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INTRODUCTION

In this note, we are going to consider Banach spaces ("B-spaces") X which are characterized by the property that every operator on a Hilbert space which factors through X is a Hilbert-Schmidt operator ("HS-operator"). We propose to call such spaces <u>Hilbert-Schmidt spaces</u>, or <u>HS-spaces</u>, for brevity.

By using Dvoretzky's theorem [3], Bellenot [1] has proved that every compact operator on a Hilbert space factors through a subspace of an arbitrary given infinite-dimensional B-space . Thus, by appealing to a result of Lindenstrauss-Pełczyński [13], we see that HS-operators factor through every prescribed infinite-dimensional B-space. Consequently, factorization through an HS-space of infinite dimension actually characterizes HS-operators among bounded operators on a Hilbert space. This may serve to justify our terminology.

Our aim is to give several characterizations of HS-spaces, to derive some of their general properties, to present a few examples, and to touch upon their relations to some classes of B-spaces which have been studied extensively in the recent literature.

NOTATION

As for B-spaces, our terminology and notation will be standard. We shall also use results on ideals of operators between B-spaces. Here all details can be found in A. Pietsch's monograph [18]; the basic theory is also contained in [8]. Frequently, we will be concerned with quotients of ideals; we therefore recall the definition. If $\mathcal{A}$ and $\mathcal{B}$ are ideals, then the component of the ideal $\mathcal{A}^{-1}\circ\mathcal{B}$ ("left quotient") for a pair (X,Y) of B-spaces consists of all operators $T\in\mathcal{L}(X,Y)$ such that, for every B-space Z and all $S\in\mathcal{A}(Y,X)$, we have $ST\in\mathcal{B}(X,Z)$. Similarly, the "right quotient" $\mathcal{A}\circ\mathcal{B}^{-1}$ is defined. Note that the identity I_X of a B-space X belongs to $\mathcal{A}^{-1}\circ\mathcal{B}(X,Y)$ (we shall simply write $I_X\in\mathcal{A}^{-1}\circ\mathcal{B}$) iff $\mathcal{A}(X,\cdot)\subset\mathcal{B}(X,\cdot)$ holds; the dot is to substitute an arbitrary B-space. Similarly, $I_X\in\mathcal{A}\circ\mathcal{B}^{-1}$ iff $\mathcal{B}(\cdot,X)\subset\mathcal{A}(\cdot,X)$.

Under favourable enough conditions, $\mathcal{A}^{-1}\circ\mathcal{B}$ and $\mathcal{A}\circ\mathcal{B}^{-1}$ can be considered as a sort of adjoint of some other ideal which simplifies a lot of the manipulations with such ideals. We do not repeat the details here; the reader is referred to Jarchow-Ott [9].

We shall in particular consider the ideals $\mathcal{K},\mathcal{P}_p,\mathcal{I}_p,\mathcal{N}_p$ $(0<p<\infty)$ of compact, p-summing, p-integral, and p-nuclear operators, further the ideals Γ_r $(0<r\leq\infty)$ of all operators $X \longrightarrow Y$, where X and Y are B-spaces, whose composition with the canonical (evaluation) map $Y \longrightarrow Y''$ factors through an $\mathcal{L}_r(\mu)$-space, and also the largest extension, $\mathcal{P}_{2,2,2}$, of HS-operators to an ideal of operators between B-spaces. Notice that $\mathcal{P}_{2,2,2} = \Gamma_2^{-1}\circ\mathcal{P}_2\circ\Gamma_2^{-1}$.

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GENERALITIES

The following characterizations of HS-spaces are easily obtained from well-known results on 2-summing operators.

1. Proposition: For every B-space X the following are equivalent:

(i) X is a HS-space.

(ii) $I_X\in\mathcal{P}_{2,2,2}$.

(iii) X* is a HS-space.

(iv) $\mathcal{L}(X,\ell_2) = \mathcal{P}_2(X,\ell_2)$.

(v) $\mathcal{L}(\ell_2,X) = \mathcal{P}_2^{dual}(\ell_2,X)$.

(vi) For every B-space Y containing X every $S\in\mathcal{L}(X,\ell_2)$ admits an extension $\tilde{S}\in\mathcal{L}(Y,\ell_2)$.

(vii) For every B-space Y containing X and every $S\in\mathcal{L}(X,\ell_2)$ there is a constant C such that $\sum_{i=1}^{k}\|Sx_i\|^2 \leq C\cdot\sum_{j=1}^{m}\|y_j\|^2$ holds for all sequences $(x_i)_{i\leq k}$ in X and $(y_j)_{j\leq m}$ in Y such that

$$\sum_{i=1}^{k} |\langle b,x_i\rangle|^2 \leq \sum_{j=1}^{m} |\langle b,y_j\rangle|^2.$$

(viii) For every B-space Y containing X every $S \in \mathcal{L}(\ell_2,X^*)$ admits a lifting $\hat{S} \in \mathcal{L}(\ell_2,Y^*)$; equivalently, every weak ℓ_2-sequence in X* can be lifted to a weak ℓ_2-sequence in Y*.

Here (i) through (v) are obvious, and (vi) and (viii) stem from the fact that $\mathcal{L}_1$-spaces and $\mathcal{L}_\infty$-spaces are HS-spaces, cf. Grothendieck [6] and Lindenstrauss-Pełczyński [13]. (vii) is due to Maurey [15].

We continue by giving some further examples.

2. Examples: The following two statements are dual to each other; they have been proved by Kisliakov [10] and Pisier [19].

(a) If R is a reflexive subspace of an $\mathcal{L}_1$-space X, then X/R is a HS-space.

(b) If Z is a subspace of an $\mathcal{L}_\infty$-space Y such that Y/Z is reflexive, then Z is a HS-space.

(c) A recent result of Bourgain's [2] yields that the disk algebra A and the space H_∞ of bounded analytic functions on $\{z \in \mathbb{C} \,|\, |z|<1\}$ are HS-spaces.

(d) If X and Y are infinite-dimensional B-spaces sucht that $\mathcal{N}_1(X,Y) = \mathcal{K}(X,Y)$, or $X\otimes_\varepsilon Y = X\otimes_\pi Y$, then X and Y are HS-spaces. This can be seen by appealing to Dvoretzky's theorem [3]; compare also with Pisier [22].

In [23], Pisier has shown that every B-space E of cotype 2 is contained in a B-space Z such that $Z\otimes_\varepsilon Z = Z\otimes_\pi Z$ holds and both, Z and Z* are of cotype 2. This surprising result answers in the negative several problems on B-spaces and nuclear locally convex spaces raised by Grothendieck [5] and others.

That a B-space Z with $Z\otimes_\varepsilon Z = Z\otimes_\pi Z$ must be a HS-space can be seen without reference to Dvoretzky's theorem. In fact, if $S: X \longrightarrow Y$ is a bounded linear operator between B-spaces X and Y such that $S\otimes S: X\otimes_\varepsilon Y \longrightarrow Y\otimes_\pi Y$ is continuous, then the adjoint of $S\otimes S$ can be considered as the map $\mathcal{L}(Y,Y^*) \longrightarrow \mathcal{I}_1(X,X^*): V \longrightarrow S^*VS$. In particular, for all $A\in\mathcal{L}(Y,\ell_2)$ and all $T\in\mathcal{L}(\ell_2,X)$, $(AST)^*AST$ belongs to $\mathcal{I}_1(\ell_2,\ell_2) = \mathcal{N}_1(\ell_2,\ell_2)$, i.e. AST is a HS-operator, and consequently we have $S\in\mathcal{P}_{2,2,2}(X,Y)$.

Let X be a HS-space and Y a (closed) subspace of X. By 1, Y [X/Y] is a HS-space if, and only if, every weak ℓ_2-sequence in Y*[in X/Y] can be lifted to a weak ℓ_2-sequence in X*[in X**]. But we do not know an intrinsic characterization of Y to ensure the HS-property for Y or X/Y.

On the other hand, HS-spaces enjoy the following "three space property":

3. <u>Proposition</u>: Let Y be a subspace of a B-space X. If Y and X/Y are HS-spaces, then so is X.

To prove this, let $S\in\mathcal{L}(X,\ell_2)$ be given. By hypothesis, $T:=S/Y$ is 2-summing, hence $T=\tilde{T}/Y$ for some $\tilde{T}\in\mathcal{P}_2(X,\ell_2)$. Since $S-\tilde{T}$ vanishes on Y, $S-\tilde{T}=A\circ Q$ for some $A\in\mathcal{L}(X/Y,\ell_2)$, Q being the quotient map $X \longrightarrow X/Y$. Again by hypothesis, $A\in\mathcal{P}_2(X/Y,\ell_2)$, hence $S-\tilde{T}$ and S are in $\mathcal{P}_2(X,\ell_2)$.

Moreover it follows from Heinrich [7] that every B-space which is "finitely dual representable" in a HS-space is itself a HS-space. In particular, ultrapowers of HS-spaces are again HS-spaces.

RELATIONS TO OTHER CLASSES OF B-SPACES

We know that, on Hilbert spaces, $\mathcal{P}_{2,2,2}$ just yields the HS-operators. By considering the appropriate norms for identity operators on finite-dimensional Hilbert spaces, we see immediately that no infinite-dimensional HS-space X can contain uniformly complemented the ℓ_2^n's. By Pisier [21], this means that X cannot be K-convex, i.e. it must contain the ℓ_1^n's uniformly, or else:

4. <u>Proposition</u> An infinite-dimensional HS-space cannot have any type $p>1$.

Being K-convex, superreflexive B-spaces cannot be HS-spaces unless their dimension is finite. On the other hand, the following is open:

5. <u>Problem</u>: Do there exist reflexive HS-spaces of infinite dimension ?

Let X be a reflexive HS-space. Then one easily checks the equation $\mathcal{L}(X,\ell_2) = \mathcal{N}_2(X,\ell_2)$. Thus, if we denote by $\zeta(X,X^*)$ the locally convex topology generated by all continuous hilbertian seminorms on X, then we get a Schwartz topolgy [8] which, by Bellenot's result [1] quoted in the introduction, has the following property: For every infinite-dimensional B-space Y there is a set M such that $[X,\zeta(X,X^*)]$ is linearly homeomorphic to a subspace of the product Y^M. When do these informations lead to the conclusion that X must be finite-dimensional ?

HS-spaces are closely related to B-spaces X which satisfy GT ("Grothendieck's theorem"), i.e. $\mathcal{L}(X,\ell_2) = \mathcal{P}_1(X,\ell_2)$. Actually, the spaces in 2(a),2(d), and the duals of the spaces in 2(b),2(c) satisfy GT.

Every B-space satisfying GT is of course a HS-space and the difference between these two classes is easy to detect. Let $\mathcal{M}_{2,1} := \mathcal{P}_2^{-1} \circ \mathcal{P}_1$ be the ideal of "(2,1)-mixing operators" [18]. The identity of a B-space X belongs to $\mathcal{M}_{2,1}$ iff $\Gamma_\infty(\cdot,X) = \mathcal{P}_2(\cdot,X)$ holds. In fact, using notation and results of [9], we may write $\mathcal{M}_{2,1} = \mathcal{P}_2^{-1} \circ \mathcal{P}_1 = [\Gamma_\infty \circ \Gamma_2 \circ \Gamma_\infty]^\Delta = \Gamma_\infty^{-1} \circ [\Gamma_\infty \circ \Gamma_2]^\Delta = ([\Gamma_\infty \circ \Gamma_2]^\Delta)^{inj}$. On the other hand, $\Gamma_2^{-1} \circ \mathcal{P}_1 = [\Gamma_\infty \circ \Gamma_2]^\Delta$. Whence:

6. Proposition: A B-space X satisfies GT iff it is a HS-space and I_X belongs to $\mathcal{M}_{2,1}$.

It also follows that a subspace of a B-space satisfying GT again satisfies GT iff it is a HS-space.

As it is well-known, $I_X \in \mathcal{M}_{2,1}$ holds for every B-space X of cotype 2. Consequently, HS-spaces of cotype 2 satisfy GT. Let $\mathcal{C}_2$ and $\mathcal{P}_\gamma$ be the ideals of cotype 2 operators and of γ-summing operators, respectively, cf. Linde-Pietsch [12]; from this, also a proof of the relation $\mathcal{C}_2 = \mathcal{P}_2 \circ \mathcal{P}_\gamma^{-1}$ can be deduced. Using this we get:

7. Proposition: A B-space X is a HS-space of cotype 2 iff I_X is in $\mathcal{D}_2 \circ \mathcal{P}_\gamma^{-1}$.

Here $\mathcal{D}_2$ is the largest extension to an ideal of operators between B-spaces of the trace class operators on Hilbert space. It is known that $\mathcal{D}_2 = \mathcal{P}_2^{dual} \circ \mathcal{P}_2$ holds.

Proof of 7: If X is a HS-space of cotype 2, then $I_X = I_X^2 \in (\Gamma_2^{-1} \circ \mathcal{P}_2) \circ (\mathcal{P}_2 \circ \mathcal{P}_\gamma^{-1}) = (\Gamma_2^{-1} \circ \mathcal{P}_2) \circ (\mathcal{P}_2^{-1} \circ \mathcal{P}_\gamma^\Delta) \subset \Gamma_2^{-1} \circ \mathcal{P}_\gamma^\Delta = \mathcal{D}_2 \circ \mathcal{P}_\gamma^{-1}$, cf. [9]. Conversely, if $I_X \in \mathcal{D}_2 \circ \mathcal{P}_\gamma^{-1}$, then our assertion follows from $\mathcal{D}_2 \circ \mathcal{P}_\gamma^{-1} \subset \mathcal{P}_2 \circ \mathcal{P}_\gamma^{-1} = \mathcal{C}_2$ and $\mathcal{D}_2 \circ \mathcal{P}_\gamma^{-1} = \Gamma_2^{-1} \circ \mathcal{P}_\gamma^\Delta \subset \Gamma_2^{-1} \circ \mathcal{P}_2^\Delta = \Gamma_2^{-1} \circ \mathcal{P}_2$. □

This concerns in particular Pisier's spaces Z in 2(d) and their duals, and also H^*_∞ and A*, by Bourgain [2].
A and H_∞ do not satisfy GT. It suffices to check $I_A \notin \mathcal{M}_{2,1}$. In fact, otherwise $\mathcal{P}_2(A,\cdot) = \mathcal{P}_{1/2}(A,\cdot)$ would follow (Maurey [14]) and every operator $\mathcal{L}(A,\ell_2)$ would be nuclear (Kisliakov[11], Bourgain [2]). But the Paley projections yield non-compact operators in $\mathcal{L}(A,\ell_2)$.

Finally, let us consider the ideal GL $:= \mathcal{P}_1^{-1} \circ \Gamma_1$. Let X be a B-space. By Gordon-Lewis [4], $I_X \in$ GL holds whenever X** is complemented in a Banach lattice. From $\mathcal{P}_1^{-1} \circ \Gamma_1 = [\mathcal{P}_1^d \circ \mathcal{P}_1]^\Delta = \Gamma_\infty \circ (\mathcal{P}_1^{dual})^{-1}$ we infer that $I_X \in$ GL and $I_{X^*} \in$ GL are equivalent properties. Compare also with Pisier [20] and Reisner [24].

8. Proposition: A B-space X satisfies GT and has $I_X \in$ GL iff $I_X \in \Gamma_2^{-1} \circ \Gamma_1$.

This is also quite easy. If X satisfies GT and has $I_X \in$ GL, then $I_X = I_X^2 \in (\Gamma_2^{-1} \circ \mathcal{P}_1) \circ (\mathcal{P}_1^{-1} \circ \Gamma_1) \subset \Gamma_2^{-1} \circ \Gamma_1$. Conversely, if $I_X \in \Gamma_2^{-1} \circ \Gamma_1$, then $I_X \in \mathcal{P}_1^{-1} \circ \Gamma_1$ since $\mathcal{P}_1 \subset \Gamma_2$, whereas $I_X \in \Gamma_2^{-1} \circ \mathcal{P}_1$ follows from $\mathcal{L}(\mathcal{L}_1,\mathcal{L}_2) = \mathcal{P}_1(\mathcal{L}_1,\mathcal{L}_2)$.

9. Remarks (i) Part of 8 can also be obtained from observing that $I_X \in \mathrm{GL} \cap \mathcal{M}_{2,1}$ is equivalent with $I_X \in \mathcal{P}_2^{-1} \circ \Gamma_1$. Note that GL and $\mathcal{M}_{2,1}$ are both injective, so that the property "identity in GL $\cap\, \mathcal{M}_{2,1}$" is inherited by subspaces. In particular, every subspace of an $\mathcal{L}_1$-space enjoys this property; see also [20].
(ii) By considering the canonical map $H_\infty \longrightarrow H_1$, Pełczyński [17] proved that neither A nor H_∞ (nor their duals) do have the above GL-property. Another proof (for A) is as follows: Suppose $I_{A^*} \in$ GL. Since A* satisfies GT, $I_{A^*} \in \Gamma_2^{-1} \circ \Gamma_1$, hence $I_A \in \mathcal{P}_1^{-1} \circ \mathcal{D}_2$. In particular, every 1-summing operator $A \longrightarrow \ell_2$ must be nuclear, which again is not true for Paley projections.
(iii) Let Z be an infinite-dimensional B-space such that both, Z and Z*, satisfy GT(see e.g. 2(d)). Then $I_Z \notin$ GL. In fact, if Z* satisfies GT, then $I_Z \in \Gamma_1^{-1} \circ \mathcal{P}_2$ (compare with [19]). Thus $I_Z \in$ GL implies $\mathcal{P}_1(Z,\ell_2) = \Gamma_1(Z,\ell_2) = \mathcal{N}_1(Z,\ell_2)$, as in (ii). If now, in addition, Z satisfies GT, then even $\mathcal{L}(Z,\ell_2) = \mathcal{N}_1(Z,\ell_2)$ follows, hence $I_Z \in \mathcal{D}_2$, or dim $Z<\infty$.
We do not know if this is also true if we only require Z* to satisfy GT.

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