P. Goddard

Infinite dimensional Lie algebras: representations and applications

In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 9. pp. 73–107.

Persistent URL: http://dml.cz/dmlcz/701391

Terms of use:

© Circolo Matematico di Palermo, 1985

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

INFINITE DIMENSIONAL LIE ALGEBRAS: REPRESENTATIONS AND APPLICATIONS

P. Goddard

1. Introduction

In these notes, based on four lectures given at the Srni Winter School, certain infinite dimensional Lie algebras, the groups to which they correspond, their representations and some of their applications in theoretical physics are described. The topics and treatment were chosen so as to compliment those covered by David Olive in his lectures at Srni (OLIVE).

The current interests in these algebras in mathematics and theoretical physics both date from the latter half of the 1960's. In a relatively short period it has become apparent that they provide a relationship between apparently disconnected, or tenuously related areas, such as: sporadic simple groups in finite group theory; the theory of modular forms; "completely integrable" dynamical systems; (possibly) gauge field theory; string theories of elementary particles; conformally invariant field theories; and the theory of critical phenomena in two-dimensional statistical systems. Each of these subjects has established connections with the next in the list, forming the links in chain joining disparate areas of mathematics and physics. The common feature underlying these connections seems to be the occurrence of infinite dimensional algebras.

By way of introduction, we give an example of an infinite dimensional Lie algebra and one of the ways it occurs in theoretical physics. In some ways the simplest of the algebras we shall discuss is the Virasoro algebra (VIRASORO),

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{1}{12} c\, m(m^2-1)\,\delta_{m,-n}\,, \tag{1.1}$$

$m, n \in \mathbb{Z}$, where c is a central element,

$$[L_m, c] = 0 \ . \tag{1.2}$$

In any irreducible representation c is effectively a number and c will also be used to denote this number. We shall be concerned with unitary representations, that is representations in which the hermiticity condition

$$L_n^\dagger = L_{-n} \tag{1.3}$$

holds. In particular we shall discuss highest weight representations, that is ones in which a basis for the representation space can be generated from a highest weight state $|h\rangle$, i.e. one satisfying

$$L_n|h\rangle = 0 \ , \quad n > 0 \ , \tag{1.4a}$$

and

$$L_0|h\rangle = h|h\rangle \ , \tag{1.4b}$$

by application of operators L_{-n} , $n > 0$. Such representations are characterized by two numbers (h,c) .

Two dimensional conformal quantum field theories have two such commuting Virasoro algebras $\{L_n\}$ and $\{\bar{L}_n\}$ (corresponding to analytic transformations on $z = x + iy$ and $\bar{z} = x - iy$, respectively). The representations of the two Virasoro algebras possess a common value of c , characteristic of the field theory concerned and describing the anomalous breaking of conformal invariance. Basic fields, and the states they create from the vacuum, have values of h and $\bar{h}$ (the eigenvalues of L_0 and $\bar{L}_0$, respectively) associated with them: $h + \bar{h}$ gives the scaling dimension of the field and $h - \bar{h}$ its spin (POLYAKOV; BELAVIN, POLYAKOV and ZAMOLODCHIKOV). Critical exponents in associated statistical mechanical systems are simple linear combinations of these scaling dimensions. Certain important statistical models have their critical behaviour described by unitary quantum field theories, and so unitary representations of the Virasoro algebras $\{L_n\}$ and $\{\bar{L}_n\}$.

Not all values of the pair (h,c) correspond to unitary representations. There is a continuum $(c \geq 1)$ of values of c and a discrete series of values in the interval $0 \leq c < 1$. For each value of c in the discrete series, there are only finitely many possibilities for h, and hence for critical exponents. This provides a sort of group theoretic explanation of the rational critical exponents found in these models (FRIEDAN, QUI and SHENKER).

The second section of these notes defines the algebras, starting from the groups to which they correspond and discusses their central extensions. The third section gives some elements of their representation theory. The fourth section describes some instances of their occurrence in field theory.

2. Algebras, Groups and Central Extensions

We introduce the infinite dimensional Lie algebras we are to consider by describing first the corresponding groups.

(a) Affine Kac-Moody algebras

Consider an ordinary finite-dimensional Lie group G. We construct a new and much larger group $\mathcal{G}$ by taking suitably smooth maps from the circle $S^1 \to G$. We shall represent S^1 as the unit circle in the complex plane,

$$S^1 = \{z \in C : |z| = 1\} , \tag{2.1}$$

writing $z = e^{i\phi}$, $0 \leq \phi < 2\pi$, and denote a typical map as

$$z \to g(z) \in G . \tag{2.2}$$

Given two such maps $g_1, g_2 : S^1 \to G$ we can define an obvious group structure on $\mathcal{G}$ by pointwise multiplication, i.e. the product of g_1 and g_2 is $g_1 \cdot g_2$ where

$$g_1 \cdot g_2(z) = g_1(z) g_2(z) . \tag{2.3}$$

Clearly this operation makes $\mathcal{G}$ into an infinite dimensional Lie group. It is called the loop group of G.

Let us now construct the Lie algebra of $\mathcal{G}$. Suppose T^a, $1 \leq a \leq d$, is a basis for the Lie algebra of G, with

$$[T^a, T^b] = if^{ab}{}_c T^c , \tag{2.4}$$

so that

$$g = \exp[-T^a \theta_a] \tag{2.5}$$

is a typical element of G; θ_a, $1 \leq a \leq d$ are parameters for the Lie group. Maps $S^1 \to G$ can be described by d functions $\theta_a(z)$ defoned on S^1. Thus

$$\mathcal{G} = \{g(z) = \exp[-T^a\theta_a(z)]\} . \tag{2.6}$$

For elements near the identity,

$$g \cong 1 - iT^a\theta_a , \tag{2.7}$$

$$g(z) \cong 1 - iT^a\theta_a(z) . \tag{2.8}$$

Making a Fourier (or Laurent) expansion of $\theta_a(z)$,

$$\theta_a(z) = \sum_{h=-\infty}^{\infty} \theta_a^{-n} z^n \tag{2.9}$$

we see that, if we introduce generators

$$T_n^a = T^a z^n \tag{2.10}$$

for $\mathcal{G}$, the θ_a^n, $1 \leq a \leq d$, $n \in \mathbb{Z}$, provide an infinite set of parameters for $\mathcal{G}$, with

$$g(z) \cong 1 - i \sum_{n,a} T_{-n}^a \theta_a^n \tag{2.11}$$

From eq.(2.11) we see that $\mathcal{G}$ has the Lie algebra

$$[T_m^a, T_n^b] = if^{ab}{}_c T_{m+n}^c , \tag{2.12}$$

This is the (untwisted) affine Kac-Moody algebra associated with the Lie algebra (2.4) (KAC 1968; MOODY), the algebra of the group of maps $S^1 \to G$.

Note that the operators T_o^a, $1 \leq a \leq d$, generate a subalgebra isomorphic to the Lie algebra of G. It corresponds to the subgroup of $\mathcal{G}$ defined by constant maps $S^1 \to G$, which is isomorphic to G, of course.

Suppose G is a compact group and $\{T^a\}$ a basis of hermitian generators,

$$T^{a\dagger} = T^a . \tag{2.13}$$

For z on the unit circle S^1, $z^* = z^{-1}$, so that

$$T_n^{a\dagger} = T_{-n}^a \ . \tag{2.14}$$

A representation of (2.12) satisfying this hermiticity condition will be called unitary as, for real θ_a^n, $g(z)$ will then be unitary for $|z| = 1$.

(b) The Virasoro algebra

Consider smooth one-to-one maps $S^1 \to S^1$ under composition:

$$\gamma_1 \circ \gamma_2(z) = \gamma_1(\gamma_2(z)) \ . \tag{2.15}$$

These form a group which we will denote by $\mathcal{V}$. Notice that, although we can regard S^1 as a Lie group U(1), this infinite-dimensional group $\mathcal{V}$ is different from the loop group $\mathcal{G}$ of maps $S^1 \to U(1)$ because the multiplication law is different in the two cases, being composition in the former case and pointwise multiplication in the latter. In particular $\mathcal{V}$ is non-abelian, but, because U(1) is abelian, $\mathcal{G}$ is also.

To calculate the Lie algebra of $\mathcal{V}$ consider the faithful representations defined by its action on functions $S^1 \to V$, where V is some vector space, the action being defined by

$$D_\gamma f(z) = f(\gamma^{-1}(z)) \tag{2.16}$$

for a function $f : S^1 \to V$. For an element

$$\gamma(z) = ze^{-i\varepsilon(z)} \ , \tag{2.17}$$

close to the identity in $\mathcal{V}$,

$$\gamma^{-1}(z) \cong z + iz\varepsilon(z) \tag{2.18}$$

so that

$$D_\gamma f(z) \cong f(z) + i\varepsilon(z)\, z \frac{d}{dz} f(z) \ . \tag{2.19}$$

Making a Fourier (or Laurent) expansion,

$$\varepsilon(z) = \sum_{n=-\infty}^{\infty} \varepsilon_{-n} z^n , \tag{2.20}$$

we are led to introduce generators

$$L_n = - z^{n+1} \frac{d}{dz} , \quad n \in \mathbb{Z} , \tag{2.21}$$

which satisfy the Lie algebra

$$[L_m, L_n] = (m-n) L_{m+n} . \tag{2.22}$$

A representation of this algebra will be called unitary if the hermiticity condition

$$L_n^\dagger = L_{-n} \tag{2.23}$$

holds.

An important finite-dimensional subgroup of $\mathcal{V}$ consists of Möbius transformations

$$z \to \gamma(z) = \frac{az+b}{b^* z+a^*} \tag{2.24}$$

which clearly maps $S^1 \to S^1$ and, provided that $|a|^2 > |b|^2$, this map $\gamma(z)$ covers S^1 once positively as z goes round S^1 positively. We can then rescale a,b so that $|a|^2 - |b|^2 = 1$; then

$$v = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \in SU(1,1) . \tag{2.25}$$

The generators of this SU(1,1) subgroup are easily seen to be $\{L_{-1}, L_o, L_1\}$. Actually there is an infinity of SU(1,1) subgroups which can be obtained by doing Möbius transformations on z^n :

$$z \to \gamma(z) = \left(\frac{az^n+b}{b^* z^n+a^*}\right)^{1/n} \tag{2.26}$$

The generators of this subgroup are $\{\frac{1}{n} L_{-n}, L_o, \frac{1}{n} L_n\}$.

(c) <u>the interrelation between Virasoro and Kac-Moody algebras</u>.

Virasoro and Kac-Moody algebras can naturally be considered as interrelating. To see this, consider a faithful representation of the group G in the vector space V . A map $\gamma : S^1 \to S^1$ in $\mathcal{V}$ is represented on functions as in eq.(2.16)

$$\gamma v(z) = v(\gamma^{-1}(z)) , \tag{2.27}$$

and the action of $g : S^1 \to G$ is

$$gv(z) = g(z)v(z) . \tag{2.28}$$

With this action, $\mathcal{V}$, $\mathcal{G}$ form the factors of a semidirect product; we define

$$(\gamma,g) = \gamma g \tag{2.29}$$

so that

$$(\gamma,g)v(z) = g(\gamma^{-1}(z))v(\gamma^{-1}(z)) . \tag{2.30}$$

Then

$$(\gamma_1,g_1)(\gamma_2,g_2) = (\gamma_1 \circ g_2, (g_1 \circ \gamma_2) \cdot g_1) . \tag{2.31}$$

Since the generators of $\mathcal{V}$ and $\mathcal{G}$ are represented by operators

$$L_n = - z^{n+1} \frac{d}{dz} , \quad T^a_n = z^n T^a , \tag{2.32}$$

the semidirect product has the algebra,

$$[T^a_m, T^b_n] = if^{ab}{}_c T^c_{m+n} , \tag{2.33a}$$

$$[L_m, T^a_n] = -nT^a_{m+n} , \tag{2.33b}$$

$$[L_m, L_n] = (m-n)L_{m+n} . \tag{2.33c}$$

The infinite dimensional groups $\mathcal{V}$ and $\mathcal{G}$ may be thought of as the groups of general coordinate transformations and gauge transformations, respectively, on the one-dimensional space S^1.

(d) <u>central extensions</u>

If in a classical theory one has a group of transformations G with Lie algebra

$$[T^a, T^b] = if^{ab}{}_c T^c , \tag{2.34}$$

the corresponding generators will, under suitable circumstances, satisfy the Poisson bracket relations,

$$[T^a, T^b]_{P.B.} = f^{ab}{}_c T^c . \tag{2.35}$$

Following Dirac's quantisation procedure, the corresponding quantum commutator is

$$[T^a, T^b] = i\hbar f^{ab}{}_c T^c + O(\hbar^2) . \tag{2.36}$$

The unspecified terms of order $O(\hbar^2)$ have to be chosen so that the Jacobi identities are satisfied. The simplest possibility is that the $O(\hbar^2)$ terms are c-numbers, multiples of the identity. What we have then is a central extension of the original Lie algebra (2.34), rescaled by a factor $\hbar$.

A central extension of (2.34) is an algebra with basis T^a, $1 \leq a \leq d$, C^j, $1 \leq j \leq M$, of the form

$$[T^a, T^b] = if^{ab}{}_c T^c + k^{ab}{}_j C^j , \tag{2.37}$$

$$[T^a, C^j] = [C^i, C^j] = 0 . \tag{2.38}$$

The additional structure constants $k^{ab}{}_j$ are constrained by Jacobi identities:

$$[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0 \tag{2.39}$$

leading to

$$f^{ab}{}_c k^{ec}{}_j + f^{bc}{}_e k^{ea}{}_j + f^{ca}{}_e k^{eb}{}_j = 0 , \tag{2.40a}$$

together with

$$k^{ab}{}_j = -k^{ba}{}_j . \tag{2.40b}$$

Eqs.(2.40) constitute a set of linear equations and the dimension of the space of solutions gives the number of independent central extensions. However some of these are in a sense trivial because they can be removed by redefinition of the generators

$$T^a \rightarrow T^a - \xi^a{}_j C^j \tag{2.41}$$

under which

$$k^{ab}{}_j \to k^{ab}{}_j + if^{ab}{}_c \xi^c{}_j \tag{2.42}$$

In the quantum mechanical context, such a redefinition is by a term of order $\hbar$, which is insignificant in the classical limit. Thus one really wants to know the space of solutions of eq.(2.40) modulo transformations of the form (2.42).

For semi-simple Lie algebras all solutions to eqs. (2.40) can be removed by (2.42) so that all central extensions are essentially trivial. In the case of both the affine Kac-Moody algebra associated with a simple Lie group and the Virasoro algebra, this quotient spacer is one-dimensional:

$$[T^a_m, T^b_n] = if^{ab}{}_c T^c_{m+n} + km\delta_{m,-n}\delta^{ab} , \tag{2.43}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n} . \tag{2.44}$$

The central elements k, c may be thought of as numbers in any irreducible representation.

3. Representation Theory

(a) Kac-Moody algebras.

We now consider irreducible unitary representations of the Kac-Moody algebra (2.43), i.e. representations in a complex space with a positive definitive inner product, with respect to which

$$T^{a\dagger}_n = T^a_{-n} . \tag{3.1}$$

To discuss such representations it is convenient to use a Cartan-Weyl basis. Such a basis for the original simple Lie algebra (2.3.4) takes the form

$$[H^a, H^b] = 0 , \tag{3.2a}$$

$$[H^a, E^\alpha] = \alpha^a E^\alpha , \tag{3.2b}$$

$$\begin{aligned}[E^\alpha, E^\beta] &= \varepsilon(\alpha,\beta)E^{\alpha+\beta}, \text{ if } \alpha+\beta \text{ is a root} ,\\ &= \frac{2}{\alpha^2}\alpha.H, \text{ if } \alpha = -\beta ,\\ &= 0 \quad \text{otherwise.}\end{aligned} \tag{3.2c}$$

The H^a, $1 \leq a \leq r$, form a Cartan subalgebra, r is the rank, and the r-dimensional vectors α the roots of the Lie algebra. Written in this basis, the algebra (2.43) takes the form

$$[H_m^a, H_n^b] = km\delta^{ab}\delta_{m,-n} , \tag{3.3a}$$

$$[H_m^a, E_n^\alpha] = \alpha^a E_{m+n}^\alpha , \tag{3.3b}$$

$$\begin{aligned} [E_m^\alpha, E_n^\beta] &= \varepsilon(\alpha,\beta) E_{m+n}^{\alpha+\beta}, \text{ if } \alpha+\beta \text{ is a root} , \\ &= \frac{2}{\alpha^2}(\alpha.H_{m+n} + km\delta_{m,-n}), \text{ if } \alpha = -\beta , \\ &= 0 \quad \text{otherwise} . \end{aligned} \tag{3.3c}$$

We shall consider specifically highest weight representations, that is ones that can be built up from vacuum vectors ψ satisfying

$$E_n^\alpha \psi = H_n^a \psi = 0 , \quad n > 0 . \tag{3.4}$$

The space of solutions to these equations clearly is a representation space of G, invariant under E_o^α, H_o^a. Since different subspaces of it invariant under G will generate disjoint invariant subspaces for the whole algebra. Thus, for an irreducible representation of a Kac-Moody algebra, the vacuum space (3.4) must be irreducible.

We can take a basis for the vacuum space (3.4) consisting of vectors ψ_λ which are simultaneous eigenvectors of the Cartan subalgebra H_o^a,

$$H_o^a \psi_\lambda = \lambda^a \psi_\lambda , \tag{3.5}$$

so that the vectors λ are weights for G. The theory of unitary representations of simple groups tells us that

$$2\frac{\alpha.\lambda}{\alpha^2} \in \mathbb{Z} \tag{3.6}$$

because it is the eigenvalue of $2I_o$, where

$$I_o = \alpha.H_o/\alpha^2 , \quad I_\pm = E_o^{\pm\alpha} , \tag{3.7}$$

form an SU(2) subalgebra.

We now consider what are the restrictions on the possibilities for the vacuum representation and the values of the central element k. To this end we calculate

$$\begin{aligned}||E_{-1}^{-\alpha}\psi_\lambda||^2 &= \langle\psi_\lambda, E_1^{\alpha}E_{-1}^{-\alpha}\psi_\lambda\rangle \\ &= \langle\psi_\lambda, [E_1^{\alpha}, E_{-1}^{-\alpha}]\psi_\lambda\rangle \\ &= \frac{2}{\alpha^2}(\alpha.\lambda + k)||\psi_\lambda||^2 \ . \end{aligned} \tag{3.8}$$

Since $||\psi_\lambda||^2 > 0$ and $||E_{-1}^{-\alpha}\psi_\lambda||^2 \geq 0$, we find

$$\frac{2k}{\alpha^2} \geq -\frac{2\alpha.\lambda}{\alpha^2} \tag{3.9}$$

Applying this with $-\alpha$ replacing α as well, we obtain

$$\frac{2k}{\alpha^2} \geq \left|\frac{2\alpha.\lambda}{\alpha^2}\right| \tag{3.10}$$

for each root α of G . This shows that $k \geq 0$ for a highest weight representation. But further we can show that the left hand side of (3.10) is an integer; consider

$$\begin{aligned}||(E_{-1}^{-\alpha})^N\psi_\lambda||^2 &= \langle(E_{-1}^{-\alpha})^{N-1}\psi_\lambda,\ E_1^{\alpha}(E_{-1}^{-\alpha})^N\psi_\lambda\rangle \\ &= N\left(\frac{2\alpha.\lambda}{\alpha^2} + \frac{2k}{\alpha^2} - N+1\right)||(E_{-1}^{-\alpha})^{N-1}\psi_\lambda||^2 \ . \end{aligned} \tag{3.11}$$

As in the usual arguments of angular momentum theory $(E_{-1}^{-\alpha})^N\psi_\lambda$ must vanish for some positive integer N , or otherwise for sufficiently large N it would have a negative norm. Thus

$$\frac{2\alpha.\lambda}{\alpha^2} + \frac{2k}{\alpha^2} \tag{3.12}$$

must be an integral and, because of (3.6),

$$\frac{2k}{\alpha^2} \in \mathbb{Z} \ . \tag{3.13}$$

For a simple Lie algebra, there are at most two values of α^2 ; if there are both long roots and short roots let ψ be a long root. Then, as $\psi^2/\alpha^2 \in \mathbb{Z}$, where α is any other root, the constraint (3.13) is satisfied if it holds for $\alpha = \psi$. The non-negative integer $2k/\psi^2$ is called the level of the representation. The possible vacuum representations at this level satisfy the inequality

(3.10) for all weights λ of the representation and roots α of G. The representation of the Kac-Moody algebra is fully determined by the level and the vacuum representation. For any non-negative integer $2k/\psi^2$ and vacuum representation satisfying (3.10) a representation of the Kac-Moody algebra exists.

(If we take a particular basis of simple roots for G, we can label the vacuum representation by its highest weight λ_o, with respect to this basis. It is then enough to apply the constraint

$$k \geq \alpha.\lambda_o \tag{3.14}$$

for all positive roots α ; the representation is determined by k and λ_o.)

(b) Virasoro algebra

We shall now consider the unitary representations of the algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \tag{3.15}$$

We shall seek highest weight representations again, this defined as irreducible representations generated from a highest weight vector ψ_o satisfying

$$L_n\psi_o = 0 , \quad n > 0 \tag{3.16a}$$

$$L_o\psi_o = h\psi_o . \tag{3.16b}$$

It is easy to see that for such representations

$$h \geq 0 , c \geq 0 , \tag{3.17}$$

because

$$\begin{aligned} ||L_{-n}\psi_o||^2 &= \langle\psi_o, L_nL_{-n}\psi_o\rangle \\ &= \langle\psi_o, [L_n, L_{-n}]\psi_o\rangle \\ &= \{2nh + \frac{c}{12}n(n^2-1)\}||\psi_o||^2 \end{aligned} \tag{3.18}$$

and taking $n = 1$ we have $h \geq 0$ and n large $c \geq 0$. However unitary representations do not exist for all values of c and h satisfying (3.17).

A highest weight unitary irreducible representation will be spanned by vectors of the form

$$L_{-1}^{n_1} L_{-2}^{n_2} \ldots L_{-r}^{n_r} \psi_o \tag{3.19}$$

Each of these is an eigenvector of the hermitian operator L_o with eigenvalue

$$h + \sum_j j n_j \equiv h+N \text{ , say .} \tag{3.20}$$

For different values of N, these vectors are orthogonal. For a given value of N, the matrix of scalar products between the various vectors (3.19) can be calculated as a function of (c,h) using eqs. (3.16) and the hermiticity condition $L_n^\dagger = L_{-n}$ (up to a factor of $\|\psi_o\|^2$, which we can take to be 1).

For the first three values of N this calculation proceeds as follows

$N = 0$
$$\langle\psi_o,\psi_o\rangle = 1 \text{ .} \tag{3.21}$$

$N = 1$
$$\begin{aligned} \langle L_{-1}\psi_o, L_{-1}\psi_o\rangle &= \langle\psi_o, L_1 L_{-1}\psi_o\rangle \\ &= 2h \text{ .} \end{aligned} \tag{3.22}$$

$N = 2$ Set $\psi_2 = L_{-2}\psi_o$, $\psi_1 = L_{-1}^2\psi_o$, then

$$\begin{pmatrix} \langle\psi_2,\psi_2\rangle & \langle\psi_1,\psi_2\rangle \\ \langle\psi_2,\psi_1\rangle & \langle\psi_1,\psi_1\rangle \end{pmatrix} = \begin{pmatrix} 4h + \frac{1}{2}c & 6h \\ 6h & 8h^2 + 4h \end{pmatrix} . \tag{3.23}$$

The determinant of this matrix is

$$4h\{\tfrac{1}{2}c + (c-5)h + 8h^2\} \text{ .} \tag{3.24}$$

If the representation is unitary, each of these matrices must be positive semidefinite. Conversely, if the matrices of scalar products of the vectors (3.19) are positive for each N, they can be used to define an inner product on the space with basis (3.19) with respect to which $L_n^\dagger = L_{-n}$. (If any of the matrices are positive semidefinite, but actually positive definite, i.e. has some zero eigenvalues, the corresponding space can be consistently converted into a positive definite space by taking the quotient by the subspace of null states.)

This problem was analysed by FRIEDAN, QIU and SHENKER. The matrices corresponding to $N \leq 2$ will be positive semidefinite if $h \geq 0$ and

$$\tfrac{1}{2}c + (c-5)h + 8h^2 \geq 0 \tag{3.25}$$

Further we know we shall need $c \geq 0$ from (3.18). These conditions leave us with the region of the first quadrant of the (c,h) plane on and outside the curve given by equality in (3.25); this is illustrated in Fig. 1.

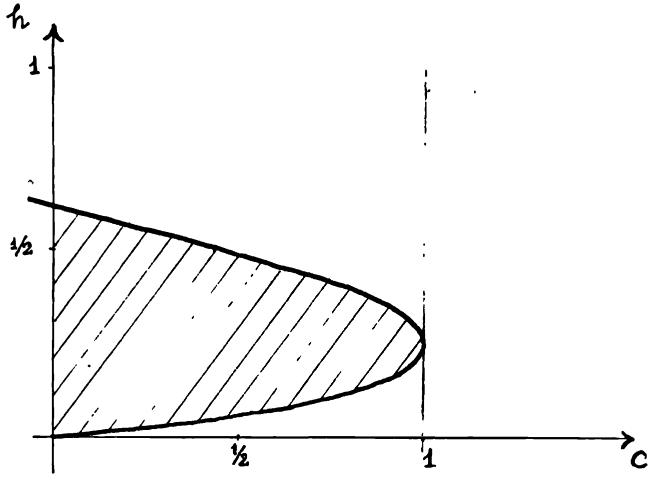

Fig. 1 : The interior of the shadedregion is forbidden.

To progress beyond the first few levels one needs a general formula and FRIEDAN, QIU and SHENKER exploited the formula given by KAC (1978) for the determinant of the matrix $M_N(c,h)$ of the scalar products of the vectors for c given value of N. The number $\pi(N)$ of such vectors grows quickly with N; it is given by the partition function

$$\prod_{n=1}^{\infty} (1-q^n)^{-1} = \sum_{N=0}^{\infty} \pi(N)\, q^N \tag{3.26}$$

The formula of Kac for this $\pi(N) \times \pi(N)$ determinant is

$$\det M_N(c,h) = \prod_{k=1}^{N} \eta_k(c,h)^{\pi(N-K)} \tag{3.27}$$

Here

$$\eta_k(c,h) = \prod_{pq=k} [h - h_{p,q}(c)] \ , \tag{3.28}$$

where p and q range over the positive integers and

$$h_{p,q}(c) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \tag{3.29}$$

with m a parameter defined in terms of c by

$$c = 1 - \frac{6}{m(m+1)} \ . \tag{3.30}$$

The formula (3.27) was proved by FEIGIN and FUCHS and more recently a proof using techniques from relativistic string theory has been given by THORN.

The determinants of eq.(3.27) are easily seen to be positive if $c > 1$ and $h \geq 0$. For large h the matrices $M_N(c,h)$ are manifestly positive definite. Hence it follows that they are positive semi-definite throughout $c \geq 1$, $h \geq 0$. The only remaining region is $0 \leq c < 1$, and one might be tempted to try to reject all of this, except for $c = 0$, $h = 0$ which corresponds to the trivial representation. But there are simple and well-known representations for $c = \frac{1}{2}$, with $h = 0, \frac{1}{16}, \frac{1}{2}$. (These occur in the RAMOND and NEVEU-SCHWARZ dual models.) Actually the careful analysis of FRIEDAN, QIU and SHENKER showed that this is the first non-trivial term in an infinite sequence of values of c for which unitary representations might exist:

$$c = 1 - \frac{6}{m(m+1)} \ , \quad m = 2,3,\ldots \ , \tag{3.31}$$

For a given c in this sequence, the possible values of h are given by eq.(3.29) with $1 \leq p \leq m-1$; $q \leq q \leq p$. The first few possibilities thus noted are:

$c = 0 \quad h = 0$

$c = \frac{1}{2} \quad h = 0, \frac{1}{16}, \frac{1}{2}$

$$c = \frac{7}{10} \qquad h = 0,\ \frac{3}{80},\ \frac{1}{10},\ \frac{7}{16},\ \frac{3}{5},\ \frac{3}{2}$$

$$c = \frac{4}{5} \qquad h = 0,\ \frac{1}{40},\ \frac{1}{15},\ \frac{1}{8},\ \frac{2}{5},\ \frac{21}{40},\ \frac{2}{3},\ \frac{7}{5},\ \frac{13}{8},\ 3\ . \tag{3.32}$$

All of these representations have now been found explicitly (GODDARD and OLIVE; GODDARD, KENT and OLIVE) and the methods used will be described in section 4 of this article and in the contribution of Adrian Kent to this meeting (KENT).

4. Current Algebras and σ-Models

We shall now look at some instances where these infinite dimensional Lie algebras occur in theories studied in physics. We only have time to look at some simple cases. Important and topic areas such as the rôle they play in theories of relativistic strings will have to be omitted, although this area in particular has held a pivotal position in the interchange of ideas and results between the mathematical and physical studies of infinite dimensional Lie algebras.

(a) current algebra representation of Kac-Moody algebras

To provide our first instance of a Kac-Moody algebra occurring in a physical theory, we consider a rather trivial example: a free real fermion field moving in one space and one time dimension. To describe such a field, we use a real representation of the γ-matrices in two dimensions:

$$\gamma^o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad \gamma^5 \equiv \gamma^o\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.1}$$

and field ψ with two real spinor components which, following WITTEN, we write as

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \tag{4.2}$$

for reasons that will become apparent. The system is described by the action

$$S = \frac{1}{2} \int d^2x\ \bar{\psi} i \gamma^\mu \partial_\mu \psi \tag{4.3}$$

where $\bar{\psi} = \psi^T\gamma^o$. This leads to the Dirac equation of motion

$$\gamma^{\mu}\partial_{\mu}\psi = 0 \qquad (4.4)$$

where, as usual, $\partial_{\mu} = \partial/\partial x^{\mu}$, $x^{o} = t$, $x' = x$. Eq.(4.4), when written out in components, is equivalent to

$$(\partial_{o} + \partial_{1})\psi_{-} = (\partial_{o} - \partial_{1})\psi_{+} = 0 \ . \qquad (4.5)$$

These Weyl equations say that ψ_{+} and ψ_{-} are only functions of $t+x$ and $t-x$ respectively:

$$\psi_{+} \equiv \psi_{+}(t+x) \ , \ \psi_{-} \equiv \psi(t-x) \qquad (4.6)$$

Each of ψ_{+} and ψ_{-} is a Weyl (i.e. an eigenvector of γ_{s}) and Majorana (i.e. real) spinor. (Such spinors only exist in space-time dimensions $8n+2$, where n is an integer; see GLIOZZI, OLIVE and SCHERK.)

For the moment let us just consider one of the two Weyl components of ψ, $\psi_{+}(t+x)$, say. Independently, for each of the Weyl components, we have canonical anticommutation relations

$$\{\psi(x),\psi(y)\} = \hbar\delta(x-y) \qquad (4.7)$$

for $\psi \equiv \psi_{+}$, say, with $\{\psi_{+}(x),\psi_{-}(y)\} = 0$. (Here $\{A,B\} \equiv AB + BA$.) Actually we wish to consider an elaboration of this theory in which we have an internal symmetry index i taking values from 1 to N; that is N non-interacting copies of the free real fermion theory we have been discussing. Then the anticommutation relations (3.39) are replaced by

$$\{\psi_{i}(x),\psi_{j}(y)\} = \hbar\delta(x-y)\delta_{ij} \ . \qquad (4.8)$$

Now to use these to construct a representation of the affine Kac-Moody algebra associated with a group G, take a real (not necessarily irreducible) representation of G under which $T^{a} \to iM^{a}$, where M^{a} is an $N\times N$ real antisymmetric matrix satisfying

$$[M^{a},M^{b}] = f^{ab}{}_{c}M^{c} \ . \qquad (4.9)$$

If G has such a representation, it is a subgroup of the $O(N)$ symmetry group of the theory. Associated with this symmetry are conserved currents

$$J^a_\mu = \frac{i}{2\sqrt{2}} \bar{\psi} M^a \gamma_\mu \psi \quad . \tag{4.10}$$

It is convenient to define light-cone coordinates for vectors $v = (v^0, v^1)$ by

$$v^\pm = (v^0 \pm v^1)/\sqrt{2} \ , \ v_\pm = (v_0 \pm v_1)/\sqrt{2} \ , \tag{4.11}$$

so that $v^\pm = v_\mp$ and, for two vectors v and w,

$$v^\mu w_\mu = v^+ w^- + v^- w^+ = v_+ w_- + v_- w_+ \ . \tag{4.12}$$

Then

$$J^a_\pm = \frac{i}{2} \psi^T_\pm M^a \psi_\pm \tag{4.13}$$

so that J^a_+ is only a function of $x^+ = (t+x)/\sqrt{2}$ and J^a_- is only a function of $x^- = (t-x)/\sqrt{2}$. Again we can consider independently $J^a_+(x^+)$ and $J^a_-(x^-)$ and they will commute with one another because ψ_+ and ψ_- anticommute. Let us denote either by $J^a(x)$. Then the canonical anticommutation relations (4.8) imply a two-dimensional current algebra (see e.g. ADLER and DASHEN),

$$[J^a(x), J^b(y)] = i\hbar f^{ab}{}_c J^c(y)\delta(x-y) + \frac{i\kappa_\lambda}{4\pi} \delta^{ab}\delta'(x-y)\hbar^2 \tag{4.14}$$

where the second term on the right hand side is a SCHWINGER term and the constant κ is (up to a representation independent normalisation) called the Dynkin index of the representation; it is defined by

$$\mathrm{tr}(M^a M^b) = -\kappa_\lambda \delta^{ab} \ . \tag{4.15}$$

Here λ is meant to label the representation considered. Taking the trace over the indices a,b we see that

$$c_\lambda d_\lambda = \kappa_\lambda \dim G \tag{4.16}$$

where c_λ is the value of the quadratic Casimir operator in the representation λ ,

$$(M^a)^2 = -c_\lambda 1 \ , \tag{4.17}$$

and $N = d_\lambda$ is the dimension of the representation.

Eq.(4.14) resembles a sort of continuous version of a Kac-Moody algebra (cf. eq.(2.43)). To make it discrete we impose periodic boundary conditions on the interval $0 \leq x \leq L$, so that

$$J^a_\mu(x,t) = J^a_\mu(x+L,t) \tag{4.18}$$

This implies that both $J^a_\pm(x)$ have period L ; thus, for either, we have a Fourier expansion

$$J^a(x) = \frac{1}{L}\sum_n J^a_{-n}(x)\, z^n \tag{4.19}$$

where the sum is over $n \in \mathbb{Z}$,

$$z = e^{2\pi i x/L} , \tag{4.20}$$

and

$$J^a_n = \int_0^L J^a(x)\; z^n dx . \tag{4.21}$$

Then (4.14) implies

$$[J^a_m, J^b_n] = i\hbar\, f^{ab}{}_c J^c_{m+n} + \frac{\kappa_\lambda}{2}\, m\hbar^2 \delta^{ab} \delta_{m,-n} \tag{4.22}$$

If we remove $\hbar$ by seeking (after noting that the central extension term is $O(\hbar^2)$ as in eq.(2.36)) setting

$$J^a_n = \hbar\, T^a_n , \tag{4.23}$$

we obtain

$$[T^a_m, T^b_n] = if^{ab}{}_c\; T^c_{m+n} + \frac{\kappa_\lambda}{2}\, m\, \delta^{ab}\; \delta_{m,-n} \tag{4.24}$$

which is exactly eq.(2.43) with $k = \frac{1}{2}\kappa_\lambda$. Note that it follows from section 3 that κ_λ/ψ^2 must be an integer for any real representation.

It is natural to introduce

$$T^a(z) = \sum_n T^a_{-n}\; z^n , \tag{4.25}$$

a dimensionless current, such that

$$J^a(x) = \frac{\hbar}{L} T^a(z) \ . \tag{4.26}$$

Similarly we can make the fermion field dimensionless by setting

$$\psi(x) = \sqrt{\frac{\hbar}{L}}\, H(z) \ . \tag{4.27}$$

Let us discuss now the implications of periodicity on the fermion field. There are two possibilities consistent with the periodicity of J^a_μ, i.e. eq.(4.18),

$$\underline{\text{either}} \quad \psi(x+L) = \psi(x) \quad \underline{\text{or}} \quad \psi(x+L) = -\psi(x) \ . \tag{4.28}$$

So we have an expansion

$$H_i(z) = \sum b^i_{-r}\, z^r \tag{4.29}$$

where in the case where ψ is periodic, the RAMOND (R) case, $r \in \mathbb{Z}$ and the case where ψ is antiperiodic, the NEVEU-SCHWARZ (NS) case, $r \in \mathbb{Z} + \frac{1}{2}$. The canonical anticommutation relations (4.8) are then equivalent to

$$\{b^i_r, b^j_s\} = \delta^{ij}\delta_{r,-s} \tag{4.30}$$

where <u>either</u> $r,s \in \mathbb{Z}$ (R) <u>or</u> $r,s \in \mathbb{Z} + \frac{1}{2}$ (NS). These fermionic annihilation and creation operators define a Fock space by the application of the b^i_r, $r > 0$, to a vacuum state $|0\rangle$ satisfying

$$b^i_r|0\rangle = 0 \ , \qquad r > 0 \ , \tag{4.31}$$

In the NS case we take the vacuum to be a single non-degenerate state; in the R case it must provide a representation for the Clifford algebra

$$\{b^i_0, b^j_0\} = \delta^{ij} \tag{4.32}$$

and so the vacuum states form a $2^{N/2}$ dimensional irreducible representation space for this algebra. In terms of these operators

$$T^a_n = \frac{1}{2} \sum_r b^i_r\, M_{ij}\, b^j_{n-r} \tag{4.33}$$

where the sum is <u>either</u> over $r \in \mathbb{Z}$ (R case) <u>or</u> over $r \in \mathbb{Z}+\frac{1}{2}$ (NS case).

(From such representations we can obtain all the representations of the classical groups satisfying (3.14). Let us consider a specific simple classical group G with the squared length of the longest root normalised to 2. For each highest weight λ_ϕ the maximum value of $\alpha\text{-}\lambda_o$ (for positive roots), k_o say, is the lowest level at which the vacuum representation can have highest weight λ_ϕ. If we consider the fundamental weights $\lambda_1,\ldots,\lambda_n$ of G, the corresponding lowest levels $\ell_1,\ldots,\ell_n$ all turn out to be 1 or 2; for SU(n+1) and Sp(n), each $\ell_i = 1$, whilst for SO(2n+1) and SO(2n), $\ell_i = 1$ for the vector and spinor fundamental representations, whilst $\ell_i = 2$ for the other representations.

Given an arbitrary highest weight, $\lambda_\phi = \sum n_i\lambda_i$, so that $n_i \geq 0$, one can show that $k_o = \sum n_i\ell_i$. To construct a representation in which the vacuum has highest weight λ_ϕ and level k, we can take the tensor product of n_i copies of representations with highest weight λ_i and level ℓ_i, $0 \leq i \leq n$, where $n_o = k - k_o$ and $\lambda_o = 0$ [and then take the subspace generated from the irreducible component of the vacuum with highest weight λ_ϕ]. Thus it remains to construct the representations with highest weights λ_i and levels ℓ_i. This can be done as follows: we use eq.(4.33) with M being the n-dimensional representation of SO(n). The NS case gives the scalar an n vector level 1 representation whilst the R case gives the spinor level 1 representation(s). The level 1 representations of SU(n) and Sp(n) are given by the inclusions $SU(n) \subset SO(2n)$ and $Sp(n) \subset SU(2n)$ SO(4n) and the level 2 anti-symmetric tensor given by fundamental representations of SO(n) are $SO(n) \subset SU(n) \subset SO(2n)$.)

(b) <u>Kac-Moody algebras in σ-models</u>

The role that Kac-Moody algebras play in σ-models was made clear by WITTEN, aspects of whose work we now describe. The principal σ-model associated with a group G in a two dimensional space-time is described by the action

$$\mathcal{A}_o = -\frac{1}{4\lambda^2}\int \mathrm{tr}(g^{-1}\partial_\mu g\; g^{-1}\partial^\mu g)\,d^2x \tag{4.34}$$

where $g(x,t)$ takes values in the group G, or rather some particular representation of it. Then the variation of $\mathcal{A}_o$ corresponding to some small variation δg of g is proportional to

$$- \int \mathrm{tr}(g^{-1}\partial_\mu g\; g^{-1}\partial^\mu \delta g)d^2x + \int \mathrm{tr}(g^{-1}\partial_\mu g\; g^{-1}\delta g\; g^{-1}\partial^\mu g)d^2x$$

$$= \int \mathrm{tr}\{g^{-1}\delta g \partial_\mu (g^{-1}\partial^\mu g)\}d^2x \;, \qquad (4.36)$$

where we have discarded the surface terms which come from integrating by parts. Thus the action ${}_{o}$ leads to the equations of motion

$$\partial_\mu (g^{-1}\partial^\mu g) = 0 \qquad (4.37)$$

Now, if we set

$$g^{-1}\partial^\mu g = J^{a\mu}(x,t)T^a \;, \qquad (4.38)$$

where T^a are as usual a set of generators G, eq.(4.37) amounts to the conservation equation

$$\partial_\mu J^{a\mu} = 0 \qquad (4.39)$$

But $J^\mu = g^{-1}\partial^\mu g$ also satisfies a consistency condition coming from eq.(4.38), essentially saying that the curl of a gradient is zero; since

$$\partial^\mu J^\nu = g^{-1}\partial^\mu\partial^\nu g - g^{-1}\partial^\mu g\; g^{-1}\partial^\nu g$$

$$= g^{-1}\partial^\mu\partial^\nu g - J^\mu J^\nu \;,$$

we have

$$\partial^\mu J^\nu - \partial^\nu J^\mu + [J^\mu, J^\nu] = 0 \;. \qquad (4.40)$$

Now if we aim to get again a Kac-Moody algebra for the light-cone components of J^μ, as in the free fermion cases we would seem to need $J^{a\pm}$ to be functions of only one light-cone variable, $x^\pm$ respectively. Thus we need

$$\partial_- J_+ = \partial_+ J_- = 0 \;. \qquad (4.41)$$

This condition implies not only to the equation of motion (4.39) but also $\partial^\mu J^\nu = \partial^\nu J^\mu$, violating the compatibility condition (4.40) unless $[J^\mu, J^\nu]$ vanishes automatically, i.e. unless G is abelian.

WITTEN found a way round this apparent impasse. He noted that if

$$J_+ = g^{-1}\partial_+ g \tag{4.42}$$

satisfies the first of eqs.(4.41), since

$$\begin{aligned}\partial_-(g^{-1}\partial_+ g) &= g^{-1}\partial_-\partial_+ g - g^{-1}\partial_- g\, g^{-1}\partial_+ g\\ &= g^{-1}\partial_+(\partial_- g\, g^{-1})\, g\,,\\ J_- &= (\partial_- g)\, g^{-1}\end{aligned} \tag{4.43}$$

satisfies the second of eqs.(4.41). The need to treat the two light-cone components of J_μ differently had eluded the authors of previous discussions of σ-models and their relations to Kac-Moody algebras or, equivalently, current algebras.

Of course eqs.(4.41), with definitions (4.42) and (4.43), are not the equations of motion following from the action (4.34). Consequently Witten also found it necessary to change the equations of motion. This he did by the addition of a WESS-ZUMINO term to $\mathcal{O}\!\mathcal{L}_0$, modifying the action to

$$\mathcal{O}\!\mathcal{L} = -\frac{1}{4\lambda^2}\int \mathrm{tr}(g^{-1}\partial_\mu g\, g^{-1}\partial^\mu g)d^2x + K\Gamma \tag{4.44}$$

where

$$\Gamma = \frac{1}{24\pi}\int_B \varepsilon^{\lambda\mu\nu}\mathrm{tr}(g^{-1}\partial_\lambda g\, g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g)d^3y\,, \tag{4.45}$$

and K is a constant of the same dimensions as action.

What is meant by the integral Γ requires some explanation. To do this it is simplest to suppose that, by analytic continuation, we may work in a Euclidean space-time which is a large two-dimension sphere S^2. We take this S^2 to be the boundary of a solid ball B in $\mathbb{R}^3$ whose coordinates are y and extend g in an arbitrary smooth fashion through B. Such a definition of Γ is not obviously unique. Indeed the difference between any two values of Γ obtained in this way can be represented as an integral of the form of (4.45) but with B replaced by some three-dimensional sphere S^3. Now for any such sphere, this integral is an integral multiple of 2π if g is in the defining N-dimensional representation of SO(N), whilst it is an integral multiple of π for the defining

N-dimensional representation of SU(N) or the defining 2N dimensional representation of Sp(N).

The ambiguity in the definition of Γ introduces a discrete ambiguity into the definition of $\mathcal{O}\!\mathcal{C}$ of the form $2\pi nK$, where $n \in \mathbb{Z}$ for SO(N) and $n \in \frac{1}{2}\mathbb{Z}$ for SU(N) or Sp(N). This discrete ambiguity in $\mathcal{O}\!\mathcal{C}$ causes no problems for the derivation of the classical motion because each of the possibilities for Γ will be stationary for the same paths. Quantum mechanically it potentially has a significance because there we had $\exp(i\mathcal{O}\!\mathcal{C}/\hbar)$ to be single-valued. This leads to the condition

$$K = \nu h \tag{4.46}$$

where $\nu \in \mathbb{Z}$ for SO(N) and $\nu \in 2\mathbb{Z}$ for SU(N) or Sp(N). Note that this constraint on K is quantum mechanical rather than classical.

To see how the term $K\Gamma$ modifies the equation of motion we calculate

$$\begin{aligned}\delta\Gamma &= \frac{1}{8\pi}\int_B \partial_\lambda \varepsilon^{\lambda\mu\nu}\mathrm{tr}(g^{-1}\delta g\, g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g)d^3y \\ &= -\frac{1}{8\pi}\int \varepsilon^{\mu\nu}\mathrm{tr}\{g^{-1}\delta g\, g^{-1}\partial_\mu(g^{-1}\partial_\nu g))\, d^2x \end{aligned} \tag{4.47}$$

as the boundary of B is the Euclidean space-time in which we are working. Combining this with

$$\delta\mathcal{O}\!\mathcal{C}_o = \frac{1}{2\lambda^2}\int \mathrm{tr}(g^{-1}\delta g\, \partial_\mu(g^{-1}\partial^\mu g))d^2x \tag{4.48}$$

we obtain

$$\partial_\mu\{g^{-1}\partial^\mu g - \frac{\lambda^2 K}{4\pi}\varepsilon^{\mu\nu}g^{-1}\partial_\nu g\} = 0 . \tag{4.49}$$

[Here $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$, $\varepsilon_{o1} = +1$.]

If we consider the special value of the coupling constant $\lambda^2 = 4\pi/K$, the equation of motion (4.49) reduces to

$$\partial_-(g^{-1}\partial_+ g) = 0 , \tag{4.50}$$

thus giving

$$\partial_- J_+ = \partial_+ J_- = 0 \tag{4.51}$$

as desired, with

$$J_+ = g^{-1}\partial_+ g \ , \quad J_- = (\partial_- g)g^{-1} \ . \tag{4.52}$$

If, on the other hand, we choose the special value $\lambda^2 = -4\pi/K$, we obtain instead

$$\partial_+(g^{-1}\partial_- g) \ , \tag{4.53}$$

so in this case obtaining eq.(4.51) with

$$J_+ = (\partial_+ g)g^{-1} \ , \quad J_- = g^{-1}\partial_- g \tag{4.54}$$

instead.

We shall now fix on the first possibility, $\lambda^2 = 4\pi/K$. Note that, if we take a particular value of ν in the quantum-mechanical condition (4.46) it is no longer possible to take the classical limit $h \to 0$ at fixed λ . This is similar to the position with magnetic monopoles where the electric and magnetic charge q,g have to satisfy $qg/2\pi h \in \mathbb{Z}$.

We now wish to quantise this theory to see whether $J_\pm$ will need to provide us with commuting Kac-Moody algebras as in the case of the current algebra of free fermion fields. To do this we need to calculate the Poisson brackets for the theory. This can be done in a canonical fashion and, if we absorb various dimensional constants by redefining

$$T^a J^a_+ = - ig^{-1}\partial_+ g \, \frac{\nu\hbar}{4\pi} \sqrt{2} \ , \tag{4.55}$$

we obtain the Poisson bracket relations

$$[J^a_+(x), J^b_+(y)]_{P.B.} = f^{ab}{}_c J^c_+(x)\delta(x-y) + \frac{\nu\hbar k}{4\pi}\delta_{ab}\delta'(x-y) \ , \tag{4.56}$$

There is a similar equation for $J^a_-(x)$ defined by

$$T^a J_- = -i(\partial_- g)g^{-1} \, \frac{\nu\hbar}{4\pi} \sqrt{2} \tag{4.57}$$

and

$$[J^a_+(x), J^b_-(y)]_{P.B.} = 0 \ , \tag{4.58}$$

Eq.(4.56) should be compared with eq.(4.14).

As before to obtain a Kac-Moody algebra we need to impose periodic boundary conditions on $0 \leq x \leq L$. So we define

$$J^a_m = \hbar\, T^a_m = \int_0^L J^a_+(x)\ z^n dx \tag{4.59}$$

giving the canonical commutation relations

$$[T^a_m, T^b_n] = if^{ab}{}_c T_{m+n} + \frac{\nu\kappa_\lambda}{2} m\, \delta^{ab} \delta_{m,-n} \ , \tag{4.60}$$

where we have

$$\mathrm{tr}\left(T^a T^b\right) = \kappa_\lambda\ \delta_{ab} \ . \tag{4.61}$$

So comparing again with eq.(2.43) we have a Kac-Moody algebra with $k = \frac{1}{2}\nu\kappa_\lambda$. If we take $G = SO(N)$ and $\nu = 1$ this is the same as eq.(4.24). For the N-dimensional representation of SO(N) $\kappa_\lambda/\alpha^2 = 2k/\alpha^2 = 1$, where α is a long root, so that both eqs.(4.24) and (4.60) then correspond to level 1 representations of the SO(N) Kac-Moody algebra. The only possibilities in this case are the scalar, spinor and vector representations and, up to this ambiguity, the representation spaces provided by the free fermion theory and the σ-model must be equivalent. Suggesting that, for the particular value of λ considered, the theories are equivalent.

For $G = SU(N)$ or $Sp(N)$ the lowest allowed non-zero value of ν is 2. Then $k = \kappa_\lambda$ in eq.(4.60) and this differs from eq.(4.24) superficially as there $k = \frac{1}{2}\kappa_\lambda$. But if for example we consider the N-dimensional representation of SU(N) or the 2N-dimensional representation of Sp(N) we cannot proceed directly as in (a) above because these representations are complex; to make them real we must double the dimension and this doubles k in eq.(4.24), so that $k = \kappa_\lambda$ agreeing with eq.(4.60). Since, for these representations $2\kappa_\lambda/\alpha^2 = 2k/\alpha^2 = 1$, where α is a long root, we have had one representation. Again there are only finitely many possibilities, the scalar and the N fundamental representations, and subject to this finite ambiguity, the theories appear to be equivalent. This equivalence, established by WITTEN, has been discussed further by KNIZHNIK and ZAMOLODCHIKOV.

(c) <u>the energy momentum tensor and the Virasoro algebra</u>

If the σ-model and the free fermion theory are to be equivalent under the identification of their Kac-Moody algebras, their energy momentum tensors must be the same after this identification. Indeed, if this is so, equivalence follows because the dynamics of the theories will be the same. The energy-momentum tensors also have an algebraic interest because they provide us with a representation of the Virasoro algebra.

For the fermion theory, the symmetric energy momentum tensor is

$$\theta^{\mu\nu} = \frac{1}{4} : \{\bar{\psi}\,\gamma^{\mu}\,\overleftrightarrow{\partial}^{\nu}\psi + \bar{\psi}\,\gamma^{\nu}\,\overleftrightarrow{\partial}^{\mu}\psi\} \;, \tag{4.62}$$

with an implicit sum over the internal index i . This tensor is traceless because of the equation of motion (4.4),

$$2\theta_{+-} = \theta^{\mu}{}_{\mu} = 0 \;. \tag{4.63}$$

This is necessary for the conformal invariance of the theory. Thus the only non-zero components of $\theta_{\mu\nu}$ are θ_{++} and θ_{--} . These are proportional to $\psi_{+}\partial_{+}\psi_{+}$ and $\psi_{-}\partial_{-}\psi_{-}$ respectively. Again let us fix our attention on one light-cone component. Then θ_{++} is proportional to

$$\frac{i}{2} : \frac{dH}{dz} H : \; = L(z) \equiv \sum L_{-n} z^{n} \tag{4.64}$$

where we have normal ordered with respect to the fermion oscillators to avoid divergences,

$$\begin{aligned} :b_r\, b_s: &= -\, b_s\, b_r && \text{if} \quad r > 0 \\ &= \frac{1}{2}\,[b_r, b_s] && \text{if} \quad r = 0 \\ &= b_r\, b_s && \text{if} \quad r < 0 \;. \end{aligned} \tag{4.65}$$

With L_n defined by eq.(4.64) we can calculate their commutators and we obtain the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{24}\, m(m^2-1)\,\delta_{m,-n} \tag{4.66}$$

which is just (2.44) with $c = \frac{1}{2}d$. Further

$$[L_m, T_n^a] = -n\, T_{m+n}^a \,, \tag{4.67}$$

with T_n^a defined by eq.(4.33), providing a representation of the whole semi-direct product algebra.

Note that in the one-dimensional case $d = 1$, $c = \frac{1}{2}$, we have the first of the non-trivial terms in the series (3.32). In the NS case the highest weight states are $|0\rangle$, $b_{-\frac{1}{2}}|0\rangle$, giving the $h = 0, \frac{1}{2}$ representations, whilst in the R case the only possibility is $|0\rangle$ which gives $h = \frac{1}{16}$ as then

$$L_o = \frac{1}{16} + \sum_{n>0} n d_{-n} d_n \,. \tag{4.68}$$

Now let us consider the energy-momentum tensor in the meson theory. It has the current-current form (SUGAWARA; SOMMERFIELD)

$$\theta_{\mu\nu} = J_\mu^a J_\nu^a - \frac{1}{2} g_{\mu\nu} J_\lambda^a J^{a\lambda} \,, \tag{4.69}$$

which is also traceless. The non-zero components are

$$\theta_{++} = \frac{1}{2} J_+^a J_+^a \tag{4.70}$$

and there is a similar expression for θ_{--}. In the quantum theory we again need to avoid divergences by some sort of normal ordering procedure. We do this with respect to the Fourier components of J^a by defining

$$\begin{aligned} \mathop{\circ}\limits^{\circ} T_m^a T_n^a \mathop{\circ}\limits^{\circ} &= T_m^a T_n^a \qquad m < 0 \\ &= T_n^a T_m^a \qquad m > 0 \end{aligned} \tag{4.71}$$

and then considering

$$\tilde{\mathscr{L}}_m = \frac{1}{2} \sum_n \mathop{\circ}\limits^{\circ} T_n^a T_{m-n}^a \mathop{\circ}\limits^{\circ} \tag{4.72}$$

which is proportional to the Fourier coefficients of (4.70). The $\tilde{\mathscr{L}}_m$ so defined satisfy a Virasoro algebra but up to a scaling factor; if we set

$$\mathscr{L}_m = \frac{1}{\beta} \tilde{\mathscr{L}}_m \,, \tag{4.73}$$

with

$$\beta = \frac{1}{2}(\nu\kappa_\lambda + c_\psi), \tag{4.74}$$

where $\nu\kappa_\lambda$ is as in eq.(4.60) and c_ψ is the quadratic Casimir operator of the group G in the adjoint representation

$$f^{ac}{}_d\, f^{bd}{}_c = -c_\psi\, \delta^{ab}, \tag{4.75}$$

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n} . \tag{4.76}$$

$$[\mathcal{L}_m, T^a_n] = -n\, T^a_{m+n}, \tag{4.77}$$

where, by eq.(4.16),

$$c = \frac{\nu\kappa_\lambda d_\psi}{\nu\kappa_\lambda + c_\psi} = \frac{\nu c_\lambda d_\lambda}{\nu\kappa_\lambda + c_\psi}, \tag{4.78}$$

with $d_\psi = \dim G$. These equations depend only on eqs.(4.72-4) and eq.(4.60). [For similar calculations see BARDAKCI and HALPERN, FRENKEL (1981), KAC (1983) and SEGAL (1981).]

Let us compare eqs.(4.66) and (4.76) for the N-dimensional representation of SO(N). Then, with the length squared of a long root normalised to 2, and taking $\nu = 1$ again,

$$\kappa_\lambda = 2,\; d_\psi = \frac{1}{2}N(N-1),\; d_\lambda = N,\; c_\psi = 2(N-1) \tag{4.79}$$

so that the value of c given by eq.(4.78),

$$c = \frac{1}{2}N \tag{4.80}$$

agrees with that in eq.(4.66). This means that if we take the renormalised energy-momentum tensor to be rescaled by the factor β, the theories are identical dynamically, subject to ambiguities mentioned before.

For the N-dimensional representation of SU(N) we need to take $\nu = 2$, and since

$$\kappa_\lambda = 1,\; d_\psi = N^2-1,\; d_\lambda = N,\; c_\psi = 2N \tag{4.81}$$

the value of c given by eq.(4.78) is

$$c = N-1 \tag{4.82}$$

We see there is a deficit of 1 in c relative to eq.(4.66) as we need to use the 2N-dimensional version of this representation. This deficit can be corrected for by adding a U(1) factor, enlarging SU(N) to U(N).

Similarly for the 2N-dimensional representation of Sp(N), we have to have $\nu = 2$, and

$$\kappa_\lambda = 1 \ , \ d_\psi = N(2N-1) \ , \ d_\lambda = 2N \ , \ c_\psi = 2N-3 \tag{4.83}$$

The value of c in eq.(4.78) is then

$$c = 2N - \frac{3N}{N+2} \tag{4.84}$$

This time the deficit of $3N/(N+2)$ can be made up by extending Sp(N) to Sp(N) $\times$ Sp(1) and, with this extension, the theories should be equivalent (GODDARD, KENT and OLIVE).

(d) new representations of the Virasoro algebra

We have seen that, for the defining representations of SO(N), SU(N) and Sp(N), the Virasoro algebras defined by eqs.(4.64) and (4.73) are the same if we identify the currents as in eq.(4.33). What happens for more general representations? Clearly for the two to be the same the values of the central element c must be the same, i.e.

$$\frac{2c_\lambda}{c_\psi + \kappa_\lambda} = 1 \tag{4.85}$$

where κ_λ is the value of the Dynkin index

$$\mathrm{tr}(M^a M^b) = -\kappa_\lambda \delta^{ab} \tag{4.86}$$

for the real form of the representation (i.e. $\kappa_\lambda = 2$ in each of the cases we have first discussed. The equality (4.85) does not always hold; if it fails to, the extent to which it fails produces an interesting algebraic structure (GODDARD and OLIVE, 1984).

Starting just from the algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n} \tag{4.87a}$$

$$[L_m, T^a_n] = -n \, T^a_{m+n} \tag{4.87b}$$

$$[T^a_m, T^b_n] = -if^{ab}{}_c\, T^c_{m+n} + km\delta_{m,-n}\delta^{ab} \tag{4.87c}$$

and defining $\mathscr{L}_m$ by eqs.(4.72) and (4.73) with

$$\beta = k + \frac{1}{2} c_\psi \tag{4.88}$$

we obtain the commutation relations

$$[L_m, \mathscr{L}_n] = (m-n)\mathscr{L}_{m+n} + \frac{c}{12} m(m^2-1)\ \delta_{m,-n}\ , \tag{4.89a}$$

$$[\mathscr{L}_m, \mathscr{L}_n] = (m-n)\mathscr{L}_{m+n} + \frac{c'}{12} m(m^2-1)\ \delta_{m,-n}\ , \tag{4.89b}$$

where

$$c' = \frac{kd_\psi}{\beta} = \frac{2kd_\psi}{c_\psi + 2k}\ . \tag{4.90}$$

It follows that $\mathscr{L}_m$ and $K_m = L_m - \mathscr{L}_m$ from two commuting Virasoro algebras

$$[\mathscr{L}_m, K_n] = 0 \tag{4.91a}$$

$$[K_m, K_n] = (m-n)\ K_{m+n} + \frac{c''}{12} m(m^2-1)\ \delta_{m,-n} \tag{4.91b}$$

where

$$c'' = c - c'\ , \tag{4.92}$$

$$= \frac{d_\lambda}{2}\left(1 - \frac{2c_\lambda}{\kappa_\lambda + c_\psi}\right) \tag{4.93}$$

in the particular case we have been discussing.

If the spectrum of L_0 is bounded below, as in a highest weight representation, it follows that the spectra of both K_0 and $\mathscr{L}_0$ must be bounded in this way. Hence we have $c', c'' \geq 0$. If the spectrum of L_0 is bounded below and $c'' = 0$, K_n is represented trivially and so $L_n = \mathscr{L}_n$. Hence eq.(4.83) is a necessary and sufficient condition for the energy-momentum tensors of the free fermion model and the σ-model to be equal. It is satisfied not only for the defining representations we have mentioned but also for the adjoint representation of any group, and the traceless symmetric tensor representation of SO(n) as well as some others (GODDARD and OLIVE, 1984), suggesting a boson-fermion equivalence in these cases as well.

An identity which throws light on these is that between the normal ordering of the product $J^a(z)J^a(z)$ performed in two different ways [see eqs.(4.65) and (4.71)]:

$$\frac{1}{2} \circ T^a(z)T^a(z) \circ = \frac{1}{2} : T^a(z)T^a(z) : + c_\lambda L(z) \tag{4.94}$$

with

$$T^a(z) = \sum T^a_{-n} z^n = \frac{1}{2} H(z) M^a H(z) \tag{4.95}$$

This yields the relation between the energy-momentum tensor

$$\mathscr{L}(z) = \frac{1}{2\beta} :T^a(z)T^a(z): + \frac{c_\lambda}{\beta} L(z) \tag{4.96}$$

So they are equal if and only if the normal ordered quartic product of four $H^i(z)$ fields vanishes and (4.85) holds; it can be verified that eq.(4.85) implies the vanishing of this quartic product as (we know it should since it implies $L_n = \mathscr{L}_n$). In general $\mathscr{L}_n$ and K_n will each involve a mixture of a normal ordered quartic, the normal ordered product of H and dH/dZ .

The physical interpretation of what happens when $K_n \neq 0$ is less clear but we can obtain some interesting representations of the Virasoro algebra this way. If we take the 7 dimensional representation of SO(3) or G_2 we obtain $c = \frac{7}{10}$. A slight extension of this procedure provides the $c = \frac{4}{5}$ representation from the 6 dimensional representation of Sp(3). A further extension to a construction based on a subgroup $H \subset G$ yields the whole sequence of eq.(3.31).

5. Conclusion

In these lectures the Virasoro algebra and the (untwisted affine) Kac-Moody algebra have been described together with some of what has been learnt about their representations. The aim has been to produce a simple approach accessible to theoretical physicists. There is an extensive mathematical literature and I am not competent to give all the appropriate references but amongst those which may be of particular interest to physicists are FRENKEL and KAC, FRENKEL (1981 and 1982), and FEINGOLD and FRENKEL. An extensive mathematical treatment of the representation theory, and further references are contained in the book of KAC (1983).

The fourth section of these notes was devoted to the current algebra construction of representations of Kac-Moody algebras and the relation of those to principal σ-models following the work of WITTEN. This is an example of an equivalence of a bosonic to a fermionic theory and our knowledge of such relationships has grown over the years as the result of the work of many authors (SKYRME, STREATER and WILDE, COLEMAN, MANDELSTAM). In such equivalences bosonic currents correspond to bilinear quantities in fermion fields and fermion fields are exponentials of quantities linear in boson fields. These expressions for fermions are of the form of the vertex operators first used in the theory of relativistic strings, and adapted for the construction of representations of Kac-Moody algebras by FRENKEL and KAC. There has not been time to develop here the use of vertex operators to represent Kac-Moody algebras and to tie in with the boson-fermion equivalence. For discussions of aspects of this we refer to FRENKEL (1981 and 1982) and GODDARD and OLIVE (1983), where some further developments are also discussed.

I am very grateful to the organisers of the 1985 Srni Winter School of the University of Prague for their very kind and generous hospitality. I should also like to acknowledge the many discussions about infinite dimensional Lie algebras I have had over the last few years with Werner Nahm and David Olive; many aspects of the presentation here are due to them.

REFERENCES

ADLER S. and DASHEN R. : Current Algebras and applications to particle physics (Benjamin, New York 1968).
BARDAKCI K. and HALPERN M. : Phys. Rev. D3 /1971/, 2493-2506.
BELAVIN A.A., POLYAKOV A.M. and ZAMOLODCHIKOV A.B. : Nucl. Phys. B241 /1980/, 333-380.
COLEMAN S. : Phys. Rev. D11 /1975/, 2088-2097.
FEIGIN B.L. and FUCHS D.B. : Funct. Anal. and Appl. 16 /1982/, 114-126.
FEINGOLD A.J. and FRENKEL I. : "Classical Affine Algebras", to be published in Advances in Mathematics.
FRENKEL I. (1981) : Journal of Functional Analysis 44 /1981/, 259-327.
FRENKEL I. (1982) : "Representations of Kac-Moody Algebras and dual resonance models" (preprint).
FRENKEL I. and KAC V.G. : Inventiones Math. 62 /1980/, 23-66.

FRIEDAN D., QIU Z. and SHENKER S. : in Vertex Operators in Mathematics and Physics ed. J. Lepowsky et al. (MSRI publication No.3, Springer Verlag, 1984)/419-449; Phys. Rev. Lett. 52 /1984/, 1575-1578.

GLIOZZI F., OLIVE D. and SCHERK J. : Nucl. Phys. B122 /1977/, 253-290.

GODDARD P., KENT A. and OLIVE D. : Phys. Lett. 152 /1985/, 88-92.

GODDARD P. and OLIVE D. (1983) : in Vertex Operators in Mathematics and Physics ed. J. Lepowsky et al. (MSRI publication No.3, Springer Verlag, 1984) /51-96.

GODDARD P. and OLIVE D. (1984) : DAMTP preprint 84-16, to appear in Nuclear Physics.

HUMPHREYS J.E. : Introduction to Lie Algebras and Representation Theory (Springer, Berlin 1972).

JACOB M. : Dual Theory (North Holland, Amsterdam 1974).

KAC V.G. (1968) : Matt. USSR-Izv. 2/1968/, 1271-1311.

KAC V.G. (1978) : Proceedings of the International Congress of Mathematicians, Helsinki /1978/, 299-304.

KAC V.G. (1983) : Infinite Dimensional Lie Algebras (Birkhauser, Boston 1983).

KENT A. : Lecture at this meeting.

KNIZHNIK V.G. and ZAMOLODCHIKOV A.B. : Nucl. Phys. B247 /1984/, 83-103.

MANDELSTAM S. : Phys. Rev. D11 /1975/, 3026-3030.

MOODY R. : Bull Amer. Math. Soc. 73 /1967/, 217-221.

NEVEU A. and SCHWARZ J. : Nucl. Phys. B31 /1971/, 86-112.

OLIVE D. : Lectures at this meeting.

POLYAKOV A.M. : JETP Lett. 12 /1970/, 381-382.

RAMOND P. : Phys. Rev. D3 /1971/, 2415-2418.

SCHWINGER J. : Phys. Rev. Lett. 3 /1959/, 296-297.

SEGAL G. : Commun. Math. Phys. 80 /1981/, 301-342.

SKYRME T.H.R. : Proc. R. Soc. A247 /1958/, 260-278; A252 /1959/, 236-245; A260 /1961/, 127-138; A262 /1961/, 237-245.

STREATER R.F. and WILDE I.F. Nucl. Phys. B24 /1970/ 561-575.

SOMMERFIELD C.M. : Phys. Rev. 176 /1968/, 2019-2025.

SUGAWARA H. : Phys. Rev. 170 /1968/, 1659-1662.

THORN C.B. : Nucl. Phys. B248 /1984/, 551-569.

VIRASORO M.A. : Phys. Rev. D1 /1970/, 2933-2936.

WESS J. and ZUMINO B. : Phys. Lett. 37B /1971/, 95-97.

WITTEN E. : Commun. Math. Phys. 92 /1984/ 455-472.

Department of Applied Mathematics and Theoretical Physics,
University of Cambridge,
Silver Street,
Cambridge CB3 9EW
United Kingdom