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Polya's theorem for non-entire functions

by Kunio Yoshino (*)

Abstract

Using transforms of analytic functionals with non-compact carrier, Polya's theorem concerning arithmetic entire functions is generalized to arithmetic non-entire functions.

1. Introduction

In 1920 Polya (see [7])) proved the following

Theorem. Suppose that the function $f(z)$ satisfies the following conditions :

(1) $|f(z)| \leqq Ce^{\alpha|z|}$ $(z \in C)$

(2) $f(N) \subset Z$.

If $\alpha < \log 2$, then $f(z)$ is a polynomial with rational coefficients.

Recently, this theorem has been generalized by several authors to the case of entire functions of several complex variables (see [2], [3], [4]).

In this paper , we investigate Polya's theorem for non-entire functions of several complex variables.

The following theorem is our main result.

Theorem 1. Let $f(z)$ be holomorphic in $\Gamma = \{z \in C^n : \mathrm{Re}\, z_i > 0,\ 1 \leqq i \leqq n\}$ and satisfy the following conditions :

(1) For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|f(z)| \leqq C\ e^{a(z)} \qquad (\mathrm{Re}\, z_i \geqslant \varepsilon,\ 1 \leqq i \leqq n)$$

where $a(z)$ is a convex function of homogeneous degree 1.

(2) $f(N^n) \subset Z$.

Furthermore let $L \subset C^n$ be defined by

$L = \{\zeta \in C^n : \mathrm{Re} \langle \zeta, z \rangle \leqq a(z),\ \forall z \in \Gamma\}$

and suppose that the i-th projection $L_i = pr_i(L)$ of L is contained in $\{\zeta_i \in C : |e^{\zeta_i} - 1| < 1\}$ for all i $(1 \leqq i \leqq n)$.

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Then $f(z)$ is a polynomial with rational coefficients.

To prove Theorem 1 we use the Fourier-Borel and Avanissian -Gay transforms of analytic functionals with unbounded carrier and in the sections 2 and 3 we define the Fourier-Borel and Avanissian - Gay transforms of such functionals. In section 4 we recall the definition of the transfinite diameter and its properties while in section 5, we give the proof of Theorem 1.

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2. The Fourier-Borel transform of analytic functionals with unbounded carrier

In this section we first recall the definition of the Fourier-Borel transform of analytic functionals with unbounded carriers and also mention the Ehrenpreis-Martineau type theorem due to J.W. DE. Roever. Let L be a closed convex set which is bounded in the imaginary direction in C^n and put

$$H_b(L:\varphi)=\{f(z)\in O(\overset{\circ}{L})\cap C(L):\sup_{z\in L}|f(z)e^{-\varphi(z)}|<+\infty\} ,$$

where $\varphi(z)$ is a real valued function and $O(\overset{\circ}{L})$ and $C(L)$ denote respectively the spaces of holomorphic functions defined in the interior of L, and the space of continuous functions in L.
Put

$$Q(L:K')=\lim_{\varepsilon\downarrow 0}\ \operatorname{ind}_{\varepsilon'\downarrow 0} H_b(L_\varepsilon:-h_{K'}(z)-\varepsilon'|z|)$$

where L_ε stands for the ε-neighbourhood of L and $h_{K'}(z)$ is the supporting function of the compact convex subset K' of C^n.
An element of the dual space $Q'(L:K')$ of $Q(L:K')$ is called an analytic functional with carrier L and of type $h_{K'}(z)$. Let us recall that if L is a compact convex subset of C^n then $Q'(L:K')$ concides with the space of analytic functionals $O'(L)$ in the sense of A. Martineau, and that if $L=R^n$ and $K'=\{0\}$ then $Q'(L:K')$ coincides with the space of Fourier-hyperfunctions studied by M.Sato and T. Kawai.
When the exponential function $\exp(\sum_{i=1}^n \zeta_i z_i)=\exp(\zeta z)$ belongs to $Q(L:K')$, the Fourier-Borel transform $\tilde{T}(z)$ of $T\in Q'(L:K')$ is defined as follows :

$$\tilde{T}(z)=\langle T_\zeta,\exp(\zeta z)\rangle .$$

Now let Γ be an open convex cone in C^n, let $a(z)$ be a convex function on Γ of homogeneous degree one and put

$$\Omega(a;\Gamma)=\{\zeta\in C^n : \mathrm{Re}<\zeta,z>\leqq a(z).\ \forall z\in\Gamma\}.$$

Then the following generalized Ehrenpreis-Martineau type theorem is valid :

Theorem 2 (J.W. DE Roever [9])
The Fourier-Borel transform is a linear topological isomorphism from $Q'(\Omega(a:\Gamma):\{0\})$ onto $\mathrm{Exp}(\Gamma:a)$, where $\mathrm{Exp}(\Gamma:a)=\lim\ \mathrm{proj}\ H_b(\Gamma+\varepsilon(z_0);\ a(z)+\varepsilon|z|)$ and z_0 is a fixed complex vector contained in Γ with $|z_0|=1$.

Note that the space $H_b(\Gamma+\varepsilon(z_0):a(z)+\varepsilon|z|)$ may be defined in a similar as $H_b(L:\varphi)$.
We close this section by giving two examples of $\Omega(a:\Gamma)$ in the case of $n=1$.
Take

$$\Gamma=\{z\in C:\mathrm{Re}\,z>0\}.$$

Example 1. If $a(z)=\alpha|z|$ with $\alpha>0$, then $\Omega(a:\Gamma)=$ $\{\zeta\in C:|\zeta|=\alpha\}\cup\{\zeta\in C:|\mathrm{Im}\zeta|\leqq\alpha,\mathrm{Re}\zeta\leqq 0\}$ (see Figure 1).

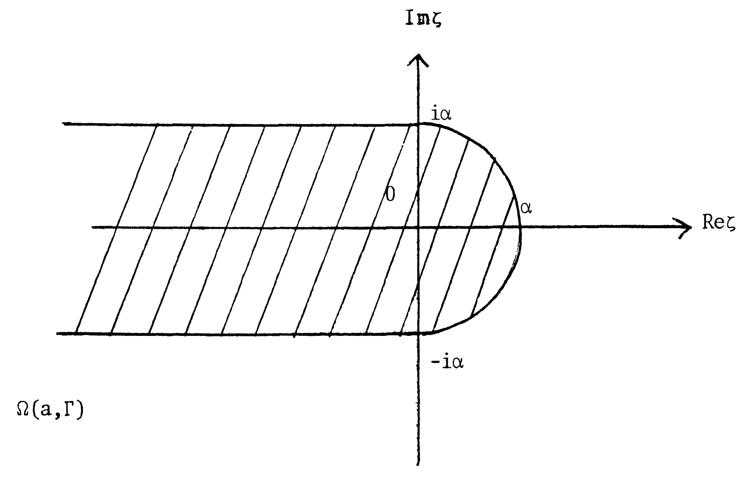


Figure 1.

Example 2 (see [8]) If $a(z)=|z|\{\cos\varphi\log(2\cos\varphi)+\varphi\sin\varphi\}$ where $z=|z|e^{i\varphi}$ $(-\pi/2<\varphi<\pi/2)$, then $\Omega(a:\Gamma)=\{\zeta\in C:|e^{\zeta}-1|<1\}$. (see Figure 2)

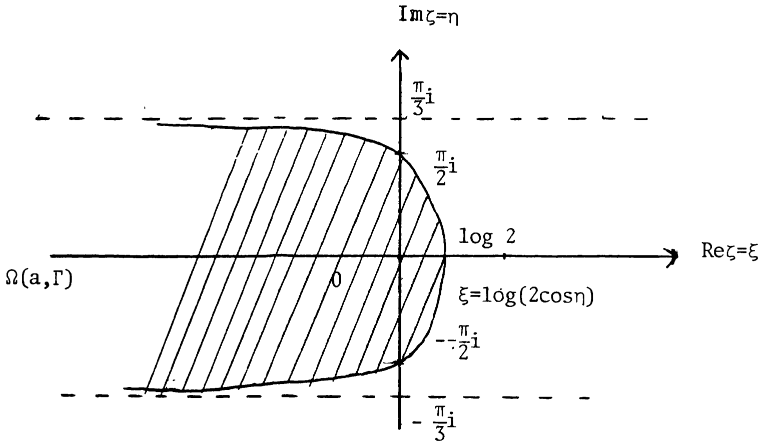


Figure 2

3. The Avanssian-Gay transform of analytic functionals with unbounded carrier

In [3], the Avanissian-Gay transform is introduced for analytic functionals with compact carrier, while in [6] and [10] it has been generalized to the case of analytic functionals with unbounded carrier. According to [3], [6] and [10]. Let us first recall the definition of the Avanissian-Gay transform of analytic functionals with unbounded carrier.

Assume that the closed convex set L is bounded in the imaginary direction and also bcunded below in the real direction. More precisely, we assume there exist $a_i\in R$ $(1\leqq i\leqq n)$ and compact sets K_i $(1\leqslant i\leqslant n)$ having a width less than 2π such that

$$L\subset \prod_{i=1}^{n}(a_i+R_+ +\sqrt{-1}K_i)$$

where $R_+=[0,\infty)$

Furthermore let $T\in Q'(L:\{0\})$; then the Avanissian-Gay transform $G_T(w)$ of T is given by

$$G_T(w) = \langle T_\zeta, \prod_{i=1}^{n} \frac{1}{1-w_i e^{\zeta_i}} \rangle$$

Some properties of $G_T(w)$ are listed in

Proposition 1. (see [3], [6], [10])

(1) $$G_T(w) \in O(\prod_{i=1}^{n} \{\mathbb{C}\setminus \exp(-L_i)\})$$

where L_i is i-th projection of L $(1 \leqq i \leqq n)$

(2) $$G_T(w) = (-1)^n \sum_{m \in N^n} T(-m) w_1^{-m_1} \ldots w_n^{-m_n} \qquad (|w_i| > e^{-a_i})$$

where $m = (m_1, \ldots, m_n) \in N^n$.

(3) Let $K_i = [k_1^{(i)}, k_2^{(i)}]$ with $k_2^{(i)} - k_1^{(i)} < 2\pi$, $1 \leqslant i \leqslant n$.

Then for all $\varepsilon > 0$ and $\varepsilon' > 0$, there exists a constant $C_{\varepsilon,\varepsilon'} > 0$, such that

$$|G_T(w)| \leqq C_{\varepsilon,\varepsilon'} |w_1|^{-\varepsilon'} \ldots |w_n|^{-\varepsilon'}$$

$(\varepsilon - k_1^{(i)} \leqq \arg \omega_i \leqq 2\pi + \varepsilon - k_2^{(i)}; 1 \leqslant i \leqslant n)$ $\quad 1 \leqq i \leqq n)$

(4)

$$\langle T, h \rangle = \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma_1 \times \ldots \times \Gamma_n} G_T(e^{-\zeta_1}, \ldots, e^{-\zeta_n}) h(\zeta) d\zeta_1 \ldots d\zeta_n$$

for all $h \in Q(\prod_{i=1}^{n} (a_i + R_+ + \sqrt{-1}K_i) : \{0\})$, hereby $\Gamma_i = \partial(a_i + R_+ + \sqrt{-1}K_i)$

Moreover

$$T(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma_1 \times \ldots \times \Gamma_n} G_T(e^{-\zeta_1}, \ldots e^{-\zeta_n}) e^{\zeta z} d\zeta_1 \ldots d\zeta_n$$

$$= \left(\frac{-1}{2\pi i}\right)^n \int_{\partial \exp(-\Gamma_1) \times \ldots \times \exp(-\Gamma_n)} G_T(w_1, \ldots w_n) w_1^{-z_1 - 1} \ldots w_n^{-z_n - 1} \, dw_1 \ldots dw_n$$

4. Transfinite diameter and the Martineau-Šeinov theorem about Laurent series of several complex variables

In this section we recall the definition of the transfinite diameter of a compact set K in the complex plane and the Martineau-Šeinov theorem about Laurent series for functions of several complex variables.

Let K be a compact set in the complex plane and put

$$V_n = \max_{\substack{z_i \in K \\ z_j \in K}} \prod_{1 \leq i < j \leq n} |z_i - z_j|$$

Then it is well known that $\tau(K) = \lim_{n\to\infty} V_n^{\frac{2}{n(n-1)}}$ exists for any compact $K \subset C$ and it is called the transfinite diameter of K (see [1] and [12]).

Some properties of the transfinite diameter of a compact set K are listed in

Proposition 2. Let K_i $(i=1,2)$ be compact subsets of C.

(1) $K_1 \subseteq K_2 \Rightarrow \tau(K_1) \leqq \tau(K_2)$

(2) $\tau(K_1) \leqq \frac{1}{2\pi}$(length of ∂K_1)

Some examples of transfinite diameters are now given (see [1] and [12]).

Example 3. If $K=\{z\in C : |z|=r\}$, then $\tau(K)=r$.
Example 4. If $K=\{z\in C : |z|=r,\ |\arg z| \leqq \alpha\}$, then $\tau(K)=r\sin\frac{\alpha}{4}$.
Example 5. If $K=[a,b]$, $a,b\in R$, then $\tau(K)=\frac{b-a}{4}$

Theorem 3. (Martineau [5] and Šeinov [11]). Suppose that $G(w)$ is holomorphic in $\prod_{j=1}^{n} (C \setminus F_j)$, where F_j is a polynomially convex compact set and $\tau(F_j)<1$ for all j $(1 \leqq j \leqq n)$. Suppose furthermore that $G(w)$ has the following Laurent expansion at infinity

$$G(w) = \sum_{v\in N^n} \frac{a_v}{w^v} \qquad (a_v \in Z).$$

Then

$$G(w) = \frac{A(w_1,\ldots,w_n)}{B_1(w_1)\ldots B_n(w_n)}$$

where $A(w_1,\ldots, w_n) \in Z[w_1,\ldots, w_n]$, $B_i(w_i) \in Z[w_i]$ and $B_i(w_i)$ are monic polynomial.

Remark 2. In theorem 3, the assumption $\tau(F_j)<1, 1\leq i\leq n$ is crucial. For instance, if n = 1 and

$$G(w) = \sum_{k=1}^{\infty} \binom{2k}{k} w^{-k} = \sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2} w^{-k}$$

$$= \sqrt{\frac{w}{w-4}} - 1,$$

then G(w) is holomorphic in the outside of the interval $[0.4]$ In view of Example 5, $\tau([0,4])=1$ and obviously G(w) is not a rational function.

5. Proof of Theorem 1.

In this section, we give the proof of Theorem 1, it is inspired by Avanissian and Gay [3].

Proof of Theorem 1.

By means of Theorem 2, there exists an analytic functional T, which is carried by L and of type {0}, such that $f(z)=\langle T_\zeta, \exp(\zeta z)\rangle=\tilde{T}(Z)$.

From the assumption, L is contained in $\prod_{i=1}^{n} \{\zeta_i : |e^{\zeta}-1|<1\}$.

Now consider the analytic functional $\check{T}$ defined as follows

$$\langle \check{T}, h\rangle = \langle T_\zeta, h(-\zeta)\rangle, \quad h \in Q(-L:\{0\}).$$

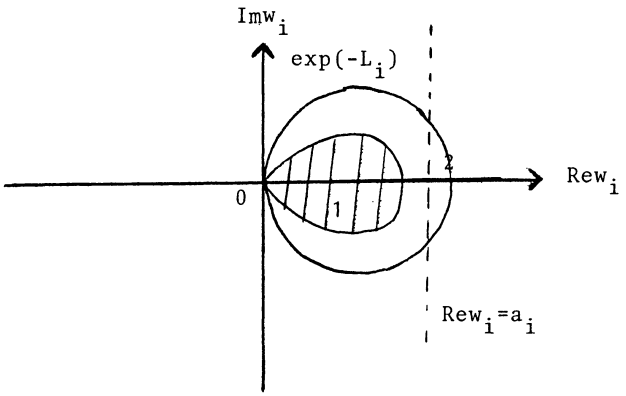
Obviously, T is carried by (-L) and of type {0}.
From Proposition 1-(2), we get :

$$G_{\check{T}}(w)=(-1)^n \sum_{m\in N^n} \check{T}(-m) w_1^{-m_1}\ldots w_n^{-m_n}$$

$$=(-1)^n \sum_{m\in N^n} T(m) w_1^{-m_1}\ldots w_n^{-m_n}$$

$$=(-1)^n \sum_{m\in N^n} f(m)w_1^{-m_1},\ldots w_n^{-m_n}$$

Remark that by means of the second assumption in Theorem 1, all $f(m)$, $m\in N^n$, belong to Z.
In virtue of Proposition 1-(1), $G_T^\vee$ is holomorphic in $\prod_{i=1}^n \{C\setminus \exp(L_i)\}$. From the assumption upon L, $\exp(L_i)$ is contained in $\{w_i\in C: |w_i-1|<1\}\cup\{0\}$.
So there exist $a_i>0$ $(1\leqq i\leqq n)$ such that $\exp(L_i)$ $\subset \{w_i\in C: |w_i-1|\leqq 1\}\cap\{w_i\in C: \mathrm{Re}w_i \leqq a_i\}$. (see Figure 3).



Call $F_i=\{w_i: |w_i-1|<1\}\cap\{w_i: \mathrm{Re}w_i\leqq a_i\}$, $1\leqq i\leqq n$.

By virtue of Proposition 2-(2) $\tau(F_i)<1$. Therefore $\tau(\exp(L_i))<1$.
Accordingly we can conclude that

$$G_T^\vee(w)=\frac{A(w_1\ldots,w_n)}{B_1(w_1)\ldots B_n(w_n)}$$

where $A(w_1,\ \ldots,\ w_n)\in Z[w_1,\ldots,\ w_n]$ and $B_i(w_i)$ are monic polynomials with integral coefficients.
The roots of $B_i(w_i)$ are algebraic integers which are contained in $\{w:\in C: |w_i-1|<1\}\cup\{0\}$ together with all their conjugate algebraic integers. But, in virtue of Proposition 1-(3), zero is not a root of $B_i(w_i)$ so that by means of C.R. Buck's lemma (See 3.2.5) in [3]), we can conclude that

$$B_i(w_i)=(w_i-1)^{m_i} \qquad (1\leqq i\leqq n).$$

Now, using the inversion formula of Proposition 1-(4),

$$f(-z)=(\tilde{\check{T}})(Z)=\left(\frac{-1}{2\pi i}\right)^{n}\int_{\partial(\exp(L_1))\times\ldots\partial(\exp(L_n))} G_T^{\vee}(w_1,\ldots,w_n)w_1^{-Z_1-1},\ldots w_n^{-Z_n-1}\, dw_1\ldots dw_n$$

$$=\left(\frac{-1}{2\pi i}\right)^{n}\int_{\partial(\exp(L_1))\times\ldots\partial(\exp(L_n))} \frac{A(w_1,\ldots,w_n)}{(w_1-1)^{m_i}\ldots(w_n-1)^{m_i}}\, w_1^{-Z_1-1}\ldots,w_n^{-Z_n-1}\, dw_1\ldots dw_n$$

whence, by means of the residue theorem

$$f(-z)=P(z_1,\ldots,z_n),\ \text{a polynomial in } z_1,\ldots,z_n.$$

But as $A(w_1,\ldots,w_n)$ belongs to $Z[w_1,\ldots,w_n]$, the coefficients of $P(z_1,\ldots,z_n)$ are rational numbers.
Hence $f(z)$ is a polynomial with rational coefficients.
Finally, we give two examples.

Example 6. Suppose that $f(z)$ is holomorphic in the right half plane $\{z\in C:\mathrm{Re}\,z>0\}$ and satisfies the following conditions :

$$|f(z)|\leqq C\,e^{\alpha|z|} \qquad (\mathrm{Re}\,z>0) \tag{1}$$

$$f(n)\in Z,\ n\in N. \tag{2}$$

Since $a(z)=\alpha|z|$ and $\Gamma=\{z\in C:\mathrm{Re}\,z>0\}$, $L=\Omega(a:\Gamma)$ is the same as in Example 1.
Therefore, if $\alpha<\log 2$, $f(z)$ is a polynomial with rational coefficients.

Example 7. Put $f(z)=\dfrac{1}{B(z,z)}=\dfrac{\Gamma(2z)}{\Gamma(z)^2}$,

where B and Γ are respectively the Bêta and Gamma functions.
This function has the following properties :

(1) $f(z)$ is holomorphic in $\{z:\mathrm{Re}\,z>-\frac{1}{2}\}$, and hence also in $\{z:\mathrm{Re}\,z>0\}$.
(2) By virtue of Stirling's formula, for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$, such that

$$|f(z)| \leq C_\varepsilon e^{(2\log 2)x+\varepsilon|y|}$$

and this for all $z=x+iy\in\{z:\mathrm{Re}\,z>0\}$.

(3) $f(n) = \binom{2n}{n}\in\mathbb{Z}$.

In this case $\Gamma=\{z\in\mathbb{C}:\mathrm{Re}\,z>0\}$ and $A(z)=2(\log 2)x$.
So $\Omega(a:\Gamma)=\{\xi\in\mathbb{R}:\xi\leq 2\log 2\}$.
Since $L=\Omega(a:\Gamma)$ is not contained in $\{\zeta\in\mathbb{C}:|e^{\zeta}-1|<1\}$ this function $f(z)$ is not a polynomial.

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