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In: Zdeněk Frolík (ed.): Proceedings of the 11th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 3. pp. [403]–410.

Persistent URL: http://dml.cz/dmlcz/701329

A GENERALIZATION OF AN EKELAND-LEBOURG THEOREM AND THE DIFFERENTIABILITY OF DISTANCE FUNCTIONS

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Ekeland and Lebourg [4] (see also [3]) proved that in a Banach space X which admits a Frechet smooth bump function under some conditions a function which is defined as pointwise infimum of a family of Frechet smooth functions is Frechet differentiable at any point of a residual subset of X. This theorem can be applied to all continuous concave functions and also to many nonconcave functions.

Our main observation is that any such "infimum function" has an "almost superdifferential" at any point and that the proof of the Ekeland-Lebourg theorem (and also of a slightly more general theorem) can be based on this property only. Using an idea from [7] we improve the Ekeland-Lebourg theorem in the case of X with a separable dual space showing that the set of nondifferentiability is even σ-porous. Using the Gregorys idea of the separable reduction (see [5], p.141) we prove that the Ekeland-Lebourg theorem holds in an arbitrary Asplund space. Note, however, that it is not known wheather there exists an Asplund space which does not admit a Frechet smooth bump function. An analogical result on the Gateaux differentiability of functions which are defined as pointwise infima is formulated. As corollaries some theorems on differentiability of distance functions are obtained.

The Ekeland-Lebourg theorem mentioned above is essentialy the following theorem.

Theorem EL. Let X be a real Banach space which admits a Frechet differentiable bump function and let $G \subset X$ be an open set. Let $\{f_\alpha, \alpha \in A\}$ be a system of functions on G for which the following conditions hold:

(i) There exists $K > 0$ such that all f_α are K-Lipschitz.

(ii) Any f_α is Frechet differentiable on G and the functions $x \to f'_\alpha(x)$, $\alpha \in A$, are equicontinuous on G.

(iii) $F(x) := \inf f_\alpha(x) > -\infty$ for $x \in G$.
Then F is Frechet differentiable at any point of a residual subset of G.

Our result which improves and generalizes the preceeding theorem is the following.

Theorem 1. Let X be a Banach space, $G \subset X$ an open set and $E \subset G$ a subset of G. Let $\{f_\alpha ; \alpha \in A\}$ be a system of functions on G such that the following conditions hold.

(i) There exists $K > 0$ such that any f_α is K-Lipschitz.

(ii) Any f_α is Frechet differentiable at any point of $G-E$ and for any $x \in G-E$ the limit
$$\lim_{h \to 0} (f_\alpha(x+hv) - f_\alpha(x))\, h^{-1}$$
is uniform with respect to $(\alpha, v) \in A \times \{v; \|v\| = 1\}$.

(iii) $F(x) := \inf f_\alpha(x) > -\infty$ for $x \in G$.

Then (a) If X is separable and E is σ-porous (resp. a first category set) then F is Frechet differentiable on G at all points except those which belong to a σ-porous set (resp. a first category set).

(b) If X is an Asplund space and $E = \emptyset$ then F is Frechet differentiable at any point of a residual subset of G.

Proof. It follows immediately from the following Lemma 1, Theorem 2 and Theorem 3.

Note 1. (i) The conditin (ii) of Theorem EL clearly implies the condition (ii) of Theorem 1 for $E = \emptyset$.

(ii) I do not know wheather the assertion (b) of Theorem 1 holds if E is an arbitrary first category set.

At first we give a brief discussion of the Dolženkoś [2] concept of σ-porous sets and then we define the notion of an almost superdifferential which is basic for our work.

Let X be a metric space. The open ball with the center $x \in X$ and the radius $r > 0$ is denoted by $B(x,r)$. Let $M \subset X$, $x \in X$, $R > 0$ be given. Then we denote the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z,r) \subset B(x,R) - M$ by $\gamma(x,R,M)$. The number
$$\limsup_{R \to 0+} \gamma(x,R,M)\, R^{-1}$$
is called the porosity of M at x.

If the porosity of M at x is positive we say that M is porous at x. A set is said to be porous if it is porous at all its points. A set is termed σ-porous if it can be written as a union of countably many porous sets. It is easy to see that any porous set is nowhere dense and therefore any σ-porous set is a first category set. Clearly any σ-porous subset of R is of Lebesgue measure zero. Using this fact, one easily notes (cf. the proof of Lemma 3.4. from [8]) that, whenever X is a Banach space, $p \in X^*$, $p \neq 0$ and $K \subset R$ is a nowhere dense set of positive Lebesgue measure, then $p^{-1}(K)$ gives an example of a first category set in X which is not σ-porous.

<u>Definition 1.</u> Let X be a Banach space and let F be a real function defined in X. We say that $g \in X^*$ is an almost superdifferential of F at $x \in X$ if

$$\limsup_{h \to 0} \left(F(x+h) - F(x) - g(h)\right) \|h\|^{-1} \leq 0 .$$

<u>Note 2.</u> (i) If $g \in X^*$ is a superdifferential of a concave function F at x, then g is an almost superdifferential of F at x.

(ii) If g is the Frechet derivative of F at x, then g is an almost superdifferential of F at x.

(iii) If we define the notion of an almost subdifferential by the natural way, then it is easy to see that F is Frechet differentiable at x iff it has at x an almost subdifferential and an almost superdifferential.

(iv) g is an almost subdifferential of F at x iff it is the ε-support of F at x (see [4]) for any $\varepsilon > 0$.

<u>Lemma 1.</u> Let $X, G, E, \{f_\alpha, \alpha \in A\}$, F be as in Theorem 1. Then F has an almost superdifferential at any $x \in G-E$.

<u>Proof.</u> Let $x \in G-E$ be fixed. Denote by $\mathcal{F}$ the filter on X with the filter basis

$$\{\{f'_\alpha(x) ; f_\alpha(x) < F(x) + \varepsilon\} ; \varepsilon > 0\} .$$

Since any function f_α is K-Lipschitz, we have $\|f'_\alpha(x)\| \leq K$, and since $\{g \in X^* ; \|g\| \leq K\}$ is w^*-compact, there exists $g \in X^*$, $\|g\| \leq K$ which is a point of accumulation of $\mathcal{F}$ in w^*-topology. We shall show that g is an almost superdifferential of F at x. Let $\omega > 0$ be given. By the condition (ii) of Theorem 1 we can choose $\delta > 0$ such that for any $\alpha \in A$ and $\|v\| = 1$

(1) $\left|\left(f_\alpha(x+hv) - f_\alpha(x)\right) h^{-1} - (v, f'_\alpha(x))\right| < \omega/2$ for any $0 < |h| < \delta$.

Let $v \in X$, $\|v\| = 1$ be fixed. Since g is a point of accumulation of $\mathcal{F}$ in w^*-topology, we can for any $\eta > 0$ choose $\alpha \in A$ such that

(2) $f_\alpha(x) < F(x) + \eta$ and $\left|(v, f'_\alpha(x) - g)\right| < \omega/2$.

Let $h \neq 0$, $|h| < \delta$ be given. By (1) we have

$$f_\alpha(x+hv) \leq f_\alpha(x) + (f'_\alpha(x), hv) + |h|\omega/2$$

and using (2) we obtain

$$F(x+hv) \leq f_\alpha(x+hv) \leq F(x) + \eta + (hv,g) + |h|\omega/2 + |h|\omega/2 .$$

Since $\eta > 0$ is an arbitrary number, we have

$$(F(x+hv) - F(x) - (hv,g))\,|h|^{-1} \leqslant \omega ,$$

which shows that g is an almost superdifferential of F at x.

The following theorem was independently proved by D.Preiss (an oral communication).

Theorem 2. Let X be a Banach space with a separable dual space and let $G \subset X$ be an open set. Let f be a Lipschitz function on G. Then the set A of the all points $x \in G$ at which f has an almost superdifferential and at which f is not Frechet differentiable is σ-porous.

Proof. Let f be K-Lipschitz on G, $K > 1$. For any $x \in A$ choose an almost superdifferential s^x. For any natural m put

$$A_m = \{ x \in A ;\ \limsup_{h \to 0} (s^x(h) - (f(x+h) - f(x)))\,\|h\|^{-1} > m^{-1} \} .$$

Clearly $A = \bigcup A_m$. Since X is separable we can choose for any m a sequence $(A_{m,k})$ such that $A_m = \bigcup A_{m,k}$ and $\|s^x - s^y\| < 1/10m$ whenever $x,y \in A_{m,k}$. We can further choose for any m,k a sequence $(A_{m,k,s,t})_{s,t=1}^{\infty}$ such that

$$A_{m,k} = \bigcup_{s,t=1}^{\infty} A_{m,k,s,t} , \quad \operatorname{diam} A_{m,k,s,t} < s^{-1} \quad \text{and}$$

$(f(x+h) - f(x) - (h,s^x))\,\|h\|^{-1} < 1/10m$ whenever $\|h\| < 1/s$ and $x \in A_{m,k,s,t}$. Now it is sufficient to show that each of the sets $A_{m,k,s,t}$ is porous. Let m,k,s,t, $x \in A_{m,k,s,t}$ and $r > 0$ be fixed. Since $x \in A_m$, we can choose $y \in B(x,r)$ such that $((y-x, s^x) - (f(y) - f(x)))\,\|y-x\|^{-1} > 1/m$.
To prove that $A_{m,k,s,t}$ is porous at x it is sufficient to show that

$$B(y, \|y-x\|/10Km) \cap A_{m,k,s,t} = \emptyset .$$

Suppose on the contrary that there exists $z \in A_{m,k,s,t}$ such that $\|y-z\| < \|y-x\|/10Km$. By the choice of y we have

$$f(y) - f(x) < (y-x, s^x) - \|y-x\|/m$$

and since f is K-Lipschitz,

$$|f(y) - f(z)| < \|y-x\|/10m .$$ Consequently we have

$$f(z) - f(x) < (y-x, s^x) - 9\|y-x\|/10m . \tag{3}$$

On the other hand, since $x,z \in A_{m,k,s,t}$, we have

$f(x) - f(z) - (x-z, s^z) < \|x-z\|/10m$ which implies

$$f(x) - f(z) < \|x-z\|/10m + (x-y, s^x) + (x-y, s^z-s^x) + (y-z, s^z). \tag{4}$$

Since $\|z-x\| < 2\|y-x\|$, $\|s^z\| \leq K$ and $\|s^z-s^x\| < 1/10m$, we obtain, adding (3) and (4) ,

$0 < 2\|y-x\|/10m - 9\|y-x\|/10m + \|y-x\|/10m + \|y-x\|/10m$

and this is a contradiction.

Theorem 3. Let X be an Asplund space, $G \subset X$ an open set and $f: G \to R$ a function which has an almost superdifferential at all points of G . Then f is Frechet differentiable at all points of a residual subset of G .

Proof. For a natural number n let D_n be the set of points $x \in G$ for which there exists a neigbourhooh U of x such that for any $y \in U$, $v \in X$, $k,h > 0$, for which $\|v\| = 1$, $y-hv \in U$, $y+kv \in U$, the inequality $|(f(y+kv) - f(y))k^{-1} + (f(y-hv) - f(y))h^{-1}| \leq 1/n$ holds. All sets D_n are obviously open and it is easy to see (using the fact that f has at any point an almost superdifferential) that f is Frechet differentiable at any point of $\bigcap D_n$. Consequently it is sufficient to prove that all D_n are dense. Suppose on the contrary that there exists n and an open set $\emptyset \neq H \subset G$ such that $H \cap D_n = \emptyset$. Using the Gregorys method of the separable reduction ([5] , p. 141) it is easy to construct a separable subspace $\tilde{X}$ such that $\tilde{H} := \tilde{X} \cap H \neq \emptyset$ and $\tilde{H} \cap \tilde{D}_n = \emptyset$, where $\tilde{D}_n$ is defined for $\tilde{f} := f/\tilde{X}$ and $\tilde{G} = G \cap \tilde{X}$ in the same way as D_n is defined for f and G . In fact, we define inductively an increasing sequence (Y_i) of separable subspaces of X . First choose a separable subspace Y_1 , $Y_1 \cap H \neq \emptyset$. Now given a subspace Y_i define a subspace Y_{i+1} as follows. Choose in $Y_i \cap H$ a countable dense subset T . For any $t \in T$ choose sequences y_j^t , v_j^t , k_j^t , h_j^t , $j=1,2,\dots$, such that $y_j^t - h_j^t v_j^t \in B(t,1/j)$, $y_j^t + k_j^t v_j^t \in B(t,1/j)$, $\|v_j^t\| = 1$ and

$$|(f(y_j^t + k_j^t v_j^t) - f(y_j^t))/k_j^t + (f(y_j^t - h_j^t v_j^t) - f(y_j^t))/h_j^t| > \\ > 1/n .$$

Then let Y_{i+1} denote the closed subspace spanned by Y_i and all points of the form y_j^t , v_j^t . Now it is sufficient to put $\tilde{X} = \overline{\bigcup Y_i}$. For a natural number m let A_m denote the set of all $x \in \tilde{X}$ such that for any $v \in \tilde{X}$, $\|v\| = 1$ and $0 < k < 1/m$, $0 < h < 1/m$, the inequality

$|(f(x+kv) - f(x))/k + (f(x-hv) - f(x))/h| > 1/n$ does not hold.

From the continuity of f follows that all A_m are closed.

Let M denote the set of all points $x \in \tilde{H}$ at which $\tilde{f}$ is Frechet differentiable. Obviously, $\tilde{f}$ has an almost superdifferential at any point of $\tilde{H}$, and since X is Asplund, $\tilde{X}$ has a separable dual space. Therefore by Theorem 2 M is a residual subset of $\tilde{H}$. Clearly $M \subset \bigcup A_m$ and therefore there exists an index m such that A_m contains a nonempty open subset $V \subset \tilde{H}$. But this is a contradiction since clearly $V \subset \tilde{D}_n$ and we know that $\tilde{H} \cap \tilde{D}_n = \emptyset$.

Since it is not difficult to prove that for an arbitrary continuous function f any point of $\bigcap D_n$ is a point of the Frechet differentiability, we have proved in fact the following assertion.

Proposition 1. Let X be a Banach space, $G \subset X$ an open set and f a continuous function on G. If for any separable subspace $\tilde{X}$, $\tilde{X} \cap G \neq \emptyset$, the function $\tilde{f} := f/\tilde{X}$ is Frechet differentiable at all points of a residual subset of $\tilde{X} \cap G$, then f is Frechet differentiable at all points of a residual subset of G.

Note 3. If we writte in the preceeding proposition "dense" instead of "residual", the new proposition also holds [6].

Let X be a Banach space and let $\emptyset \neq M \subset X$ be an arbitrary set. Then for the distance function d_M we have

$$d_M(x) = \inf \{ f_\alpha(x) \; ; \; \alpha \in M \} ,$$

where $f_\alpha(x) = \| x - \alpha \|$. If G is an open nonempty subset of $X-M$, then there exists a bounded set $A \subset M$ such that $d_A(x) = d_M(x)$ for $x \in G$. The functions f_α are 1-Lipschitz and if X has uniformly Frechet differentiable norm (a.e. if the limit $\lim_{t \to 0} (\|x+tv\| - \|x\|)/t$ is uniform with respect to $(x,v) \in S_1 \times S_1$, where $S_1 = \{ y \in X \; ; \; \|y\| = 1 \}$), then it is easy to see that for the system $\{ f_\alpha \; ; \alpha \in A \}$ the condition (ii) of Theorem 1 is satisfied. Therefore Theorem 1 yields the following propositions.

Corollary 1. Suppose that X is a Banach space with a separable dual and X has uniformly Frechet differentiable norm. Then any distance function d_M is Frechet differentiable at all points of the set $X - \overline{M}$ except those which belong to a σ-porous set.

Corollary 2. Suppose that X has uniformly Frechet differentiable norm. Then any distance function in X is Frechet differentiable at any point of a residual subset of X.

Theorem EL has the following "Gateaux" analogy.

Theorem 4. Let X be a Banach space, $G \subset X$ an open set and $E \subset G$ a subset of G. Let $\{ f_\alpha \; ; \alpha \in A \}$ be a system of real functions on G such that the following conditions hold.

(i) There exists $K > 0$ such that any f_α is K-Lipschitz.

(ii) At any $x \in G-E$ any function f_α is Gateaux differentiable and for any $x \in G-E$ and $v \in X$ the limit $\lim_{t \to 0} (f_\alpha(x+tv) - f_\alpha(x))/t$ is uniform with respect to $\alpha \in A$.

(iii) $F(x) := \inf f_\alpha(x) > -\infty$ for any $x \in G$.

Then (a) The one-sided directional derivative $D_v F(x)$ exists for any $x \in G-E$ and $v \in X$, and for any fixed $x \in G-E$ the function $v \to D_v F(x)$ is K-Lipschitz concave function on X.

(b) If on X exists a Lipschitz bump function which is uniformly differentiable at any direction (a.e. supp $f \neq \emptyset$ is a bounded set and for any $v \in X$ the limit $\lim_{t \to 0} (f(x+tv) - f(x))/t$ is uniform with respect to $x \in X$), then F is Gateaux differentiable at all points of $G-E$ except at those which belong to a first category set.

Note 4. The proof of Theorem 4 will be given in a subsequent article. The proof of (a) is straitforward and essentially known. The Asplunds method([1]) shows that (a) implies (b).

Corollary 3. Suppose that X has uniformly Gateaux differentiable norm. Then any distance function in X is Gateaux differentiable at any point of a residual subset of X.

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