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A COMMON GENERALIZATION OF BINOMIAL COEFFICIENTS, STIRLING NUMBERS AND GAUSSIAN COEFFICIENTS

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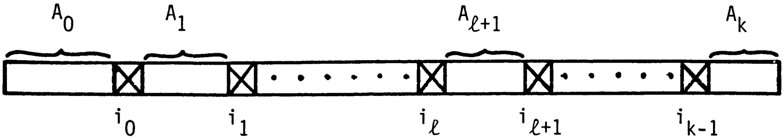
In this paper we present a common generalization of some basic enumeration problems. We show how well known recursion and inversion formulae fit into our model.

Let $A_0, A_1, A_2, \ldots$ be finite sets. For nonnegative integers n and k denote by $S_k^n(a_0, a_1, a_2, \ldots)$, where $a_i = |A_i|$, the number of words $w = (w_0, \ldots, w_{n-1})$ such that

(1) w contains k labels, say at positions $i_0, \ldots, i_{k-1}$,

(2) all entries in w before position i_0 belong to A_0, all entries in w between positions i_ℓ and $i_{\ell+1}$, where $\ell = 0, \ldots, k-2$, belong to $A_{\ell+1}$, all entries after position i_{k-1} belong to A_k.



As
$$S_k^n(\vec{a}) = \sum_{0 \le i_0 < i_1 < \ldots < i_{k-1} < n} a_0^{i_0} \cdot a_1^{i_1 - i_0 - 1} \cdot \ldots \cdot a_k^{n - i_{k-1} - 1},$$
the numbers S_k^n can obviously be defined for sequences of complex numbers.

Examples:

(1) $S_k^n(1,1,\ldots) = \binom{n}{k}$ (Binomial coefficients)

(2) $S_k^n(0,1,2,\ldots) = S_k^n$ (Stirling numbers of the second kind)

(3) $S_k^n(1,q,q^2,\dots) = \binom{n}{k}_q$ (Gaussian Binomial coefficients)

(4) $S_k^n(q,q^2,q^3,\dots) =$ number of affine k-dimensional subspaces in the n-dimensional affine space over $GF(q)$.

(5) $S_k^n(2,3,4,\dots) =$ number of Boolean sublattices $P(k)$ in $P(n)$ ($P(n) \equiv$ lattice of subsets of an m-element set) .

Theorem 1

Let n be a nonnegative integer and let $a_0,\dots,a_k$ be mutually distinct complex numbers. Then

$$S_k^n(a_0,\dots,a_k) = \frac{\det\begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ a_0 & a_1 & \dots & a_{k-1} & a_k \\ a_0^2 & a_1^2 & \dots & a_{k-1}^2 & a_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^{k-1} & a_1^{k-1} & \dots & a_{k-1}^{k-1} & a_k^{k-1} \\ a_0^n & a_1^n & \dots & a_{k-1}^n & a_k^n \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ a_0 & a_1 & \dots & a_{k-1} & a_k \\ a_0^2 & a_1^2 & \dots & a_{k-1}^2 & a_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^{k-1} & a_1^{k-1} & \dots & a_{k-1}^{k-1} & a_k^{k-1} \\ a_0^k & a_1^k & \dots & a_{k-1}^k & a_k^k \end{bmatrix}}$$

$$= \sum_{i=0}^{k} a_i^n \cdot \prod_{\substack{j=0 \\ j\neq i}}^{k} (a_i-a_j)^{-1}$$

Proof:

As the determinant occuring in the denominator is van der Monde's determinant and the determinant occuring in the numerator differs from van der Monde's determinant only in the last row, the second equality follows immediately by expanding the numerator with respect to the last row. Thus it suffices to show that

$$S_k^n(a_0,\ldots,a_k) = \sum_{i=0}^{k} a_i^n \cdot \prod_{\substack{j=0\\ j\neq i}}^{k} (a_i-a_j)^{-1}$$

We proceed by induction on k, the case $k = 0$ being obviously valid. Let us consider the case $k + 1$:

$$S_{k+1}^n(a_0,\ldots,a_{k+1}) = \sum_{-1=\mu_{-1}<\mu_0<\ldots<\mu_k<\mu_{k+1}=n} \prod_{j=0}^{k+1} a_j^{\mu_j-\mu_{j-1}-1}$$

$$\text{(distributivity)} \quad = \sum_{i=0}^{n-k-1} a_0^i \cdot S_k^{n-i-1}$$

$$\text{(by induction)} \quad = \sum_{i=0}^{n-k-1} a_0^i \cdot \sum_{\ell=1}^{k+1} a_\ell^{n-i-1} \cdot \prod_{\substack{j=1\\ j\neq \ell}}^{k+1} (a_\ell-a_j)^{-1}$$

$$\text{(changing summation)} \quad = \sum_{\ell=1}^{k+1} \prod_{\substack{j=1\\ j\neq \ell}}^{k+1} (a_\ell-a_j)^{-1} \cdot \sum_{i=0}^{n-k-1} a_0^i \cdot a_\ell^{n-i-1}$$

$$\text{(distributivity)} \quad = \sum_{\ell=1}^{k+1} a_\ell^k \cdot \prod_{\substack{j=1\\ j\neq \ell}}^{k+1} (a_\ell-a_j)^{-1} \cdot \sum_{i=0}^{n-k-1} a_0^i \cdot a_\ell^{n-k-i-1}$$

$$\text{(Cauchy-convolution)} \quad = \sum_{\ell=1}^{k+1} a_\ell^k \cdot (a_0^{n-k}-a_\ell^{n-k}) \cdot (a_0-a_\ell)^{-1} \cdot \prod_{\substack{j=1\\ j\neq \ell}}^{k+1} (a_\ell-a_j)^{-1}$$

$$= \sum_{\ell=1}^{k+1} a_\ell^k \cdot a_0^{n-k} \cdot (a_0-a_\ell)^{-1} \cdot \prod_{\substack{j=1\\ j\neq \ell}}^{k+1} (a_\ell-a_j)^{-1}$$

$$- \sum_{\ell=1}^{k+1} a_\ell^n \cdot (a_0-a_\ell)^{-1} \cdot \prod_{\substack{j=1\\ j\neq \ell}}^{k+1} (a_\ell-a_j)^{-1}$$

$$= \sum_{i=1}^{k+1} a_i^n \cdot \prod_{\substack{j=0\\ j\neq i}}^{k+1} (a_i-a_j)^{-1}$$

$$- a_0^{n-k} \cdot \sum_{i=1}^{k+1} a_i^k \cdot \prod_{\substack{j=0\\ j\neq i}}^{k+1} (a_i-a_j)^{-1}$$

Hence it remains to show that

$$0 = a_0^n \cdot \prod_{\substack{j=0\\ j\neq 0}}^{k+1} (a_0-a_j)^{-1} + a_0^{n-k} \cdot \sum_{i=1}^{k+1} a_i^k \cdot \prod_{\substack{j=0\\ j\neq i}}^{k+1} (a_i-a_j)^{-1}$$

Multiplying with $a_0^{k-n} \cdot \prod_{0\leq s<t\leq k+1} (a_s-a_t)$ yields the equivalent formulation

$$0 = \sum_{i=0}^{k+1} (-1)^i \cdot a_i^k \cdot \prod_{\substack{0\leq s<t\leq k+1\\ s\neq i\\ t\neq i}} (a_s-a_t) \quad .$$

However, the expression on the right hand side is the determinant of the following matrix (expanded with respect to the first row):

$$\begin{bmatrix} a_0^k & a_1^k & \dots & a_k^k & a_{k+1}^k \\ 1 & 1 & \dots & 1 & 1 \\ a_0 & a_1 & \dots & a_k & a_{k+1} \\ a_0^2 & a_1^2 & \dots & a_k^2 & a_{k+1}^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^k & a_1^k & \dots & a_k^k & a_{k+1}^k \end{bmatrix}$$

As the first and last row of this matrix are identical, the determinant is zero, thus completing the proof of the theorem. □

It is wellknown that e.g.

$$\lim_{q\to 1} S_k^n(1,q,\dots,q^k) = \lim_{q\to 1} \binom{n}{k}_q = \binom{n}{k} \quad ,$$

thus it is desirable to have an explicit expression for the numbers $S_k^n(a_0,\dots,a_k)$ also if some of the a_i's are equal.

<u>Notation:</u>

Let a be a complex number and let k be a positive integer. The k-tuple $(a,\dots,a)$ consisting of precisely k a's is abbreviated by "$\langle a\rangle^k$" .

Theorem 2

Let n be a nonnegative integer and let $a_0,\dots,a_\ell$ be mutually distinct complex numbers. Let $k_0,\dots,k_\ell$ be positive integers. Then

$$S^n_{(\Sigma k_i)-1}(\langle a_0\rangle^{k_0},\dots,\langle a_\ell\rangle^{k_\ell}) = \sum_{i=0}^{\ell}\sum_{\mu=0}^{k_i-1}\binom{n}{k_i-1-\mu}\cdot a_i^{n-k_i+1+\mu}\cdot\frac{1}{\mu!}\cdot\frac{\delta^\mu}{\delta^\mu a_i}\left[\prod_{\substack{j=0\\ j\neq 1}}^{\ell}(a_i-a_j)^{-k_j}\right]$$

Remark:

By definition, the numbers $S^n_k(a_0,\dots,a_k)$ are invariant under permutations of the arguments, i.e. for every permutation $\tau : \{0,\dots,k\}\to\{0,\dots,k\}$ it follows that

$$S^n_k(a_0,\dots,a_k) = S^n_k(a_{\tau(0)},\dots,a_{\tau(k)}) \quad .$$

Thus the theorem gives an explicit characterization of the numbers $S^n_k(a_0,\dots,a_k)$ in general.

Proof:

We proceed by induction on the sequence $(k_0,\dots,k_\ell)$.

The beginning of the induction, viz. $k_0 = \dots = k_\ell = 1$, has been established in theorem 1.

For the inductive step it suffices to show that the validity of the assertion for the sequence $(1,k_0,\dots,k_\ell)$ implies the validity of the assertion for $(k_0+1,k_1,\dots,k_\ell)$.

Assume that for all x which are different from $a_0,\dots,a_\ell$ it follows that

$$\begin{aligned} S^n_{\Sigma k_i}(x,\langle a_0\rangle^{k_0},\dots,\langle a_\ell\rangle^{k_\ell}) = {} & x^n\cdot\prod_{j=0}^{\ell}(x-a_j)^{-k_j} \\ & + \sum_{i=0}^{\ell}\sum_{\mu=0}^{k_i-1}\binom{n}{k_i-1-\mu}\cdot a_i^{n-k_i+1+\mu}\cdot\frac{1}{\mu!}\cdot \\ & \cdot\frac{\delta^\mu}{\delta^\mu a_i}\left[(a_i-x)^{-1}\cdot\prod_{\substack{j=0\\ j\neq i}}^{\ell}(a_i-a_j)^{-k_j}\right] \end{aligned}$$

The mapping $S^n_{\Sigma k_i}(\cdot,\langle a_0\rangle^{k_0},\ldots,\langle a_\ell\rangle^{k_\ell}) : \mathbb{C}\to\mathbb{C}$ is continous, hence

$$\lim_{x\to a_0} S^n_{\Sigma k_i}(x,\langle a_0\rangle^{k_0},\ldots,\langle a_\ell\rangle^{k_\ell}) = S^n_{\Sigma k_i}(\langle a_0\rangle^{k_0+1},\langle a_1\rangle^{k_1},\ldots,\langle a_\ell\rangle^{k_\ell}) \quad .$$

We show that

$$\lim_{x\to a_0} S^n_{\Sigma k_i}(x,\langle a_0\rangle^{k_0},\ldots,\langle a_\ell\rangle^{k_\ell}) = \sum_{\mu=0}^{k_0}\binom{n}{k_0-\mu}\cdot a_0^{n-k_0+\mu}\cdot\frac{1}{\mu!}\cdot\frac{\delta^\mu}{\delta^\mu a_0}\left[\prod_{j=1}^{\ell}(a_0-a_j)^{-k_j}\right]$$
$$+\sum_{i=1}^{\ell}\sum_{\mu=0}^{k_i-1}\binom{n}{k_i-1-\mu}\cdot a_i^{n-k_i+1+\mu}\cdot\frac{1}{\mu!}\cdot\frac{\delta^\mu}{\delta^\mu a_i}\left[(a_i-a_0)^{-k_0-1}\cdot\prod_{\substack{j=1\\ j\neq i}}^{\ell}(a_i-a_j)^{-k_j}\right] \quad .$$

From elementary calculations it follows that

$$\lim_{x\to a_0} S^n_{\Sigma k_i}(x,\langle a_0\rangle^{k_0},\ldots,\langle a_\ell\rangle^{k_\ell}) = \lim_{x\to a_0}(x-a_0)^{-k_0}\Bigg(x^n\cdot\prod_{j=1}^{\ell}(x-a_j)^{-k_j}$$
$$+(x-a_0)^{k_0}\cdot\sum_{\mu=0}^{k_0-1}\binom{n}{k_0-1-\mu}\cdot a_0^{n-k_0+1+\mu}\cdot\frac{1}{\mu!}\cdot\frac{\delta^\mu}{\delta^\mu a_0}\left[(a_0-x)^{-1}\cdot\prod_{j=1}^{\ell}(a_0-a_j)^{-k_j}\right]\Bigg)$$
$$+\sum_{i=1}^{\ell}\sum_{\mu=0}^{k_i-1}\binom{n}{k_i-1-\mu}\cdot a_i^{n-k_i+1+\mu}\cdot\frac{1}{\mu!}\cdot\frac{\delta^\mu}{\delta^\mu a_i}\left[(a_i-a_0)^{-k_0-1}\cdot\prod_{\substack{j=1\\ j\neq i}}^{\ell}(a_i-a_j)^{-k_j}\right] \quad .$$

We apply the rule of de l'Hospital to the first summand, observing that

$$\frac{\delta^{k_0}}{\delta^{k_0}x}\left[(x-a_0)^{k_0}\cdot\sum_{\mu=0}^{k_0-1}\binom{n}{k_0-1-\mu}\cdot a_0^{n-k_0+1+\mu}\cdot\frac{1}{\mu!}\cdot\frac{\delta^\mu}{\delta^\mu a_0}\left[(a_0-x)^{-1}\cdot\prod_{j=1}^{\ell}(a_0-a_j)^{-k_j}\right]\right] = 0$$

Hence

$$\lim_{x\to a_0} S^n_{\Sigma k_i}(x,\langle a_0\rangle^{k_0},\ldots,\langle a_\ell\rangle^{k_\ell}) = \frac{1}{k_0!}\cdot\lim_{x\to a_0}\cdot\frac{\delta^{k_0}}{\delta^{k_0}x}\left[x^n\cdot\prod_{j=1}^{\ell}(x-a_j)^{-k_j}\right]$$
$$+\sum_{i=1}^{\ell}\sum_{\mu=0}^{k_i-1}\binom{n}{k_i-1-\mu}\cdot a_i^{n-k_i+1+\mu}\cdot\frac{1}{\mu!}\cdot\frac{\delta^\mu}{\delta^\mu a_i}\left[(a_i-a_0)^{-k_0-1}\cdot\prod_{\substack{j=1\\ j\neq i}}^{\ell}(a_i-a_j)^{-k_j}\right]$$
$$=\sum_{\mu=0}^{k_0}\binom{k_0}{k_0-\mu}\cdot\frac{n!}{k_0!\cdot(n-k_0+\mu)!}\cdot a_0^{n-k_0+\mu}\cdot\frac{\delta^\mu}{\delta^\mu a_0}\left[\prod_{j=1}^{\ell}(a_0-a_j)^{-k_j}\right]$$

$$+\sum_{i=1}^{\ell}\sum_{\mu=0}^{k_i-1}\binom{n}{k_i-1-\mu}\cdot a_i^{n-k_i+1+\mu}\cdot\frac{1}{\mu!}\cdot\frac{\delta^\mu}{\delta^\mu a_i}\left[\prod_{\substack{j=1\\j\neq i}}^{\ell}(a_i-a_j)^{-k_j}\right]$$

$$=\sum_{\mu=0}^{k_0}\binom{n}{k_0-\mu}\cdot a_0^{n-k_0+\mu}\cdot\frac{1}{\mu!}\cdot\frac{\delta^\mu}{\delta^\mu a_0}\left[\prod_{j=1}^{\ell}(a_0-a_j)^{-k_j}\right]$$

$$+\sum_{i=0}^{\ell}\sum_{\mu=0}^{k_i-1}\binom{n}{k_i-1-\mu}\cdot a_i^{n-k_i+1+\mu}\cdot\frac{1}{\mu!}\cdot\frac{\delta^\mu}{\delta^\mu a_i}\left[\prod_{\substack{j=1\\j\neq i}}^{\ell}(a_i-a_j)^{-k_j}\right] .$$

This completes the proof of the theorem.

□

Remark:

Theorems 1 and 2 show that the numbers $S_k^n(a_0,\dots,a_k)$ are divided differences, see e.g. [6]. For a treatment based on the calculus of finite differences see the forthcoming paper [2]. In order to keep this paper self contained we continue to give elementary proofs.

For the remainder of this section let $\vec{a} = (a_0,a_1,a_2,\dots)$ denote an infinite sequence of (complex) numbers.

For convenience put $S_{-1}^n(\vec{a}) = 0$ for every nonnegative integer n .

Theorem 3 (Pascal identity for the S - numbers of the second kind)

$$S_k^{n+1}(\vec{a}) = S_{k-1}^n(\vec{a}) + a_k \cdot S_k^n(\vec{a}) \quad .$$

Proof: obvious.

□

Definition:

For nonnegative integers let the polynomial $p_k^{\vec{a}}(x) \in \mathbb{C}[x]$ be defined as follows: $p_0^{\vec{a}}(x) = 1$, viz. the polynomial which is constantly 1 ,

$$p_{k+1}^{\vec{a}}(x) = (x-a_k)\cdot p_k^{\vec{a}}(x) \quad , \quad \text{i.e.} \quad p_{k+1}^{\vec{a}}(x) = (x-a_0)\cdot(x-a_1)\cdot\ldots\cdot(x-a_k) \quad .$$

Examples:

(1) For $\vec{a} = (1,1,1,\dots)$ it is $p_k^{\vec{a}}(x) = (x-1)^k$,

(2) for $\vec{a} = (0,1,2,\dots)$ it is $p_k^{\vec{a}}(x) = [x]_k$, the falling factorial,

(3) for $\vec{a} = (0,-1,-2,\dots)$ it is $p_k^{\vec{a}}(x) = [x]^k$, the rising factorial,

(4) for $\vec{a} = (1,q,q^2,\dots)$ the polynomial $p_k^{\vec{a}}(x) = (x-1)\cdot(x-q)\cdot\dots\cdot(x-q^{k-1})$ is the k.th Gaussian polynomial.

Theorem 4 (Inversion from $(x^n)_{n\in\mathbb{N}}$ to $(p_n^{\vec{a}})_{n\in\mathbb{N}}$)

$$x^n = \sum_{k=0}^{n} S_k^n(\vec{a}) \cdot p_k^{\vec{a}}(x) \quad .$$

Lemma:

Let k be a positive integer and let $b_0,\dots,b_k$ be mutually distinct (complex) numbers. Then

$$\sum_{i=0}^{k} (-1)^i \cdot \prod_{j=0}^{i-1} \frac{b_j-b_k}{b_0-b_{j+1}} = 0 \quad .$$

Proof:

We use induction on k , the case $k = 1$ is obviously valid. Let us consider the case $k+1$:

$$\sum_{i=0}^{k+1} (-1)^i \cdot \prod_{j=0}^{i-1} \frac{b_j-b_k}{b_0-b_{j+1}} = 1 - \frac{b_0-b_k}{b_0-b_1} + \sum_{i=2}^{k+1} (-1)^i \cdot \prod_{j=0}^{i-1} \frac{b_j-b_k}{b_0-b_j}$$

$$= \frac{b_k-b_1}{b_0-b_1} + \frac{b_k-b_1}{b_0-b_1} \cdot \sum_{i=2}^{k+1} (-1)^{i-1} \cdot \frac{b_0-b_k}{b_0-b_2} \cdot \prod_{j=2}^{i-1} \frac{b_j-b_k}{b_0-b_{j+1}} = 0$$

using the inductive hypothesis on $b_0,b_2,\dots,b_k$.

□

Proof of theorem 4

We prove the particular case, where the numbers $a_0,a_1,\dots$ are mutually distinct. The remaining cases follow from continuity reasons.

According to theorem 1 we show that

$$x^n = \sum_{k=0}^{n} \left(\sum_{i=0}^{k} a_i^n \cdot \prod_{\substack{j=0 \\ j\neq i}}^{k} (a_i-a_j)^{-1} \right) \cdot \prod_{j=0}^{k-1} (x-a_j)$$

As both sides of this equation are polynomials of degree n it suffices to show that

$$a_\ell^n = \sum_{k=0}^{n} \left(\sum_{i=0}^{k} a_i^n \cdot \prod_{\substack{j=0 \\ j\neq i}}^{k} (a_i-a_j)^{-1} \right) \cdot \prod_{j=0}^{k-1} (a_\ell-a_j)$$

for every $0 \leq \ell \leq n$. Fix any such ℓ . Consider, changing the summation:

$$(*) \qquad \sum_{i=0}^{n} a_i^n \cdot \left(\sum_{k=i}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{k} (a_i-a_j)^{-1} \right) \cdot \prod_{j=0}^{k-1} (a_\ell-a_j) \quad .$$

For $i = 0,\ldots,n$ then consider each summand separately in order to find out what it contributes to the sum. We distinguish three cases:

Case 1:

$i > \ell$, then the contribution is zero, due to the factor $\prod_{j=0}^{k-1} (a_\ell-a_j)$ which vanishes as $k \geq i > \ell$.

Case 2:

$i = \ell$, then we have the contribution a_ℓ^n , viz.

$$a_\ell^n \cdot \left(\sum_{k=\ell}^{n} \prod_{\substack{j=0 \\ j\neq \ell}}^{k} (a_\ell-a_j)^{-1} \right) \cdot \prod_{j=0}^{k-1} (a_\ell-a_j) = a_\ell^n \cdot \left(\prod_{j=0}^{\ell-1} (a_\ell-a_j)^{-1} \right) \cdot \prod_{j=0}^{\ell-1} (a_\ell-a_j)$$

$$= a_\ell^n \quad .$$

Case 3:

$i < \ell$, then the contribution is zero again, viz.

$$a_i^n \cdot \left(\sum_{k=i}^{n} \left(\prod_{j=0}^{i-1} (a_i-a_j)^{-1} \right) \cdot \left(\prod_{j=i+1}^{k} (a_i-a_j)^{-1} \right) \right) \cdot \prod_{j=0}^{k-1} (a_\ell-a_j) =$$

$$= a_i^n \cdot \prod_{j=0}^{i-1} (a_i-a_j)^{-1} \cdot \prod_{j=0}^{i-1} (a_\ell-a_j) \cdot \left[\sum_{k=i}^{\ell} (-1)^k \cdot \prod_{j=i}^{k-1} \frac{a_j-a_\ell}{a_i-a_{j+1}} \right] = 0$$

as according to the lemma the sum in the square brackets vanishes.

Hence the sum (*) takes the value a_ℓ^n , thus completing the proof of the theorem. □

Definition:

The numbers $\mathscr{s}^n_k(\vec{a})$ are defined by the following identities:

$$\sum_{j\geq 0} \mathscr{s}^n_j(\vec{a}) \cdot S^j_k(\vec{a}) = \delta^n_k \qquad \text{(Kronecker-Symbol)} ,$$

resp. $$\sum_{j\geq 0} S^n_j(\vec{a}) \cdot \mathscr{s}^j_k(\vec{a}) = \delta^n_k \quad , \quad \text{resp.} \quad p^{\vec{a}}_n(x) = \sum \mathscr{s}^n_k \cdot x^k .$$

Remark: From the classical theory we know:

$$\mathscr{s}^n_k(1,\dots,1) = (-1)^{n+k} \binom{n}{k} \qquad \text{(Binomial inversion)} ,$$

$$\mathscr{s}^n_k(0,1,2,\dots,k) = s^n_k \qquad \text{(Stirling numbers of the first kind)},$$

$$\mathscr{s}^n_k(1,q,q^2,\dots,q^k) = (-1)^{n+k} \cdot q^{\binom{n-k}{2}} \cdot \binom{n}{k}_q \qquad \text{(Gaussian inversion)}$$

Theorem 5 (Pascal identity for the $\mathscr{s}$ - numbers of the first kind)

$$\mathscr{s}^{n+1}_k(\vec{a}) = \mathscr{s}^n_{k-1}(\vec{a}) - a_n \cdot \mathscr{s}^n_k(\vec{a}) ,$$

where again for convenience $\mathscr{s}^n_{-1}(\vec{a}) = 0$ for every nonnegative integer n .

Proof:

Consider the polynomial $p^{\vec{a}}_{n+1}(x) = x \cdot p^{\vec{a}}_n(x) - a_n \cdot p^{\vec{a}}_n(x)$ and compare the coefficients of x^k in the expansion $\sum_{j\geq 0} \mathscr{s}^{n+1}_j \cdot x^j =$

$$= \sum_{j\geq 0} \mathscr{s}^n_{j-1} \cdot x^j - \sum_{j\geq 0} a_n \cdot \mathscr{s}^n_j \cdot x^j .$$

□

Theorem 6

$$\mathscr{s}^n_k = (-1)^{n+k} \cdot \sum_{0\leq\mu_0<\mu_1<\dots<\mu_{n-k-1}<n} \prod_{i=0}^{n-k-1} a_{\mu_i} \qquad \text{for every } k < n .$$

Proof:

We use induction on n . For $n = 0$ there is nothing to show. Thus consider the case $n+1$. The particular case $k = n$ is treated separately, viz.

$$s_n^{n+1} = s_{n-1}^n - a_n \cdot s_n^n = - \sum_{i=0}^{n-1} a_i - a_n \cdot s_n^n = - \sum_{i=0}^{n} a_i$$

in accordance with the assertion.

Now let be $k < n$. Then

$$\begin{aligned} s_k^{n+1} &= s_{k-1}^n - a_n \cdot s_k^n \\ &= (-1)^{n+k-1} \cdot \sum_{0 \leq \mu_0 < \ldots < \mu_{n-k} < n} \prod_{i=0}^{n-k} a_{\mu_i} \\ &\qquad - (-1)^{n+k} \cdot a_n \cdot \sum_{0 \leq \mu_0 < \ldots < \mu_{n-k-1} < n} \prod_{i=0}^{n-k-1} a_{\mu_i} \\ &= (-1)^{n+k+1} \cdot \sum_{0 \leq \mu_0 < \ldots < \mu_{n-k} < n+1} \prod_{i=0}^{n-k} a_{\mu_i} \end{aligned}$$

in accordance with the assertion.

□

The connection between the matrices $[S_k^n(\vec{a})]$, $[s_k^n(\vec{a})]$ and the sequence $[p_n^{\vec{a}}]^T$ written as a column vector is given by the following two inversion formulae:

$$(*) \quad \begin{cases} [p_n^{\vec{0}}]^T = [S_k^n(\vec{a})] \cdot [p_n^{\vec{a}}]^T \quad , \\ [p_n^{\vec{a}}]^T = [s_k^n(\vec{a})] \cdot [p_n^{\vec{0}}]^T \end{cases}$$

Of course, we also could have started from a given ascending sequence of normalized polynomials $(p_n)_{n \in \mathbb{N}}$, where ascending means that p_n divides (w.r.t. the ring $\mathbb{C}[x]$) p_{n+1} . As $\mathbb{C}$ is algebraically closed then $p_{n+1}(x) = p_n(x) \cdot (x-a_n)$ for some complex number a_n . As $p_n^{\vec{a}} = p_n$ for the sequence $\vec{a}$ of roots we again obtain the inversion formulae (*) .

Particularly the sequence $\vec{0} = (0,0,\ldots)$ yields that $p_n^{\vec{0}}(x) = x^n$ for every non-negative integer n .

Nowlet $\vec{a}$ and $\vec{b}$ be two infinite sequences. From the inversion formulae (*) we immediately obtain inversion formulae for transforming the polynomials $(p_n^{\vec{a}})_{n \in \mathbb{N}}$

into $(p_n^{\vec{b}})_{n\in\mathbb{N}}$ and vice versa:

$$(**)\quad \begin{cases} [p_n^{\vec{a}}]^T = [s_k^n(\vec{a})] \cdot [S_k^n(\vec{b})] \cdot [p_n^{\vec{b}}]^T \\ [p_n^{\vec{b}}]^T = [s_k^n(\vec{b})] \cdot [S_k^n(\vec{a})] \cdot [p_n^{\vec{a}}]^T \end{cases} .$$

Let us denote the numbers occuring in the matrices $[s_k^n(\vec{a})] \cdot [S_k^n(\vec{b})]$ by $S_k^n(\vec{a},\vec{b})$. Analogous let us denote the entries of $[s_k^n(\vec{b})] \cdot [S_k^n(\vec{a})]$ by $s_k^n(\vec{a},\vec{b})$, more precisely:

Definition:

$$S_k^n(\vec{a},\vec{b}) = \sum_{j\geq 0} s_j^n(\vec{a}) \cdot S_k^j(\vec{b}) \quad \text{and} \quad s_k^n(\vec{a},\vec{b}) = \sum_{j\geq 0} s_j^n(\vec{b}) \cdot S_k^j(\vec{a}) \quad .$$

An well known example are the Lah numbers [4] , let us consider here the unsigned Lah numbers:

One immediately verifies that $s_k^n(0,-1,-2,-3,\ldots) = |s_k^n|$, the absolute Stirling numbers of the first kind. Hence

$$L'^n_k = \sum_{j\geq 0} |s_j^n| \cdot S_k^j = S_k^n((0,-1,-2,\ldots),(0,1,2,\ldots)) \quad ,$$

the signless Lah numbers,satisfy the identity

$$[x]^n = \sum_{k\geq 0} L'^n_k \cdot [x]_k \quad .$$

The following recursion for the unsigned Lah numbers is well known:

$$L'^{n+1}_k = L'^n_{k-1} + (k+n) \cdot L'^n_k \quad .$$

This recursion can be generalized as follows:

Theorem 7 (Pascal identity for $S_k^n(\vec{a},\vec{b})$ and $s_k^n(\vec{a},\vec{b})$)

$$S_k^{n+1}(\vec{a},\vec{b}) = S_{k-1}^n(\vec{a},\vec{b}) + (b_k-a_n) \cdot S_k^n(\vec{a},\vec{b})$$

$$s_k^{n+1}(\vec{a},\vec{b}) = s_k^n(\vec{a},\vec{b}) + (a_k-b_n) \cdot s_k^n(\vec{a},\vec{b}) \quad .$$

Proof:

We proceed by induction on n :

$$\begin{aligned} S_k^{n+1}(\vec{a},\vec{b}) &= \sum_{j\geq 0} s_j^{n+1}(\vec{a}) \cdot S_k^j(\vec{b}) \\ &= \sum_{j\geq 0} \left(s_{j-1}^n(\vec{a}) \cdot S_k^j(\vec{b})\right) - a_n \cdot \sum_{j\geq 0} s_j^n(\vec{a}) \cdot S_k^j(\vec{b}) \\ &= \sum_{j\geq 0} \left(s_j^n(\vec{a}) \cdot S_k^{j+1}(\vec{b})\right) - a_n \cdot S_k^n(\vec{a},\vec{b}) \\ &= \sum_{j\geq 0} \left(s_j^n(\vec{a}) \cdot S_{k-1}^j(\vec{b})\right) + b_k \cdot \sum_{j\geq 0} \left(s_j^n(\vec{a}) \cdot S_k^j(\vec{b})\right) - a_n \cdot S_k^n(\vec{a},\vec{b}) \\ &= S_{k-1}^n(\vec{a},\vec{b}) + (b_k-a_n) \cdot S_k^n(\vec{a},\vec{b}) \quad . \end{aligned}$$

The second recursion follows from the first one, as $S_k^n(\vec{a},\vec{b}) = s_k^n(\vec{b},\vec{a})$. □

The next recursions generalizes an identity for Gaussian binomial coefficients which has been discovered by Carlitz [1] .

Notation:

Let ℓ be a complex number. By $\vec{a} - \ell$ we denote the sequence $(a_0-\ell, a_1-\ell, \dots)$, i.e. ℓ is subtracted from each component of $\vec{a}$.

Theorem 8

(i) $S_k^n(\vec{a}) = \sum_j \binom{n}{j} \cdot \ell^{n-j} \cdot S_k^j(\vec{a}-\ell)$

(ii) $S_k^n(\vec{a}) = \sum_j \binom{j}{k} \cdot \ell^{j-k} \cdot S_j^n(\vec{a}-\ell)$

(iii) $s_k^n(\vec{a}) = \sum_j \binom{n}{j} \cdot \ell^{n-j} \cdot s_k^j(\vec{a}+\ell)$

(iv) $s_k^n(\vec{a}) = \sum_j \binom{j}{k} \cdot \ell^{j-k} \cdot s_j^n(\vec{a}+\ell)$.

Proof:

We prove (i) , the remaining cases can be handled analogously. Proceed by induction on n . The case $n = 0$ is obviously valid, thus let us consider the case $n + 1$:

$$
\begin{aligned}
S_k^{n+1}(\vec{a}) &= S_{k-1}^n(\vec{a}) + a_k \cdot S_k^n(\vec{a}) \\
&= \sum_{j\geq 0} \left(\binom{n}{j}\cdot \ell^{n-j} \cdot S_{k-1}^j(\vec{a}-\ell)\right) \quad + \quad a_k \cdot \sum_{j\geq 0} \binom{n}{j}\cdot \ell^{n-j} \cdot S_k^j(\vec{a}-\ell) \\
&= \sum_{j\geq 0} \left(\binom{n}{j} \cdot \ell^{n-j} \cdot (S_{k-1}^j(\vec{a}-\ell) \;+\; (a_k-\ell)\cdot S_k^j(\vec{a}-\ell))\right) \\
&\quad + \ell \cdot \sum_{j\geq 0} \binom{n}{j} \cdot \ell^{n-j} \cdot S_k^j(\vec{a}-\ell) \\
&= \ell \cdot \binom{n}{0} \cdot \ell^n \cdot S_k^0(\vec{a}-\ell) + \sum_{j\geq 1} \left(\binom{n}{j} \cdot \ell^{n-j+1} + \binom{n}{j-1} \cdot \ell^{n-j+1}\right) \cdot S_k^j(\vec{a}-\ell) \\
&= \binom{n+1}{0} \cdot \ell^{n+1} \cdot S_k^0(\vec{a}-\ell) + \sum_{j\geq 1} \binom{n+1}{j} \cdot \ell^{n+1-j} \cdot S_k^j(\vec{a}-\ell) \\
&= \sum_{j\geq 0} \binom{n+1}{j} \cdot \ell^{n+1-j} \cdot S_k^j(\vec{a}-\ell) \qquad ,
\end{aligned}
$$

completing the proof.

□

Remark:

The identity of Carlitz [1] appears considering (i) with $\vec{a} = (1,q^2,q^3,\ldots)$ and $\ell = 1$, viz. $\binom{n}{k}_q = \sum_{j\geq 0} \binom{n}{j} \cdot A_j^{(k)}$, where $A_j^{(k)} = S_k^j(0,q-1,q^2-1,\ldots)$ counts the number of k-dimensional linear subspaces W of the n-dimensional vector space $(GF(q))^n = \{(x_1,\ldots,x_n)|x_i \in GF(q)\}$, such that every projection $\pi_i : V \to GF(q)$ is surjective, where the projection π_i is defined by $\pi_i(x_1,\ldots,x_n) = x_i$ [5] .

We should mention that theorem 8 (i) applied to Stirling numbers of the second kind does not yield the familiar recursion $S_k^{n+1} = \sum_{j\geq 0} \binom{n+1}{j} S_{k-1}^j$, simply because generally the numbers S_{k-1}^j and $S_k^j(-1,0,\ldots,k-1)$ are not the same. However, the recursion $S_k^{n+1} = \sum_{j\geq 0} \binom{n+1}{j} \cdot S_{k-1}^j$ is a unique feature of the Stirling numbers of the second kind, more precisely:

Observation:

Let $a = (a_0,a_1,\ldots)$ be an infinite sequence of complex numbers such that

$$S_k^{n+1}(\vec{a}) = \sum_{j\geq 0} \binom{n+1}{j} \cdot S_{k-1}^j(\vec{a})$$

holds for every pair of nonnegative integers k and n. Then it follows that $a_i = i$, i.e. $S_k^n(\vec{a}) = S_k^n$.

Proof:

As $S_0^{n+1}(\vec{a}) = a_0^{n+1} = \binom{n+1}{0} \cdot S_{-1}^{n+1}(\vec{a}) = 0$ it follows that $a_0 = 0$. Assume by induction that $a_{k-1} = k-1$ and consider $S_k^{n+1}(\vec{a})$, viz.

$$\begin{aligned}
S_k^{n+1}(\vec{a}) &= S_{k-1}^n(\vec{a}) + a_k \cdot S_k^n(\vec{a}) \\
&= \sum_{j\geq 0} \binom{n}{j} \cdot S_{k-2}^j(\vec{a}) + \sum_{j\geq 0} \binom{n}{j} \cdot a_k \cdot S_{k-1}^j(\vec{a}) \\
&= \sum_{j\geq 0} \left(\binom{n}{j} \cdot (S_{k-2}^j(\vec{a}) + (k-1) \cdot S_{k-1}^j(\vec{a}))\right) + (a_k-k+1) \cdot \sum_{j\geq 0} \binom{n}{j} \cdot S_{k-1}^j(\vec{a}) \\
&= \left(\sum_{j\geq 1} \binom{n}{j-1} \cdot S_{k-1}^j(\vec{a})\right) + \binom{n}{0} \cdot S_{k-1}^0(\vec{a}) + \left(\sum_{j\geq 1} \binom{n}{j} \cdot S_{k-1}^j(\vec{a})\right) \\
&\quad + (a_k-k) \cdot \sum_{j\geq 0} \binom{n}{j} \cdot S_{k-1}^j(\vec{a}) \\
&= \sum_{j\geq 0} \binom{n+1}{j} \cdot S_{k-1}^j(\vec{a}) + (a_k-k) \cdot \sum_{j\geq 0} \binom{n}{j} \cdot S_{k-1}^j(\vec{a}) ,
\end{aligned}$$

hence $(a_k-k) \cdot S_k^n(\vec{a}) = 0$, which shows that $a_k = k$. This completes the proof of the observation.

□

Let us apply theorem 8 (i) to Stirling numbers of the second kind and see what happens:

Consider the sequence $(-1,0,1,2,3,\ldots)$. Call the numbers

$$S_k^n(-1,0,1,2,\ldots) = \hat{S}_k^n$$

the reduced Stirling numbers. By 8 (i) then

$$S_k^n = \sum_{j\geq 0} \binom{n}{j} \cdot \hat{S}_k^n \quad .$$

The following table contains some values of the reduced Stirling numbers:

n\k	0	1	2	3	4	5	6	7	8
0	1	0							
1	-1	1	0						
2	+1	-1	1	0					
3	-1	+1	0	1	0				
4	+1	-1	1	2	1	0			
5	-1	+1	0	5	5	1	0		
6	+1	-1	1	10	20	9	1	0	
7	-1	+1	0	21	70	56	14	1	0
8	+1	-1	1	42	231	294	126	21	1

For $k \geq 3$ the reduced Stirling numbers admit the following combinatorial interpretation:

Theorem 9

For $k \geq 3$ it follows that

$\hat{S}_k^n \equiv$ number of surjections $f : \{0,\dots,n-1\} \to \{\lambda_0,\lambda_1,\dots,\lambda_{k-1}\}$ such that there exists an even nonnegative integer $\ell < n$ satisfying

$$(*)\quad \begin{cases} f^{-1}(\lambda_0) = \{0,\dots,\ell\}\ , \\ f(\ell+1) = \lambda_1 \text{ and } f(\ell+2) = \lambda_2 \text{ and} \\ \min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_{i+1}) \quad \text{for every } i = 0,\dots,k-2 \end{cases} .$$

Proof: One immediately verifies (using Pascal identity) that

$$\hat{S}_2^n = 1 \quad \text{iff } n \geq 2 \text{ and } n \equiv 0 \pmod 2$$
$$= 0 \quad \text{iff } n < 2 \text{ or } n \not\equiv 1 \pmod 2 .$$

Now we use induction on n. The case $n = 3$ is obviously valid, hence consider $n+1$, viz. $\hat{S}_k^{n+1} = \hat{S}_{k-1}^n + (k-1) \cdot \hat{S}_k^n$.

Let $f : \{0,\dots,n\} \to \{\lambda_0,\dots,\lambda_{k-1}\}$ be any surjection satisfying (*) .

If still $f\rceil\{0,\dots,n-1\}$ acts surjectively onto $\{\lambda_0,\dots,\lambda_{k-1}\}$ there exist precisely $k-1$ possibilities for $f(n)$, viz. $\lambda_1,\dots,\lambda_k$. This explaines the right summand. If $f\rceil\{0,\dots,n-1\}$ does not act surjectively, then $f(n) = \lambda_{k-1}$. But in this case $f\rceil\{0,\dots,n-1\}$ acts surjectively onto $\{\lambda_0,\dots,\lambda_{k-2}\}$. If still $k-1 \geq 3$, $f\rceil\{0,\dots,n-1\}$ also satisfies (*). This explaines the first summand. If $k-1 = 2$ and f satisfies (*), then it follows that $f^{-1}(\lambda_1) = \{n-1\}$, i.e. $n-2$ is even and hence $\hat{S}_2^n = 1$. In both cases $\hat{S}_k^{n+1}$ turns out to be the right number.

□

We give two more examples applying theorem 8.

<u>Example 1</u> (homogenous Boolean sublattices)

Let $P(n)$ denote the Boolean lattices of subsets of an n-element set. A $P(k)$-sublattice L of $P(n)$ is a homogenous sublattice provided that $\min L = \min P(n)$. By hB_k^n we denote the number of homogenous $P(k)$-sublattices of $P(n)$.
A homogenous $P(k)$-sublattice L of $P(n)$ is determined by its atoms, viz. by k mutually disjoint and nonempty subsets $A_0,\dots,A_{k-1}$. Without restriction say that $\min A_1 < \min A_2 < \dots < \min A_k$. L can be represented by a mapping $f : \{0,\dots,n-1\} \to \{\lambda_0,\dots,\lambda_{k-1}\} \cup \{0\}$, where $f(i) = \lambda_j$ iff $i \in A_j$ and $f(i) = 0$ in all other cases. Then

$$(*) \qquad \min f^{-1}(\lambda_0) < \min f^{-1}(\lambda_1) < \dots < \min f^{-1}(\lambda_{k-1}) .$$

On the other hand, every function $f : \{0,\dots,n-1\} \to \{\lambda_0,\dots,\lambda_{k-1}\} \cup \{0\}$ satisfying (*) determines uniquely a homogenous $P(k)$-sublattice of $P(n)$. This establishes a bijection between homogenous sublattices and such functions f. Hence $hB_k^n = S_k^n(1,2,3,\dots)$.

<u>Corollary 10</u>

$$hB_k^n = \sum_j \binom{n}{j} \cdot S_k^j$$

$$\sum_k hB^n_k = \sum_k S^{n+1}_k = B_{n+1} \qquad \text{(Bell-number)} \quad .$$

Proof:

The first equality is 8 (i) . Concerning the second inequality one observes that

$$\begin{aligned}\sum_k hB^n_k &= \sum_k \sum_j \binom{n}{j} \cdot S^j_k \\ &= \sum_j \binom{n}{j} \sum_k S^j_k \\ &= \sum_j \binom{n}{j} \cdot B_k = B_{n+1} \quad ,\end{aligned}$$

where the last equality is well known.

□

Example 2 (Boolean sublattices)

By aB^n_k we denote the number of arbitrary (viz. affine) $P(k)$ - sublattices of $P(n)$. As a $P(k)$ - sublattice of P is determined by k nonempty subsets $A_0,\dots,A_{k-1}$ which have pairwise the same intersection, it follows that

$$aB^n_k = S^n_k(2,3,4,\dots) \quad .$$

Corollary 11

$$aB^n_k = \sum_j \binom{n}{j} \cdot 2^{n-j} \cdot S^j_k$$

$$\sum_k aB^n_k = B_{n+2} \quad .$$

Proof: proceed as before.

□

We conclude with an application of the inversion formula for the s - numbers of the first kind, deriving a recursion formula for Mac Mahon numbers.

Example (Mac Mahon numbers)

The Mac Mahon numbers B_k^n, where $n \geq 1$ and $0 \leq k \leq \binom{n}{2}$, are defined by the following identities:

$$(*) \qquad \sum_{k=0}^{\binom{n}{2}} B_k^n \cdot q^k = \prod_{i=1}^{n} \frac{1-q^i}{1-q} \quad .$$

Foata [3] gives a combinatorial interpretation for the numbers B_k^n.
Let $[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\ldots+q^{n-1}$ be the q-analogue of the nonnegative integer n. Then (*) can be rewritten as

$$(**) \qquad \sum_{k=0}^{\binom{n}{2}} B_k^n \cdot q^k = \prod_{i=1}^{n} [i]_q \quad .$$

Consider any sequence $\vec{a} = (a_0, a_1, \ldots)$ of complex numbers such that for every nonnegative integer n the numbers $a_{\binom{n}{2}}, a_{\binom{n}{2}+1}, \ldots, a_{\binom{n}{2}+n-1}$ are the $(n+1)$-st roots of unity different from 1 . Say

$$\vec{a} = (e^{\frac{1}{2}\cdot\sqrt{-1}\cdot 2\cdot\pi}, e^{\frac{1}{3}\cdot\sqrt{-1}\cdot 2\cdot\pi}, e^{\frac{2}{3}\cdot\sqrt{-1}\cdot 2\cdot\pi}, e^{\frac{1}{4}\cdot\sqrt{-1}\cdot 2\cdot\pi}, \ldots)$$

i.e.

$$a_{\binom{n}{2}+j} = e^{\frac{1+j}{n+1}\cdot\sqrt{-1}\cdot 2\cdot\pi} \quad \text{for every } n \geq 1 \text{ and } 0 \leq j < n .$$

Then

$$(***) \qquad [n+1]_x = 1+x+\ldots+x^n = \prod_{i=\binom{n}{2}}^{n-1} (x-a_i)$$

and thus

$$P^{\vec{a}}_{\binom{n}{2}}(q) = \prod_{i=1}^{n} [i]_q \quad .$$

According to the definition of the s-numbers of the first kind it follows that

$$\sum_{k=0}^{\binom{n}{2}} s_k^{\binom{n}{2}} \cdot q^k = \prod_{i=1}^{n} [i]_q \quad ,$$

hence $B_k^n = s_k^{\binom{n}{2}}$ for every $n \geq 1$.

From theorem 6 we have the following explicit characterization for the numbers B_k^n:

$$B_k^n = (-1)^{\binom{n}{2}+k} \cdot \sum_{0\leq\mu_0<\ldots<\mu_{\binom{n}{2}-k-1}<\binom{n}{2}} \prod_{i=0}^{\binom{n}{2}-k-1} a_i \quad ,$$

where the complex numbers a_i have been defined above.

However, the Pascal-identity for the s - numbers of the first kind (theorem 5) yields a recursion for the Mac Mahon numbers:

<u>Theorem</u> (Recursion for Mac Mahon numbers)

$$B_k^{n+1} = B_{k-1}^{n+1} + B_k^n - B_{k-1-n}^n \quad ,$$

where we put $B_0^0 = 1$ and $B_k^n = 0$ if k or n (or both) are negative.

<u>Proof:</u> using induction it follows from theorem 5 that

$$s_k^{n+m} = \sum_{i=0}^{m} (-1)^i \Big(\sum_{0\leq\mu_0<\ldots<\mu_{i-1}<m} \prod_{\nu=0}^{i-1} a_{n+\mu_\nu} \Big) \cdot s_{k-m+i}^n$$

From (***) it follows that

$$(-1)^i = \sum_{0\leq\mu_0<\ldots<\mu_{i-1}<n} \prod_{\nu=0}^{i-1} a_{\binom{n}{2}+\mu_\nu} \quad ,$$

hence

$$B_k^{n+1} = s_k^{\binom{n+1}{2}} = s_k^{\binom{n}{2}+n} = \sum_{i=0}^{n} s_{k-n+1}^n$$

$$= \sum_{i=0}^{n} B_{k-i}^n \quad ,$$

and the desired recursion follows immediately.

□

References:

[1] L. Carlitz: On abelian fields, Trans. AMS, 35, 1933, 122 - 136 .

[2] W. Dahmen und B. Voigt: The use of divided differences for basic enumeration problems, in preparation .

[3] D. Foata: Distributions eulériennes et mahoniennes sur le groupe des permutations, in Higher Combinatorics, ed. M. Aigner, 1977, 27 - 49 .

[4] I. Lah: Eine neue Art von Zahlen, ihre Eigenschaften und Anwendungen in der math. Statistik, Mitteilungsblatt f. math. Statistik, 7, 1955, 203 - 212 .

[5] S.C. Milne: Mappings of subspaces into subsets, JCT(A), 33, 1982, 36 - 47 .

[6] N.E. Nörlund: Vorlesungen über Differenzenrechnung, Chelsea, New York 1959 (reprint) .

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