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NOTE ON PACKING COVERING AND TURÁN NUMBERS

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The aim of this communication is to give a brief summary of work on packing and covering to be published in detail elsewhere [4]. We also give some related remarks concerning Turán numbers.

Let $2 \le \ell < k < n$ be positive integers, a family $\mathcal{F}$ of $k$ element subsets of $n$ set $V$ is called $\ell$-sparse if every two members of $\mathcal{F}$ intersect in less than $\ell$ elements (i.e. if every $\ell$-element subset of $V$ is in at most one member of $\mathcal{F}$.) On the other hand we say that $\mathcal{F}'$ is $\ell$-dense if every $\ell$-element subset of $V$ is contained in at least one member of $\mathcal{F}'$. It is wellknown (see e.g. [2,3]) that

$$|\mathcal{F}| \le \frac{\binom{n}{\ell}}{\binom{k}{\ell}} \le |\mathcal{F}'| \qquad (1)$$

for any $\ell$-sparse family $\mathcal{F}$ and $\ell$-dense family $\mathcal{F}'$, $\mathcal{F}, \mathcal{F}' \subset [V]^k$. Denote by $M(n,k,\ell)$ the minimal number of elements of $\ell$-dense family $\mathcal{F}' \subset [V]^k$ and by $m(n,k,\ell)$ the maximal number of elements of $\ell$-sparse family $\mathcal{F} \subset [V]^k$. It follows immediately from (1) that

$$m(n,k,\ell) \le \frac{\binom{n}{\ell}}{\binom{k}{\ell}} \le M(n,k,\ell) \qquad (2)$$

In 1963 P.Erdös and H.Hanani [1] conjectured that both

$$\left.\begin{aligned} M(n,k,\ell) &= \frac{\binom{n}{\ell}}{\binom{k}{\ell}}(1+o(1)) \\ \text{and} \qquad & \\ m(n,k,\ell) &= \frac{\binom{n}{\ell}}{\binom{k}{\ell}}(1+o(1)) \end{aligned}\right\} \qquad (3)$$

holds. Here and below $o(1)$ is a function tending to zero as $n$

tends to infinity .

They proved (3) for $\ell=2$ and all $k$ and for $\ell=3$, $k=p$ or $p+1$, where $p$ is prime power.

It was further shown by Erdös and Spencer [2] that

$$M(n,k,\ell) \leq \frac{\binom{n}{\ell}}{\binom{k}{\ell}}\left(1+\log\binom{k}{\ell}\right)$$

The numbers $M(n,k,\ell)$ and $m(n,k,\ell)$ are related to Turán number $T(n,k,\ell)$, [6] – for $2 \leq \ell < k < n$ denote by $T(n,k,\ell)$ the smallest $q$ such that there exists a family $\mathcal{F}$ of $q$ $\ell$-subsets of an $n$-set $V$ with no independent set of size $k$ .

It was noted in [2], that

$$m(n,k,\ell) \geq \frac{T(n,k,\ell)}{\binom{k}{\ell}}$$

The functions $M, m$ have been also studied by J.Schönheim [5].

We can prove (3) for all $2\leq\ell<k<n$ and thus the following holds;

Theorem : Let $2\leq\ell<k<n$ be positive integers.Then

$$M(n,k,\ell) = \frac{\binom{n}{\ell}}{\binom{k}{\ell}}(1+o(1))$$

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The proof of this theorem is going to appear in [4].Our Theorem has the following

Corollary: Let $2\leq\ell<k<n$ be positive integers,then

$$T(n,n-\ell,k-\ell) = \frac{\binom{n}{\ell}}{\binom{k}{\ell}}(1+o(1))$$

Proof : Take an $\ell$-dense family $\mathcal{F}$ of $k$-sets of an $n$-set $V$ (i.e. $\mathcal{F}\subset[V]^{k}$) such that

$$|\mathcal{F}| = \frac{\binom{n}{\ell}}{\binom{k}{\ell}}(1+o(1))$$

Consider the system $\mathcal{T} = \{V-F;\ F \in \mathcal{F}\}$ .Clearly $\mathcal{T} \subset [V]^{n-k}$ and moreover every $n-\ell$ subset of $V$ contains some element of $\mathcal{T}$

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