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In: Zdeněk Frolík (ed.): Proceedings of the 10th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1982. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 2. pp. [249]–262.

Persistent URL: http://dml.cz/dmlcz/701279

ON TWO PROBLEMS OF MICE

A.Pultr and J.Úlehla

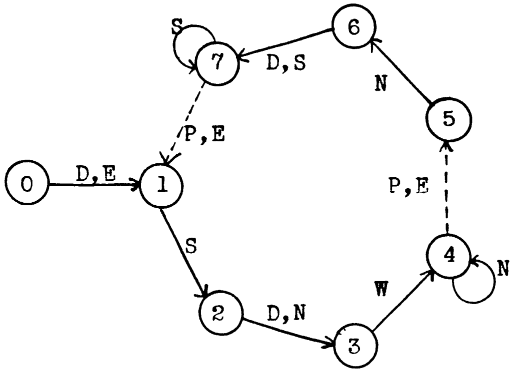
Abstract: This paper deals with two problems concerning the behaviour of mice /finite automata/ in environments. First of them was formulated by M.S.Paterson [FCT Computing Problem Book] . The question was whether one can design a pair of mice, each of them equipped with two pebbles, such that if they are dropped in the plane in different times and in different places they will eventually find each other. We answer the question in the affirmative. We will give here a short informal description of the solution. Longer and more rigorous treatment will appear in [Pultr/Úlehla] .

The second problem has become known as the "Mouse in the First Octant Problem". It was formulated by L.Budach [FCT Computing Problem Book] . /Cf. also [Karpiński/van Emde Boas] ./ The problem asks to describe behaviours of very simple mice in a non-homogeneous environment. The environment is a cone which arises from the first octant of square lined paper by glueing the diagonal and the x-axis together. /One has to stretch the x-axis first to match the lattice points on both sides./ There arises a kind of singularity along the glueing and it seems to be the reason why the behaviour is "hard" predicable. We add few remarks to the discussion of the problem.

Acknoledgement: We are very indebted to M.Karpiński for making us acquinted with the problems and for very valuable discussions.

Paterson's Problem

Mice with two pebbles: A mouse can be described as a bicolored directed graph with /possibly/ labeled arrows. The vertices of a graph correspond to inner states of a mouse. /See Fig.1./ A mouse starts its life in some point of the planar lattice of points with integer coordinates /the point a in our example/, it starts in the initial state - 0 , and with two pebbles in its pocket. It looks whether there is a pebble lying on the same point and it follows the instruction along the dotted arrow if there is a pebble and along



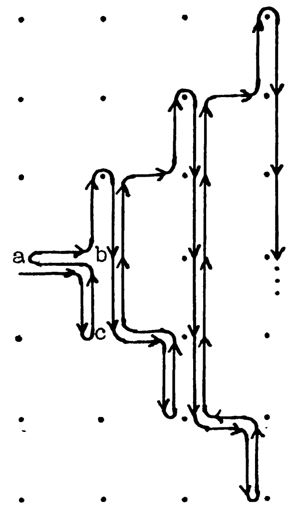


Fig.1. An example of a mouse with two pebbles and its trajectory.

the full arrow if there is not a pebble. In our example the mouse deposits /D/ a pebble on the point a , moves a step east /E/ and enters the state 1 . In each next step it again looks whether there is a pebble or not and according to the information it /possibly/ handles the pebbles, and /possibly/ moves, and enters a new state. In our example the mouse continues: It moves south /S/ and enters the state 2 , it deposits its second pebble in the point c and moves north /N/ and west /W/. Then it is again in the point a , now in the state 4 , and it finds a pebble there. So it follows a dotted arrow, that is, it picks /P/ a pebble, moves east and enters the state 5 . A mouse continues moving unless there hapens to be no correct /dotted or full/ arrow prescribing next step. In this case a mouse halts.

Paterson's problem: The problem asks whether there is a pair of mice such that

if they are started arbitrarily and independently in time and space they will eventually meet each other.

Why just 2,2 pebbles? It seems to be general /among mousy theorists/ knoledge that one can fool any pair of mice one having less then 2 pebbles and the other less then 3 pebbles. That is, one can put them in different places in the plane in such a way that they will never meet. /There is no need to use different times as well./ On the other side if a mouse has 3 pebbles it can simulate Turing machine, and hence find its friend sitting idly where it was dropped. /Note that because of the time difference the active member of the team has not only to search the plane but it has to visit each point again and again./

Partner's pebbles: Now we have a crowd of two mice and four pebbles moving around the plane. So a mouse can meets its own pebbles, the other mouse and the other mouse's pebbles. A mouse can by no means react to the meating with the other one. /Only we as outside observers will recognize that they met and solved the task./ But we can adopt at least two conventions concerning the partner's pebbles. Either a mouse react to a partners's pebble in the same way as to its own pebble /dotted arrow/ or it ignores it /full arrow/. The former case is more natural, the latter one is more easily solvable.

Ignorance of the partner's pebbles: In the case mice ignore the partner's pebbles the question whether a pair of mice will always meet can be translated into a question about their trajectories. A trajectory of a mouse is a sequence /finite or infinite/ of its successive positions in the plane when it was started in the point $(0,0)$. Thus the trajectory of the mouse on the figure 1 starts:

$$f(0) = (0,0)$$
$$f(1) = (1,0)$$
$$f(2) = (1,-1)$$
$$f(3) = (1,0)\ldots .$$

Now the problem can be reformulated to the question whether the following holds:

/1/
$$\exists f,g\ \forall t_f,t_g,v_f,v_g\quad t_f t_g = 0 \Rightarrow \exists t\ (\ f(t_f+t) + v_f = g(t_g+t) + v_g\ )\ )\ ,$$

where f,g ranges over mousy trajectories

t_f, t_g, t ranges over N /nonnegative integers/

v_f, v_g, v ranges over Z^2 /pairs of integers/ .

/1/ can be rewritten to

/2/ $\exists f,g\ \forall t_f,t_g,v\ (\ t_f t_g = 0 \Rightarrow \exists t\ (\ f(t_f+t) - g(t_g+t) = v\ )\ )$.

If g is mousy trajectory $-g$ is mousy trajectory as well. Thus /2/ is equivalent to

/3/ $\exists f,g\ \forall t_f,t_g,v\ (\ t_f t_g = 0 \Rightarrow \exists t\ (\ f(t_f+t) + g(t_g+t) = v\ )\ )$

and /3/ again can be rewritten to

/4/ $\exists f,g\ \forall t_f,t_g\ (\ t_f t_g = 0 \Rightarrow [f(t_f+.) + g(t_g+.)](N) = Z^2\ )$.

That is to say, that $f + g$ covers the plane even under time delays.

Solution: Here we present a pair of mice A,B such that the sum of corresponding trajectories f,g covers the plane under time delays. /See Fig.2./

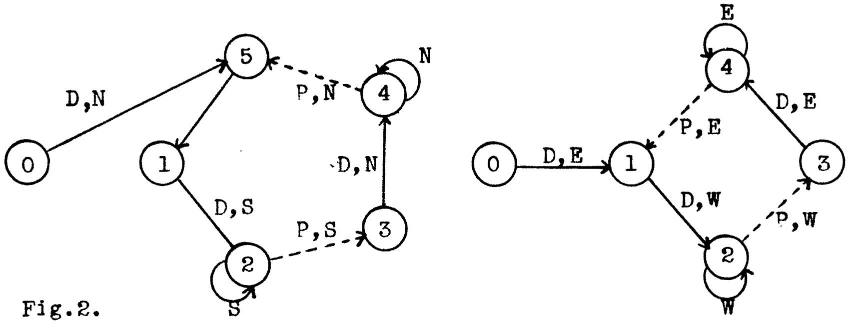
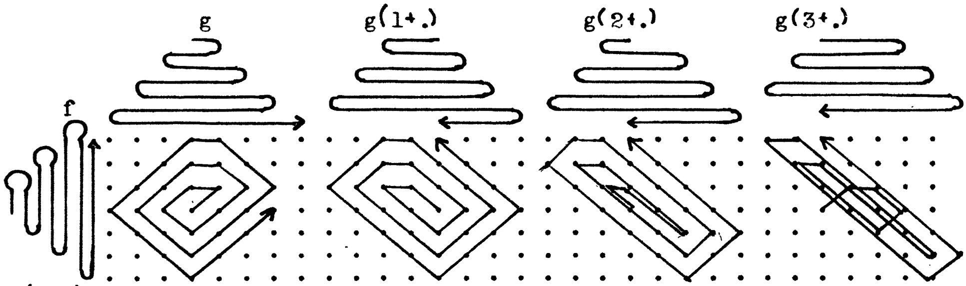


Fig.2.

f + g: The figure 3 shows few examples of sums of f,g to convince the reader that for any time delay /with $t_f t_g = 0$ /, after some initial mess lasting a time which is a quadratic function of the time delay, a trajectory reaches the right lover corner /for $t_f = 0$ / or left lover corner /for $t_g = 0$ / and then it stars to create an anti/clockwise spiral.



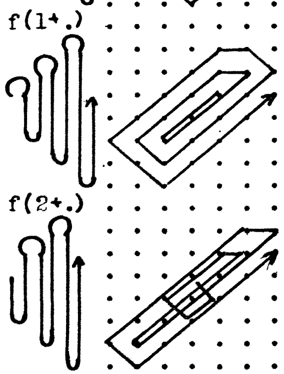


Fig.3. Sums of f,g under several time delays.

Open problem: We use this left space to formulate a problem concerning the space itself:

Does there exists a pair of mousy trajectories such that they cover the plane even under delays which fall into this left space, that is if we do not require one of the t_f, t_g to be 0 .

Further improvements: By modification of the above pair of mice we can construct a pair of mice such that they will always eventually meet even if /some or all/ following conditions hold:

a/ They react to the partner's pebbles as to their own.

b/ They are not equipped with compasses, that is their moves are prescribed by: forward, backward, left and right /referring to the previous move/.

c/ They do not know which paw is left, that is we can switch an orientation of one of them.

Mouse in the First Octant Problem

Definitions: In this part difficulty of the problem is created by non-homogeneous environment. The mice world here will be the first octant of the plannar lattice of points with integer coordinates:

$$FO = \{(x,y) \; ; \; 0 \leq y < x\} \; .$$

Mouse in this part will be very simple. It is only a nonempty sequence over N,E

$$M = v_0 v_1 \dots v_{n-1} \; .$$

Where

$$N = (0,1) \quad E = (1,0) \; .$$

The numbers $0,1,\dots,n-1$ are called inner states. For a notational convinience we put for arbitrary integer i

$$v_i = v_{i \; \underline{mod} \; n} \; .$$

A mouse creates its trajectory m in FO as follows:

/1/ $$m(0) = (1,0)$$

/2/ $$m(i+1) = m(i) + v_i$$

unless $m(i+1)$ defined by /2/ does not lie in FO . In this case it has to lie on the diagonal, $m(i+1) = (x,x)$ for some x , and we put

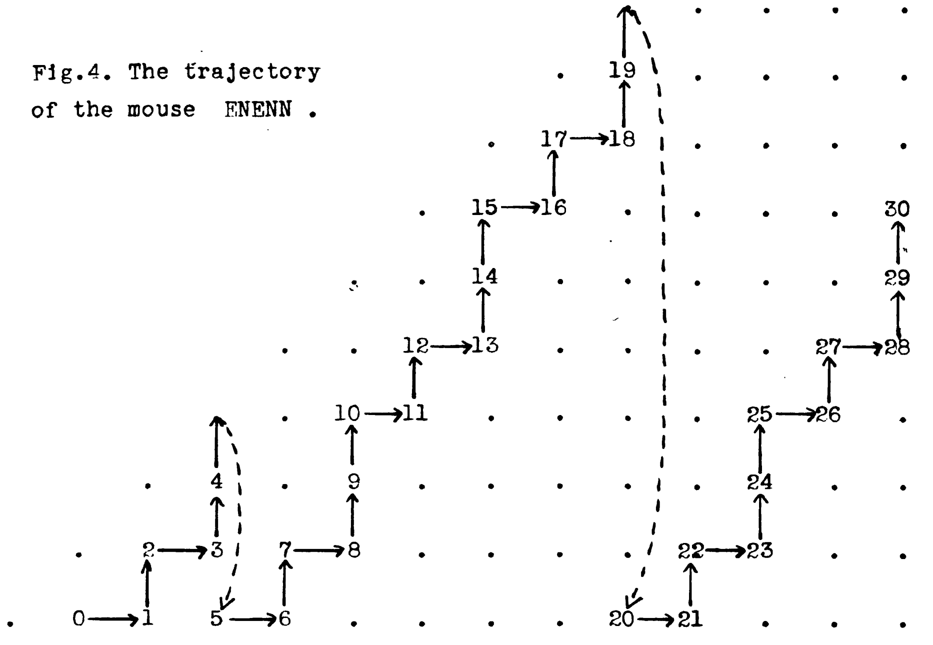
/3/ $$m(i+1) = (x,0) \; .$$

Further we say in this case that a mouse hitted the diagonal in the time $i+1$, in the state $(i+1) \; \underline{mod} \; n$, in the point x .

See figure 4 for the initial part of the trajectory of the mouse ENENN . The mouse hits the diagonal for the first time in time 5 , in state $0 = 5 \; \underline{mod} \; 5$ and in the point 3 .

Formulation of the problem: Now the problem asks to deside for a given mouse and a distinguished state in it whether the mouse will ever hit the diagonal in the distinguishe state, or to show that the problem is in general undecidable.

Fig.4. The trajectory of the mouse ENENN .



Notation, observations and a convention: Let us denote a vector a mouse walks from time i till time j unless it hits the diagonal: For integers i,j , $i \leq j$

$$v(i,j) = \sum_{k=i}^{j-1} v_k .$$

We have immediatly

$$v(i,i+n) = v(j,j+n)$$

for any i and j . Let us further denote

$$(b,a) = v(0,n) ,$$

the projections

$$v = (v^x, v^y)$$

and the depth from diagonal

$$D(v) = v^x - v^y .$$

Thus a mouse hits the diagonal in time t if

$$D(m(t-1) + v_{t-1}) = 0 .$$

We can now eliminate the case

$$a \leq b .$$

As was already mentioned [Karpiński/van Emde Boas] in this case a mouse can hit the diagonal only during the first period of its life. Indeed, if we take time t , $t \geq n$, we have

$$D(m(t-1) + v_{t-1}) = D(m(t-n) + (b,a))$$

unless a mouse hitted the diagonal in some of the times $t-n+1$, $t-n+2,\dots,t-1$. But as falling down can only increase the depth we have

$$D(m(t-1)+v_{t-1}) \geq D(m(t-n)+(b,a)) \quad .$$

Now since D is linear and $m(t-n)$ lies in the FO /hence $D(m(t-n)) > 0$ / and $D(b,a) = b-a \geq 0$ /if $a \leq b$ /, we obtain

$$D(m(t-n)+(b,a)) > 0 \; .$$

Hence our mouse will not hit the diagonal in any time t , $t \geq n$.

So we can easily decide any mouse with $a \leq b$; therefore we restrict our attention to the mice with

$$a > b$$

in the following. Finally let us put

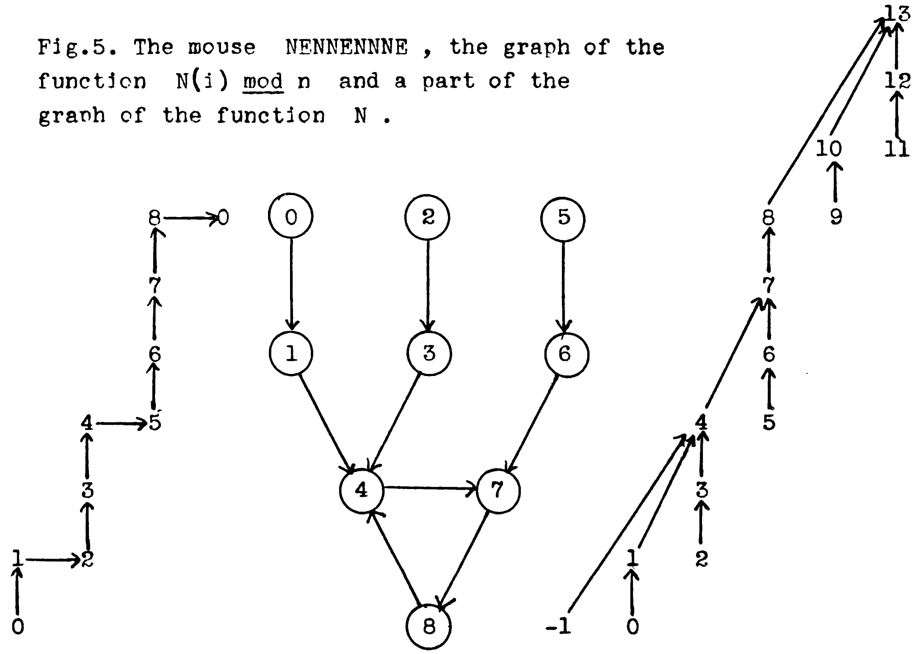
$$c = a-b \; .$$

A step nearer: Because of the importance of hitting the diagonal we should watch a depth of a mouse from the diagonal: If a mouse is in the time i in depth d it is not important to watch its behaviour in deeper positions but it is important to know the time - $N(i)$, when it emerge in the depth $d-1$ - a step nearer to the diagonal. More formally we put for an integer i

$$N(i) = \min \{ j > i \; ; \; D(v(i,j)) = -1 \} \; .$$

Figure 5 shows a part of the mapping N for the mouse NENNENNNE .

Fig.5. The mouse NENNENNNE , the graph of the function $N(i)$ mod n and a part of the graph of the function N .



Corectness and basic properties of N :

A.Proposition: /1/ N is total

/2/ $i < N(i) \leq i + n$

/3/ $i \leq j < N(i) \Rightarrow N(j) \leq N(i)$

/4/ $N(i+n) = N(i) + n$.

Proof: We have

$$D(v(i,i)) = 0$$
$$D(v(i,j+1)) = D(v(i,j)) + D(v_j) = D(v(i,j)) \pm 1$$
$$D(v(i,i+n)) = D(b,a) = b-a < 0$$

which give /1/ and /2/ . If we take $i \leq j < N(i)$ we have

$$-1 = D(v(i,N(i)) = D(v(i,j)) + D(v(j,N(i))) .$$

We also have

$$D(v(i,j)) \geq 0 .$$

Hence

$$D(v(j,N(i))) < 0$$

and we will get /3/ in the same way as /2/. The fourth line follows immediatly from equalities

$$v_i = v_{i+n} . \quad \%$$

Now we can study a sequence

$$p_0 = 0, \ p_1 = N(0), \ p_2 = NN(0), \dots, \ p_i = N^i(0), \dots .$$

Let us further denote f an integer uniquely determined by

$$p_{f-1} < n \leq p_f .$$

Then we have:

B.Proposition: $p_f - n \in \{p_0, p_1, \dots, p_{f-1}\}$.

Proof: If it is not so, we have a unique k among $0,1,\dots,f-1$ with

$$p_{f-1} < p_k + n < p_f < p_{k+1} + n$$

and

$$N(p_{f-1}) = p_f \quad \text{and} \quad N(p_k+n) = p_{k+1} + n$$

which contradict proposition A. %

So $p_f - n$ equals to, say, p_g and the preceding two lemmas leads that the sequence $p_0, p_1, p_2, \dots$ can be written:

$$p_0 < p_1 < \dots < p_{g-1} < p_g < p_{g+1} < \dots < p_{f-1} < n \leq p_g + n <$$
$$< p_{g+1} + n < \dots < p_{f-1} + n < 2n \leq p_g + 2n \dots$$

Now we can count down the length of cycle $p_g, p_{g+1}, \dots, p_{f-1}$ because

$$-c = D(b,a) = D(v(p_g, p_g + n) = D(v(p_g, p_{g+1})) + D(v(p_{g+1}, p_{g+2})) +$$
$$+ \dots + D(v(p_{f-2}, p_{f-1})) = -1 + -1 + \dots + -1 .$$

Thus we can name

$$p_g = q_0,\ p_{g+1} = q_1, \ldots, p_{f-1} = q_{c-1} \cdot$$

$q_0, q_1, \ldots q_{c-1}$ are called essential states. /Cf. [Karpiński/van Emde Boas] ./

Note also that if a distinguished state is not an essenial one we can easily decide whether a mouse will ever hit the diagonal in it. Indeed, it will either appear among $p_0, p_1, \ldots, p_{g-1}$ during the first period of mouse life or the mouse will never hit the diagonal in it, because it is not nearer to the diagonal than immediately preceding essential state. Thus we can assume in the following that our distinguished state will be among essential states.

Esential states:

C.Proposition: If a mouse is sitting in a point x,y in an essential state q_i it will next hit the diagonal in the state

$$q_j = q_{(i+x-y) \underline{\text{ mod }} c}$$

in the point

$$\bar{x} = x + ((x-y) \underline{\text{ over }} c) * b + v^x(q_i, q_j) ,$$

where we extend v for $q_i < q_j$ by

$$v(q_i, q_j) = v(q_i, q_j + n) .$$

/These formulae are small generalizations of very similar ones in [Karpiński/van Emde Boas] ./

Proof: The depth of x,y is $x - y$. A mouse has to decrease its depth by $x - y$ to reach the diagonal. Each transition to a new essential state decreases the depth of the mouse by 1 . Thus

$$q_j = N^{(x-y)}(q_i) = q_{(i+x-y) \underline{\text{ mod }} c} \cdot$$

Performing $x - y$ transitions between its essential states the mouse will run through

$$(x-y) \underline{\text{ over }} c$$

full periods, each adding b to the x-coordinate of the mouse. Then there remains

$$(x-y) \underline{\text{ mod }} c$$

transitions from the state q_i to the state q_j adding the final

$$v^x(q_i, q_j)$$

to the x-coordinate. %

Thus a mouse is essentially described by giving a non-empty sequence of vectors

$$v(q_0,q_1),v(q_1,q_2),\dots,v(q_{c-2},q_{c-1}),v(q_{c-1},q_0)$$

from $\{(x,x+1)\}$. It is not described totally in this way because we do not know in which point a mouse will appear in a state q_0 for the first time. On the other side we see easily that this question depends only upon $v_{q_{c-1}},v_{q_{c-1}+1},\dots,v_{n-1},v_0,v_1,\dots,v_{q_0}$. We describe the possible first appearences of q_0 by the following two lemmas.

<u>D.Lemma</u>: If $v(q_{c-1},q_0) = (r,r+1)$ and $m(q_0) = (x,y)$ then

$$x \le r \ ,$$

unless $r = 0$ in which case $x = 1$.

<u>Proof</u>: If $r = 0$ then $q_{c-1} = c-1$ and $q_0 = 0$. Thus $m(q_0) = (1,0)$.

If $r > 0$ then $v_{q_{c-1}} = E$. Hence

$$v^x(0,q_0) = v^x(q_{c-1},q_0) - v^x(q_{c-1},0) \le r-1$$

and finally

$$x \le 1 + (r-1) = r \ . \quad \%$$

<u>E.Lemma</u>: For arbitrary non-negative integers $u_0,u_1,\dots,u_{c-1}$ and for any x,y such that

$$u_{c-1} > 0 \Rightarrow 0 \le y < x \le u_{c-1}$$

$$u_{c-1} = 0 \Rightarrow (x,y) = 1,0$$

there exists a mouse with

$$v(q_i,q_{i+1}) = (u_i,u_i+1) \quad \text{for} \quad i = 0,1,\dots,c-2$$

$$v(q_{c-1},q_0) = (u_{c-1},u_{c-1}+1)$$

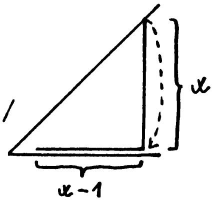
$$m(q_0) = (x,y) \ .$$

<u>Proof</u>: If $x = 1$, we will put $r_0 = 0$. If $x > 1$ and $y = 0$, we will put

$$v_0 = v_1 = \dots = v_{x-2} = E \quad /(x-1)\text{ times}/$$

$$v_{x-1} = v_x = \dots = v_{2x-2} = N \quad /x\text{ times}/$$

$$r_0 = 2x-1 \ .$$



If $x > 1$ and $y > 0$ then we will put

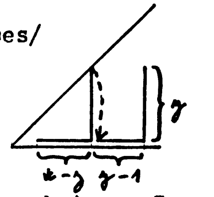
$$v_0 = v_1 = \dots = v_{x-y-1} = E \quad /(x-y)\text{ times}/$$

$$v_{x-y} = v_{x-y+1} = \dots = v_{2x-2y} = N \quad /(x-y+1)\text{ times}/$$

$$v_{2x-2y+1} = v_{2x-2y+2} = \dots = v_{2x-y-1} = E \quad /(y-1)\text{ times}/$$

$$v_{2x-y} = v_{2x-y+1} = \dots = v_{2x-1} = N \quad /y \text{ times}/$$

$$r_0 = 2x \; .$$



Adding no more information here we can see that

$$m(r_0) = (x,y) \; .$$

/r's are candidates for q's./ In the first case it is obvious. In the second one mouse starts moving $x - 1$ steps east reaching the point $(x,0)$ then it moves x steps north, hits the diagonal in time $2x - 1$ and falls down to be again in the point $(x,0)$ in time $2x - 1$. In the third case it starts as in the second one /with x changed to $x-y$ 1 / and reaches the point $(x-y+1,0)$ after $2x-2y+1$ steps. Then it continues by $y - 1$ steps to the point $(x,0)$ and it adds finally y steps north to reach (x,y) in time $2x$.

We continue by putting

$$r_i = r_{i-1} + 2u_{i-1} + 1$$

for $i = 1,2,\dots,c - 1$; and

$$v_{r_i} = v_{r_i+1} = \dots = v_{r_i+u_i-1} = E \quad /u_i \text{ times}/$$

$$v_{r_i+u_i} = v_{r_i+u_i+1} = \dots = v_{r_i+2u_i} = N \quad /(u_i+1) \text{ times}/$$

for $i = 0,1,\dots,c-2$. Now there remains to define moves between r_{c-1} and $n-1$. We put

$$v_{r_{c-1}} = v_{r_{c-1}+1} = \dots = v_{r_{c-1}+u_{c-1}-x} = E \quad /(u_{c-1}-x+1) \text{ times}/$$

$$v_{r_{c-1}+u_{c-1}-x+1} = v_{r_{c-1}+u_{c-1}-x+2} = \dots = v_{n-1} = N \quad / \; u_{c-1}+x-r_0 \text{ tim.}/$$

It remains to prove that

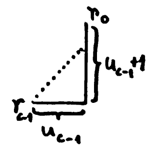
$$N(r_i) = r_{i+1} \quad \text{for} \quad i=0,1\dots,c-2$$

which is evident, and

$$N(r_{c-1}) = r_0 + n \; .$$

To prove the last equation we start with the case $x = 1$. Then we have

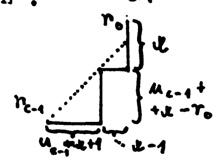
$$u_{c-1} - x + 1 = u_{c-1}$$



and
$$u_{c-1} + x - r_0 = u_{c-1}+1$$

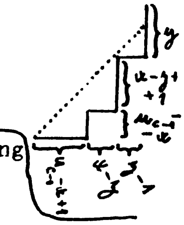
and
$$N(r_{c-1}) = r_{c-1} + 2u_{c-1} + 1 = n = r_0 + n \; .$$

If further $x > 1$ and $y = 0$ we have



$$u_{c-1} + x - r_0 = u_{c-1} - x + 1$$

and
$$N(r_{c-1}) = r_0 + n \; .$$



If finally $x > 1$ and $y > 1$ we have

$$u_{c-1} + x - r_0 = u_{c-1} - x$$

and again

$$N(r_{c-1}) = r_0 + n \ .$$

2 number theoretical problems: We say that a mapping

$$\varphi : Z^2 \longrightarrow Z^2$$

is a mousy mapping if there exists a non-empty sequence $u_0, u_1, \ldots$ $\ldots, u_{c-1}$ of non-negative integers and

$$(\varphi(x,q))^x = x + (x \underline{\text{ over }} c) * b + r(q,(x+q) \underline{\text{ mod }} c) \ ,$$

$$(\varphi(x,q))^y = (x+q) \underline{\text{ mod }} c$$

where

$$b = \sum_{i=0}^{c-1} u_i$$

and for $0 \leq i,j < c$

$$r(i,j) = \sum_{k=i}^{j-1} u_k \qquad \text{if } \ i \leq j$$

$$r(i,j) = b - \sum_{k=j}^{i-1} u_k \ \text{ if } \ i > j \ .$$

Problem 1: For a given mousy mapping φ and a given integer s, $0 \leq s < c$, decide

$$\exists k > 0 \quad (\varphi^k(1,0))^y = s \quad .$$

Problem 2: For a given mousy mapping φ and a given integer s, $0 \leq s < c$, and a given positive integer x decide

$$\exists k > 0 \quad (\varphi^k(x,0))^y = s \quad .$$

It easily follows from the preceding discussion that the Problem 1 can be translated into the Mouse in the First Octant Problem and the Mouse in the First Octant Problem can be translated into the Problem 2. We have tried to describe the position of the Mouse in the First Octant Problem between the Problem 1 and the Problem 2 by studying where a mouse can appear in an essential state / q_0 / for the first time. It has been claimed [Karpiński/van Emde Boas] the Problem 1 is equivalent to the Mouse in the First Octant Problem. We do not see any evidence of it.

Now we prove that two special cases of the Problem 2 are solvable.

The case c divides b : Let φ be a mousy mapping with c divides b. Let us put

$$(x_1, q_1) = \varphi^1(x_0, q_0) \ ,$$

where x, q are arbitrary. Then the sequence

$$(x_0 \ \underline{\text{mod}}\ c, q_0)\ ,\ (x_1 \ \underline{\text{mod}}\ c, q_1),\ \ldots$$

is ultimately periodic. Hence also

$$q_0, q_1, q_2, \ldots$$

is ultimately periodic. Moreover the length of the period is shorter than or equal to c^2 and the first full period has to appear among first c^2 items of the sequence. This holds becouse of the following easy proposition:

F.Proposition: If c divides b then

$$(\varphi(x,q))^x \ \underline{\text{mod}}\ c = (x \ \underline{\text{mod}}\ c + r(q,\ (x \ \underline{\text{mod}}\ c + q)\ \underline{\text{mod}}\ c))\ \underline{\text{mod}}\ c$$

$$(\varphi(x,q))^y = (x \ \underline{\text{mod}}\ c + q)\ \underline{\text{mod}}\ c\ .$$

Proof:

$$(\varphi(x,q))^y = (x + q)\ \underline{\text{mod}}\ c = (x \ \underline{\text{mod}}\ c + q)\ \underline{\text{mod}}\ c\ .$$

And similarly

$$(\varphi(x,q))^x \ \underline{\text{mod}}\ c = (x + (x \ \underline{\text{over}}\ c) * b + \ldots)\ \underline{\text{mod}}\ c = (x \ \underline{\text{mod}}\ c + \ldots)\ \underline{\text{mod}}\ c\ . \quad \%$$

Thus we have a function

$$\psi : c \times c \longrightarrow c \times c$$

where $c = \{0,1,\ldots,c-1\}$

such that for any i

$$(x_{i+1} \ \underline{\text{mod}}\ c, q_{i+1}) = \psi(x_i \ \underline{\text{mod}}\ c, q_i)\ .$$

So we can deside this case.

The case $c = 2$: We can moreover suppose c does not divide otherwise we can use the preceding paragraph . Let φ and (x_i, q_i) be the same as in the preceding paragraph. We have only two essential states here and we will prove that both appear infinitely often among q´s. We will use two lemmas which hold even if $c \neq 2$.

G.Lemma: If c devides x_i then

$$x_{i+1} = a * x_i/b$$

$$q_{i+1} = q_i$$

where $a = b + c$.

Proof: $(\varphi(x_i,q_i))^y = (x_i + q_i)\ \underline{\text{mod}}\ c =$

$$= (x_i \ \underline{\text{mod}}\ c + q_i)\ \underline{\text{mod}}\ c = q_i \ \underline{\text{mod}}\ c = q_i\ .$$

$$(\varphi(x_i,q_i))^x = x_i + (x_i \ \underline{\text{over}}\ c) * b + r(q_i,q_i) =$$

$$= x_i + x_i/c * b = (c * x_i + b * x_i)/c = a * x_i/c\ . \quad \%$$

H.Lemma: If c does not divide x_i then

$$q_{i+1} \neq q_i\ .$$

Proof:

$$(\varphi(x_i,q_i))^y = (x_i + q_i) \text{ mod } c = (x_i \text{ mod } c + q_i) \text{ mod } c \neq q_i .\%$$

Now we restrict our attention to $c = 2$. In this case we watch the maximal power of 2 which devides x_i , say

$$p_i = \max \{ k \ ; \ 2^k \text{ divides } x_i \} \ .$$

Now the preceding two lemmas guarantees that if we take arbitrary i then

$$q_{i+p_i+1} \neq q_i \ .$$

Thus both essential states will appear infinitely many times among q's.

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