: X \to Y$ by $f(x) = \lim \mathfrak{F}_x$ then f is obviously a selection for F. To show that f satisfies the desired measurability condition let $U \in \mathfrak{G}(Y)$ be given. Then there is a family $(V_i)_{i \in I}$ in $\mathfrak{B}$ with $\kappa(\mathfrak{B},U) =$ card I and $U = \bigcup_{i \in I} V_i = \bigcup_{i \in I} \overline{V_i}$. We claim that $f^{-1}(U) = \bigcup_{i \in I} \varphi(V_i)$,

hence $f^{-1}(U) \in \mathcal{H}_{\kappa(B,U)}$. If x belongs to $f^{-1}(U)$ then we can find an $i \in I$ with $f(x) \in V_i$. Since $\mathfrak{F}_x$ converges to $f(x)$ there exists a $B \in \mathfrak{F}_x$ with $B \subset V_i$, hence $x \in \varphi(B) \subset \varphi(V_i)$ which shows $f^{-1}(U) \subset \bigcup_{i \in I} \varphi(V_i)$.

Conversely let $x \in \bigcup_{i\in I} \varphi(V_i)$ be given. Then, for some $i \in I$,

$x \in \varphi(V_i)$ which implies $V_i \in \mathfrak{F}_x$ and, therefore, $f(x) = \lim \mathfrak{F}_x \in \overline{V}_i \subset U$, which proves $\bigcup_{i\in I} \varphi(V_i) \subset f^{-1}(U)$.

The above theorem looks rather technical in its assumptions. Its usefulness becomes visible only in its applications some of which we will now discuss (others can be found in [23]). As a first application we shall reprove Sion's selection theorem (cf. Sion [54]). For its formulation let us recall that a topological space is called stronlgy Lindelöf iff each of its open subspaces is Lindelöf.

4.8 Corollary: (Sion [54])
Let $(X,\mathfrak{A})$ be a measurable space, Y a strongly Lindelöf regular Hausdorff space of weight less than or equal to $\aleph_1$, and $F: X \to \mathfrak{K}^*(Y)$ $\mathfrak{A}$-measurable. Then there exists an $\mathfrak{A}$-$\mathfrak{B}(Y)$-measurable selection for F.

Proof: With the notation of Theorem 4.7 let $\mathfrak{X} = \mathfrak{A}$ and let $\mathfrak{B}$ be a base for the topology of Y with card $\mathfrak{B} \leq \aleph_1$. As a σ-field $\mathfrak{X}$ is obviously κ-complete for all $\kappa <$ card $\mathfrak{B}$. The application of Theorem 4.7 with F^{-1} in the place of Φ yields a selection f for F with $f^{-1}(U) \in \mathfrak{X}_{\kappa(\mathfrak{B},U)}$. Since Y is strongly Lindelöf we have $\kappa(\mathfrak{B},U) \leq \aleph_0$, hence $\mathfrak{X}_{\kappa(\mathfrak{B},U)} = \mathfrak{A}$; and the corollary is proved.

4.9 Remark:
Sion [54] has used the above corollary to prove a selection result for certain correspondences with K-analytic graphs. In particular the Jankov-von Neumann theorem (Cor. 2.6) can be deduced from Sion's result.

As a second application of Theorem 4.7 we shall prove a general selection theorem for upper semi-continuous compact-valued correspondences which is again the starting point for many other selection results. The crucial notion in the following considerations is that of a lifting as defined below.

4.10 Definition:
Let X be a set, $\mathfrak{A} \subset \mathfrak{P}(X)$ a field, and $\mathfrak{N} \subset \mathfrak{A}$ an ideal. Then a map $\Theta: \mathfrak{A} \to \mathfrak{A}$ is called a lifting w.r.t. $\mathfrak{N}$ iff the following conditions

are satisfied:

(i) Θ is a Boolean homomorphism

(ii) For all $A, A' \in \mathfrak{A}$ with $A \Delta A' \in \mathfrak{n}$ the equality $\Theta(A)=\Theta(A')$ holds.

(iii) For $A \in \mathfrak{A}$ the set $\Theta(A) \Delta A$ belongs to $\mathfrak{n}$.

If X carries a topology τ with $\tau \subset \mathfrak{A}$ then a lifting $\Theta: \mathfrak{A} \to \mathfrak{A}$ w.r.t. $\mathfrak{n}$ is called strong iff

(iv) $\Theta(A) \subset A$ for all τ-closed sets $A \subset X$, or equivalently, $U \subset \Theta(U)$ for all $U \in \tau$.

4.11 Proposition:

Let X be a topological space, $\mathfrak{n}$ the ideal of nowhere dense subsets of X, and

$\mathfrak{A} = \{U \Delta N \mid U \in \mathfrak{G}(X) \text{ and } N \in \mathfrak{n}\}$.

Then $\mathfrak{A}$ is a field and there exists a strong lifting $\Theta: \mathfrak{A} \to \mathfrak{A}$ w.r.t. $\mathfrak{n}$

Such a strong lifting has the following properties:

(i) $\Theta(\mathfrak{A})$ is a complete field.

(ii) If $(A_i)_{i \in I}$ is a family in $\mathfrak{A}$ and $A := \bigcup_{i \in I} \Theta(A_i)$ then $\mathring{A} \subset A \subset \mathring{\bar{A}}$. In particular A belongs to $\mathfrak{A}$.

Proof: Using the fact that $\bar{U} \setminus U$ is nowhere dense for any open set $U \subset X$ it follows by standard calculations that $\mathfrak{A}$ is a field. It is also easy to check that for $U, U' \in \mathfrak{G}(X)$ and $N, N' \in \mathfrak{n}$ the equality $U \Delta N = U' \Delta N'$ holds if and only if $\bar{U} = \bar{U}'$. Now define $\delta: \mathfrak{A} \to \mathfrak{A}$ by $\delta(U \Delta N) = \mathring{\bar{U}}$. Then δ has the following properties:

a) $\delta(\emptyset)=\emptyset$, $\delta(X)=X$

b) $\delta(A \cap A') = \delta(A) \cap \delta(A')$

c) $A \Delta A' \in \mathfrak{n} \Rightarrow \delta(A) = \delta(A')$

d) $A \Delta \delta(A) \in \mathfrak{n}$ for all $A, A' \in \mathfrak{A}$

e) $U \subset \delta(U)$ for all $U \in \mathfrak{G}(X)$

For $x \in X$ define $\mathfrak{A}_x = \{A \in \mathfrak{A} \mid x \in \delta(A)\}$. Then $\mathfrak{A}_x$ is a filter in $\mathfrak{A}$, which according to a) and c) does not contain any nowhere dense set. Hence there is an ultrafilter $\mathfrak{U}_x$ in $\mathfrak{A}$ with $\mathfrak{A}_x \subset \mathfrak{U}_x$ and $\mathfrak{U}_x \cap \mathfrak{n} = \emptyset$. Define $\Theta: \mathfrak{A} \to \mathfrak{P}(X)$ by $\Theta(A) = \{x \in X \mid A \in \mathfrak{U}_x\}$. Then Θ is a Boolean homomorphism with $\delta(A) \subset \Theta(A) \subset (\delta(A^c))^c$. According to d), $\delta(A^c) \setminus \delta(A)$ belongs to $\mathfrak{n}$ which implies $\Theta(A) \in \mathfrak{A}$ and $\Theta(A) \Delta A \in \mathfrak{n}$. It is also easy to check that $\Theta(A)=\Theta(A')$ for all $A, A' \in \mathfrak{A}$ with $A \Delta A' \in \mathfrak{n}$. Since $U \subset \delta(U) \subset \Theta(U)$ for $U \in \mathfrak{G}(X)$ we, therefore, know that Θ is a strong lifting w.r.t. $\mathfrak{n}$.

Because Θ is a Boolean homomorphism we know that $\Theta(\mathfrak{A})$ is a field. Let $(A_i)_{i \in I}$ be an arbitrary family in $\mathfrak{A}$ and define $A = \bigcup_{i \in I} \Theta(A_i)$. Then

$\bigcup_{i\in I} \delta(A_i)$ is an open set with $\overline{\bigcup_{i\in I} \delta(A_i)} \supset A \supset \bigcup_{i\in I} \delta(A_i)$. This proves $A \in \mathfrak{A}$ and $\mathring{A} \subset A \subset \overline{\mathring{A}}$. Obviously $\Theta(A)$ is the supremum of $(\Theta(A_i))_{i\in I}$ in $\Theta(\mathfrak{A})$, and the proof is completed.

We are now ready to prove the following result.

4.12 Corollary:

Let X be a topological space, Y a regular Hausdorff space, and $F: X \to \mathcal{K}^*(Y)$ u.s.c. Then there exists a selection f for F with

$$\widehat{f^{-1}(U)}^{\circ} \subset f^{-1}(U) \subset \overline{\widehat{f^{-1}(U)}^{\circ}}$$

for all $U \in \mathfrak{G}(Y)$. In particular f is $\mathfrak{A}$-$\mathfrak{G}(Y)$-measurable.

Proof: Let $\mathfrak{A}$ and $\mathfrak{n}$ be as in Prop. 4.11 and let $\Theta: \mathfrak{A} \to \mathfrak{A}$ be a strong lifting w.r.t. $\mathfrak{n}$. With the notation of Theorem 4.7 let $\mathcal{H} = \Theta(\mathfrak{A})$ and let $\mathfrak{B}$ be any base for the topology of Y. According to Prop. 4.11, $\mathcal{H}$ is complete. Define $\Phi: \mathfrak{F}(Y) \to \mathcal{H}$ by $\Phi(A) = \Theta(F^{-1}(A))$. Since Θ is strong and F is u.s.c. we have $\Phi(A) \subset F^{-1}(A)$ for all $A \in \mathfrak{F}(Y)$. Thus Theorem 4.7 yields the existence of a selection f for F with $f^{-1}(U) \in \mathcal{H}_{K(\mathfrak{B},U)}$ for all $U \in \mathfrak{G}(Y)$. It follows from condition (ii) in Prop. 4.11 that

$$\widehat{f^{-1}(U)}^{\circ} \subset f^{-1}(U) \subset \overline{\widehat{f^{-1}(U)}^{\circ}} .$$

Let us recall that for a topological space X the collection $\{U\Delta K \mid U\in\mathfrak{G}(X)$, K of first category in X$\}$ forms a σ-field $\mathfrak{B}_p(X)$. Then the following result is an immediate consequence of Corollary 4.12.

4.13 Corollary:

Let X be a topological space, Y a regular Hausdorff space, and $F: X \to \mathcal{K}^*(Y)$ u.s.c. Then there exists a $\mathfrak{B}_p(X)$-$\mathfrak{B}(Y)$-measurable selection for F.

As a further consequence of the above corollary we obtain a result on continuous selections due to Hasumi [28]. Let us recall that a topological space X is called extremally disconnected if $\overline{U}$ is open for every open set $U \subset X$.

4.14 Corollary: (Hasumi [28])

Let X be an extremally disconnected space, Y a regular Hausdorff space, and $F: X \to \mathcal{K}^*(Y)$ u.s.c. Then there exists a continuous selection for F.

Proof: According to Corollary 4.12 there exists a selection f for F with

$$\mathring{\widehat{f^{-1}(U)}} \subset f^{-1}(U) \subset \overline{\mathring{\widehat{f^{-1}(U)}}} \tag{1}$$

for every $U \in \mathfrak{G}(Y)$. Since (1) implies $\overline{f^{-1}(U)} \subset \mathring{\widehat{f^{-1}(U)}} \subset \overline{f^{-1}(U)}$ and since $\mathring{\widehat{f^{-1}(U)}}$ is open due to the fact that X is extremally disconnected we know that

$$\overline{f^{-1}(U)} \text{ is open} \tag{2}$$

for all $U \in \mathfrak{G}(Y)$. By complementation we deduce from (1) that

$$\overline{\mathring{f^{-1}(A)}} \subset f^{-1}(A) \subset \overline{f^{-1}(A)} \tag{3}$$

for all $A \in$ (Y).

Now let $U \in \mathfrak{G}(Y)$ be given. Then there exists a family $(V_i)_{i\in I}$ in $\mathfrak{G}(Y)$ with

$$U = \bigcup_{i\in I} V_i = \bigcup_{i\in I} \overline{V}_i \tag{4}$$

Since, due to (2), $\overline{f^{-1}(V_i)}$ is open we have

$$f^{-1}(V_i) \subset \overline{f^{-1}(V_i)} \subset \overline{\mathring{f^{-1}(V_i)}} \subset \overline{\mathring{f^{-1}(\overline{V}_i)}} \tag{5}$$

From (3) we obtain

$$f^{-1}(\overline{V}_i) \supset \overline{\mathring{f^{-1}(\overline{V}_i)}} \tag{6}$$

Combining (4), (5), and (6) yields

$$f^{-1}(U) = \bigcup_{i\in I} \overline{\mathring{f^{-1}(\overline{V}_i)}} \in \mathfrak{G}(X).$$

Our next aim is to apply Theorem 4.7 in a measure theoretic context. For this purpose we have to collect some definitions and facts about measure spaces.

4.15 Definitions:

a) A measure space $(X,\mathfrak{A},\mu)$ is said to be strictly localizable iff there exists a collection ϑ of pairwise disjoint sets from $\mathfrak{A}$ of strictly positive finite measure such that
   (i) $A \in \mathfrak{A} \Leftrightarrow \forall D \in \vartheta: A \cap D \in \mathfrak{A}$
   (ii) $\mu(A) = \sum_{D\in\vartheta} \mu(A\cap D)$

b) A measure space $(X,\mathfrak{A},\mu)$ is said to be complete iff $B \in \mathfrak{A}$ for all $B \subset X$ with $B \subset A$ for some μ-nullset $A \in \mathfrak{A}$.

c) $(X,\mathfrak{A},\mu,\tau)$ is said to be a topological measure space iff $(X,\mathfrak{A},\mu)$ is a measure space and τ is a topology on X with $\tau \subset \mathfrak{A}$.

d) A measure space $(X,\mathfrak{A},\mu)$ (resp. topological measure space $(X,\mathfrak{A},\mu,\tau)$ has a <u>lifting</u> (resp. <u>strong lifting</u>) iff there exists a lifting (resp. strong lifting) $\theta:\mathfrak{A}\to\mathfrak{A}$ w.r.t. the σ-ideal of μ-nullsets.

<u>4.16 Remarks</u>:

(i) Obviously any σ-finite measure space is strictly localizable. The same is true for Radon measure spaces.(see, for instance, Schwarz [53], p.46, Thm.13).

(ii) Maharam [44] (for the σ-finite case) and Ionescu-Tulcea [31] have shown that every complete strictly localizable measure space has a lifting.

(iii) For a topological measure space $(X,\mathfrak{A},\mu,\tau)$ with $\mu(U) > 0$ for $U \in \tau\setminus\{\emptyset\}$ the existence of a strong lifting is known in each of the following cases:

a) (X,τ) is second countable and $(X,\mathfrak{A},\mu)$ is finite (see Graf [21]).

b) (X,τ) is an analytic space and $(X,\mathfrak{A},\mu)$ is σ-finite (see Schwarz [53], p.132, Lemma 21).

c) (X,τ) is a metrizable locally compact space, μ is a Radon measure on (X,τ), and $\mathfrak{A}$ is the σ-field of μ-measurable sets (Ionescu-Tulcea [33], p.129).

d) (X,τ) is a locally compact group, μ is the Haar (Radon) measure on (X,τ) and $\mathfrak{A}$ is the σ-field of μ-measurable sets (Ionescu-Tulcea [32]).

e) (X,τ) is the quotient of a locally compact group G with respect to a closed subgroup H, μ is a quasi-invariant Radon measure on X (i.e. $\forall\, g \in G\ \forall B \in \mathfrak{B}(X)$: $\mu(B)=0 \Leftrightarrow \mu(g,B)=0$), and $\mathfrak{A}$ is the σ-field of μ-measurable sets (Kupka [36]).

The following corollary was conjectured by Christensen and independently proved by the author [22, 23] , and Talagrand [55]. It is also possible that H. v. Weizsäcker, who communicated Christensen's conjecture to the author, has an unpublished proof for the same result.

<u>4.17 Corollary</u>: (Graf [23], Talagrand [55])
Let $(X,\mathfrak{A},\mu,\tau)$ be a strictly localizable complete topological measure space which has a strong lifting Θ. Moreover, let Y be a regular Hausdorff space and $F:X \to \mathcal{K}^*(Y)$ u.s.c. Then there exists an

$\mathfrak{A}$-$\mathcal{B}(Y)$-measurable selection for F.

Proof: With the notation of Theorem 4.7 let $\mathcal{X}=\theta(\mathfrak{A})$, $\mathcal{B}=\mathcal{G}(Y)$. Since $\mathcal{X}$ is Boolean isomorphic to $\mathfrak{A}/\mu^{-1}(0)$ we know that $\mathcal{X}$ is complete. Define $\Phi: \mathfrak{F}(Y) \to \mathcal{X}$ by $\Phi(A)=\theta(F^{-1}(A))$. Then Φ is a U-homomorphism with $\Phi(A) \subset F^{-1}(A)$ for all $A \in \mathfrak{F}(Y)$. According to Theorem 4.7 there exists a selection f for F with $f^{-1}(U) \in \mathcal{X}_{K(\mathcal{B},U)}$ for all $U \in \mathcal{G}(Y)$. Due to a generalized version of a theorem of Maharam (see Ionescu Tulcea [33], p. 55) we have $\mathcal{X}_{K(\mathcal{B},U)} \subset \mathfrak{A}$ and the proof of the theorem is finished.

The following result of Losert [40] shows that there is a close connection between the existence of measurable selections and the existence of some kind of lifting. To state this result let us recall that the Baire σ-field $\mathcal{B}a(X)$ is the σ-field generated by all continuous real-valued functions on X (see also [33], p.169, Thm.2)

4.18 Proposition: (Losert [40])

Let X be a completely regular Hausdorff space and $\mu: \mathcal{B}(X) \to \mathbb{R}_+$ a measure. Then the following conditions are equivalent:

(i) For every completely regular Hausdorff space Y and every u.s.c. $F:X \to \mathcal{K}^*(Y)$ there exists a $\mathcal{B}(X)_\mu$-$\mathcal{B}a(Y)$-measurable selection for F.

(ii) There exists a Boolean homomorphism $\theta:\mathcal{B}(X)_\mu \to \mathcal{B}(X)_\mu$ such that $\theta(A)=\theta(B)$ for all $A,B \in \mathcal{B}(X)_\mu$ with $\mu(A\Delta B)=0$ and $\theta(C) \subset C$ for all $C \in \mathfrak{F}(X)$.

Proof:

(i) $\Rightarrow$(ii): Let Y be the Stone representation space of $\mathcal{B}(X)_{/\mu^{-1}(0)}$, i.e. Y is the set of all ultrafilters in $\mathcal{B}(X)$ which do not meet $\mu^{-1}(0)$ with its Boolean topology. Define $F:X \to \mathcal{K}^*(Y)$ by $F(x)=\{y \in Y \mid y \text{ converges to } x\}$. Then F is u.s.c.. There exists, therefore, a $\mathcal{B}(X)_\mu$-$\mathcal{B}a(Y)$-measurable selection f for F. The map $\theta:\mathcal{B}(X)_\mu \to \mathcal{B}(X)_\mu$ defined by $\theta(B)=f^{-1}(\{y \in Y \mid B \in y\})$ is a Boolean homomorphism with the desired properties.

(ii) $\Rightarrow$(i): Let Y be a completely regular Hausdorff space, $F:X \to \mathcal{K}^*(Y)$ u.s.c. With the notation of Theorem 4.7 let $\mathcal{X}=\theta(\mathcal{B}(X)_\mu)$, $\mathcal{B}=\mathcal{G}(Y)$. Then $\mathcal{X}$ is isomorphic to $\mathcal{B}(X)_{/\mu^{-1}(0)}$ and, therefore, complete. The map $\Phi:\mathfrak{F}(Y) \to \mathcal{X}$ defined by $\Phi(A)=\theta(F^{-1}(A))$ is a $\cup$-homomorphism with $\theta(F^{-1}(A)) \subset F^{-1}(A)$ for all $A \in \mathfrak{F}(Y)$. Hence, Theorem 4.7 yields the existence of a selection f for F with $f^{-1}(U) \in \mathcal{X}_{K(\mathcal{B},U)}$ for every $U \in \mathfrak{G}(Y)$. For open $\mathfrak{F}_\sigma$-sets $U \subset Y$ one has $\kappa(\mathfrak{B},U) \leq \aleph_0$ and, therefore, $f^{-1}(U) \in \mathfrak{B}(X)_\mu$. Thus f has the desired properties.

4.19 Remark: (Losert [40])
Using the above proposition and a rather involved construction Losert [40] has shown that there is a compact space X, a Radon measure μ on X, a compact space Y, and a continuous surjective map $p: Y \to X$ which has no $\mathfrak{B}(X)_\mu$ -$\mathfrak{B}a(Y)$-measurable section.

It was Kupka [36] who first realized that Corollary 4.17 is very useful in the framework of topological group theory.

4.20 Corollary: (Kupka [36])
Let G be a locally compact group, H a closed subgroup of G, μ a quasi-invariant Radon measure on G/H , and $\mathfrak{A}$ the σ-field of μ-measurable sets in G/H. Then there exists an $\mathfrak{A}$-$\mathfrak{B}(G)$-measurable section f for the quotient map $\pi: G \to G/H$ such that f(K) is relatively compact for every compact subset $K \subset G/H$

Proof: It is well-known that there exists a closed subset M of G such $M \cap \pi^{-1}(K)$ is a non-empty compact subset of G for every compact subset K of G/H (see, for instance, Bourbaki [5], p.51 Prop.8). Define $F: G/H \to \mathfrak{K}^*(M)$ by $F(x)= \pi^{-1}(x) \cap M$. One can see that F is u.s.c. Thus, Corollary 4.17 combined with Remark 4.16 (iii) a) yields the desired result.

In this context Kupka [36] posed the following problem.

4.21 Problem:
In the situation of the above corollary, does there always exist a Lusin μ-measurable section for π ?

As application of Corollary 4.20 we obtain the following result.

4.22 Application:
Every closed subgroup H of an amenable locally compact group G is amenable.

Proof: Let $m: L^\infty(G) \to \mathbb{R}$ be a left-invariant mean. Let π,μ, and $\mathfrak{A}$ be as in Corollary 4.20. Let f be an $\mathfrak{A}$-$\mathfrak{B}(G)$-measurable section for π . Define $T: \mathfrak{C}_b(H) \to L^\infty(G)$ by $T(h)(x)= h(x^{-1} f(\pi(x)))$. Then T is pôsitive and linear with T1=1. Therefore m'=moT is a left-invariant mean on H.

In [36] Kupka also asked the question whether in the situation of Corollary 4.20 there always exists a Baire measurable section for π. $f:G/H \to G$ is called Baire measurable iff it is $\mathcal{B}a(G/H)$-$\mathcal{B}a(G)$-measurable. The following result is a partial answer to Kupka's question.

4.23 Proposition: (Graf-Mägerl [25])
Let G be a compact group, H a closed subgroup of G and $\pi:G \to G/H$ the quotient map. Then π has a Baire measurable section.

By embedding G into the product of compact metrizable groups and a transfinite induction argument the proof of the above proposition can be reduced to proving a result which is again a corollary of the main theorem of this section. Before we can formulate this corollary we need to say what the Bockstein separation property for a topological space is.

4.24 Definition:
A topological space X is said to possess the Bockstein separation property (BSP) iff any two disjoint open sets in X can be separated by open $\mathfrak{F}_\sigma$-sets.

4.25 Remark:
a) Bockstein [4] has shown that arbitrary products of Polish spaces have the BSP.
b) Due to a result of Pelczyński ([48], Thm.7.5 and Cor.5.11) every compact group has the BSP.

Our next lemma gives a characterization of spaces with the BSP.

4.26 Lemma:
A topological space X has the BSP if and only if the closure of every open set in X is a G_δ-set.

Proof:
"$\Rightarrow$": Let X have the BSP and let $U \subset X$ be open. Then there are open $\mathfrak{F}_\sigma$-sets V and W in X with $W \cap V = \emptyset$, $U \subset V$ and $X \setminus \overline{U} \subset W$. Obviously $\overline{U} \cap W = \emptyset$ and, therefore, $X \setminus \overline{U} = W$ which shows that $\overline{U}$ is a G_δ-sets.
"$\Leftarrow$": Now let $U,V \in \mathfrak{G}(Y)$ be disjoint. Def. $V' := X \setminus \overline{U}$ and $U' = X \setminus \overline{V'}$

Then V' and U' are disjoint open $\mathfrak{F}_\sigma$-sets with $V \subset V'$ and $U \subset X \setminus \overline{V'} = U'$.

Now we are in the position to prove the following corollary of Theorem 4.7.

4.27 Corollary: (Graf-Mägerl [25])
Let X be topological space with BSP, $\mathfrak{A}$ the σ-field generated by the open $\mathfrak{F}_\sigma$-sets in X, Y a separable metric space, and $F: X \to \mathfrak{K}^*(Y)$ u.s.c. Then there exists an $\mathfrak{A}$-$\mathfrak{B}(Y)$-measurable selection for F.

Proof: With the notation of Theorem 4.7 let $\mathfrak{X} = \mathfrak{A}$ and let $\mathfrak{B}$ be any countable base for the topology of Y. Then $\mathfrak{X}$ is obviously $\underset{\sigma}{\kappa}$-complete for all $\kappa <$ card $\mathfrak{B}$. The map $\Phi: \mathfrak{F}(Y) \to \mathfrak{X}$ defined by $\Phi(A) = \widehat{F^{-1}(A)}$ is a $\cup$-homomorphism with $\Phi(A) \subset F^{-1}(A)$ for all $A \in \mathfrak{F}(Y)$. Hence, according to Theorem 4.7, there exists a selection f for F with $f^{-1}(U) \in \mathfrak{X}_{\kappa(\mathfrak{B},U)}$ for all $U \in \mathfrak{G}(Y)$. Since $\kappa(\mathfrak{B},U) \leq \aleph_0$ we have $\mathfrak{X}_{\kappa(\mathfrak{B},U)} = \mathfrak{A}$ and the conclusion of the theorem follows.

4.28 Problem:
It would be interesting to know whether the topological spaces X with the BSP are characterized by the fact that any u.s.c. compact-valued correspondence from X to any separable metric space admits a selection as described in Corollary 4.27.

In most of the results proved in this section the correspondences have been upper semi-continuous and compact-valued. This is a rather severe restriction. But, using an idea quoted in Dellacherie ([10]. p.217), we shall show how more general results can be obtained from our special ones.

4.29 Proposition:
Let X and Y be Hausdorff spaces and let μ be a finite Radon measure on X. Assume that for every $K \in \mathfrak{K}(X)$ and for all $G: K \to \mathfrak{K}^*(Y)$ u.s.c. there exists a $\mathfrak{B}(K)_\mu$-$\mathfrak{B}(Y)$-measurable selection for G.
If $F: X \to \mathfrak{P}^*(Y)$ satisfies

$$\mu(X) = \sup\{\, \mu(\pi_X(K)) \mid K \subset \operatorname{Gr} F,\ K \text{ compact}\}$$

then there exists a $\mathfrak{B}(X)_\mu$-$\mathfrak{B}(Y)$-measurable selection for F.

Proof: According to the assumptions about F there exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of Gr(F) with $\lim_{n\to\infty} \mu(\pi_X(K_n)) = \mu(X)$, i.e.

$N := X \setminus \bigcup_{n\in\mathbb{N}} \pi_X(K_n)$ is a μ-nullset. Since the map $\pi_X(K_n) \to Y$,

$x \to \{y \in Y \mid (x,y) \in K_n\}$ is an u.s.c. compact-valued correspondence we know that there exists a $\mathcal{B}(\pi_X(K_n))_\mu$-$\mathcal{B}(Y)$ measurable map $f_n: \pi_X(K_n) \to Y$ with $(x, f_n(x)) \in K_n$ for all $x \in \pi_X(K_n)$. Obviously $f: X \to Y$ defined by

$$f(x) = \begin{cases} f_n(x), & x \in K_n \setminus \bigcup_{\nu=1}^{n-1} K_\nu \\ \in F(x) & \text{arbitrary, } x \in N \end{cases}$$

is a $\mathcal{B}(X)_\mu$-$\mathcal{B}(Y)$-measurable selection for F.

4.30 Corollary:

Let X be a Hausdorff space, μ a Radon measure on X, and $\mathfrak{A}$ the σ-field of μ-measurable sets. Assume that $(X,\mathfrak{A},\mu)$ has a strong lifting. Moreover, let Y be a Hausdorff space and let $F: X \to \mathfrak{P}^*(Y)$ have a graph which is capacitable w.r.t. every capacity on $X \times Y$. Then F has an $\mathfrak{A}$-$\mathcal{B}(Y)$-measurable selection.

Proof: The corollary follows immediately from Corollary 4.17 combined with Prop. 4.29.

4.31 Remark:

$F: X \to \mathfrak{P}^*(Y)$ satisfies the assumption of Corollary 4.30 if Gr(F) is a K-analytic set contained in some $\mathfrak{K}_\sigma$-set (see Choquet [8], p.155, Theorem 9.5).❙ One major application of measurable selection theorems is the proof of existence and uniqueness of preimages of a given measure (see, for instance, Landers-Rogge [38], Lubin [41], Ershov [13], [14], Lehn-Mägerl [39], Graf [22], [24]). In [22] it is shown how one can derive the following rather general theorem (a special case of which has been proved by Edgar [12]) from Corollary 4.17.

4.32 Proposition: (Edgar [12], Graf [22])

Let $(X,\mathfrak{A},\mu)$ be a finite measure space, Y a Hausdorff space, and $p: Y \to X$ $\mathcal{B}(Y)$-$\mathfrak{A}$-measurable such that

(i) $\mu(X) = \sup\{\mu^*(p(K)) \mid K \subset Y,\ K \text{ compact}\}$

and (ii) $\forall K \in \mathfrak{K}(Y)$: $\mu^*(p(K)) = \inf\{\mu^*(p(U)) \mid K \subset U,\ U \text{ open}\}$.

Then there exists an $\mathfrak{A}_\mu$-$\mathcal{B}(Y)$-measurable map $f: X \to Y$ such that:

a) $\forall A \in \mathfrak{A}$: $\mu(A \Delta f^{-1}p^{-1}(A)) = 0$

b) $\nu = \mu \circ f^{-1}$ is a Radon measure on Y with $\nu \circ p^{-1} = \mu$.

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