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**Generalized inverses of elliptic systems of
differential operators with constant coefficients and
related REDUCE programs for explicit calculations**

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In this paper it is shown how the theory of generalized inverses for closed densely defined linear operators $f : H_1 \rightarrow H_2$, H_1 and H_2 being Hilbert spaces, may be applied to the case where $f = f(D)$ is an elliptic matrix differential operator with constant coefficients. For $f(D)$ the gradient operator in \mathbb{R}^3 an example is worked out and the explicit solution is constructed by means of a REDUCE program.

1. Introduction.

Let H_1, H_2 be Hilbert spaces and let $f : H_1 \rightarrow H_2$ be a closed densely defined linear operator with domain $\text{dom}(f)$, kernel $\eta(f)$ and range $R(f)$. Furthermore call $\mathcal{C}(f) = \text{dom}(f) \cap \eta(f)^\perp$; then $\text{dom}(f) = \mathcal{C}(f) \oplus \eta(f)$ and f admits a generalized inverse f^{-1} with $\text{dom}(f^{-1}) = R(f) \oplus R(f)^\perp$ and $R(f^{-1}) = \mathcal{C}(f)$. As is well known $f^{-1} : H_2 \rightarrow H_1$ is also a closed densely defined linear operator. Moreover $L = f^* f$ is a non-negative self-adjoint operator and f admits the polar decomposition $f = R\sqrt{L}$ whereby $R : H_1 \rightarrow H_2$ is a partial isometry called the elementary operator associated with f . Denoting by M the spectral measure associated with L , f and f^{-1} admit the following spectral decomposition (see [2])

$$f = \int_0^{+\infty} \sqrt{t} \, dRM \quad , \quad f^{-1} = \int_0^{+\infty} \frac{1}{\sqrt{t}} \, dMR^*$$

whereby RM and MR^* are so-called generalized spectral measures w.r.t. R (see also [3]).

Now assume that $L_r = L|_{\eta(f)^\perp}$ is a positive definite operator having a pure point spectrum $(\lambda_j)_{j \in \mathbb{N}}$; then, if $\langle Lf, f \rangle \geq C\|f\|^2$ for all

$f \in \text{dom}(L_R)$, there exists an orthonormal basis $(f_j)_{j \in \mathbb{N}}$ of eigenvectors of L_R with

$$L f_j = \lambda_j f_j, \quad j \in \mathbb{N}$$

$$0 < C \leq \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_j \leq \dots$$

and

$$\lim_{j \rightarrow \infty} \lambda_j = +\infty.$$

The following theorem may then easily be proved :

Theorem. Let f be a closed densely defined operator from H_1 into H_2 , such that, if $L = f^* f$, $L_R = L|\eta(f)|^{-1}$ is a self-adjoint positive definite operator having a pure point spectrum $(\lambda_j)_{j \in \mathbb{N}}$. Furthermore, let $(f_j)_{j \in \mathbb{N}}$ be a corresponding orthonormal basis consisting of eigenvectors of L_R , let R be the elementary operator associated with f and let $g_j = R f_j$, $j \in \mathbb{N}$. Then

- (i) $(g_j)_{j \in \mathbb{N}}$ is an orthonormal basis for $R(R) = R(f)$
- (ii) $\text{dom } f = \{f \in H_1 : \sum_{j \in \mathbb{N}} \lambda_j |\langle f, f_j \rangle|^2 < +\infty\}$
- (iii) $\text{dom } f^* = \{g \in H_2 : \sum_{j \in \mathbb{N}} \lambda_j |\langle g, g_j \rangle|^2 < +\infty\}$
- (iv) $\text{dom}(f^{-1}) = H_2$ and $\text{dom}(f^{*-1}) = H_1$
- (v) $f f = \sum_{j \in \mathbb{N}} \sqrt{\lambda_j} \langle f, f_j \rangle R f_j$, $f \in \text{dom}(f)$
- (vi) $f^* g = \sum_{j \in \mathbb{N}} \sqrt{\lambda_j} \langle g, g_j \rangle R^* g_j$, $g \in \text{dom}(f^*)$
- (vii) $f^{-1} g = \sum_{j \in \mathbb{N}} \frac{1}{\sqrt{\lambda_j}} \langle g, g_j \rangle R^* g_j$, $g \in H_2$
- (viii) $f^{*-1} f = \sum_{j \in \mathbb{N}} \frac{1}{\sqrt{\lambda_j}} \langle f, f_j \rangle R f_j$, $f \in H_1$
- (ix) f^{-1} and f^{*-1} are compact operators.

Corollary. For each $j \in \mathbb{N}$, $g_j = \frac{1}{\sqrt{\lambda_j}} f f_j$ and $f_j = \frac{1}{\sqrt{\lambda_j}} f^* g_j$. Moreover

$$R f = \sum_{j \in \mathbb{N}} \langle f, f_j \rangle g_j, \quad f \in H_1$$

and

$$R^* g = \sum_{j \in \mathbb{N}} \langle g, g_j \rangle f_j, \quad g \in H_2$$

Remark. In this context the results and examples of M.R. Hestenes in [6] should also be mentioned.

2. Elliptic systems of differential operators

In what follows $f = f(D)$ stands for an elliptic system of differential operators with constant coefficients in \mathbb{R}^n , i.e.

$$f(D) = [f_{jk}(D)] , \quad j = 1, \dots, M ; k = 1, \dots, N$$

whereby

$$(i) f_{jk}(D) = \sum_{|\alpha| \leq r_{jk}} c_{jk\alpha} \partial_{x_1^{\alpha_1} \dots x_n^{\alpha_n}}^{|\alpha|}$$

$$\text{with } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n , \quad |\alpha| = \sum_{j=1}^n \alpha_j , \quad c_{jk\alpha} \in \mathbb{C} , \quad r_{jk} \in \mathbb{N}$$

(ii) If $r = \max r_{jk}$,

$$\overset{\circ}{f}_{ij}(D) = \sum_{|\alpha|=r} c_{jk\alpha} \partial_{x_1^{\alpha_1} \dots x_n^{\alpha_n}}^r$$

and

$$\overset{\circ}{f}(D) = [\overset{\circ}{f}_{ij}(D)] ,$$

then the equation

$$\overset{\circ}{f}(iy) \overset{\circ}{f} = 0 , \quad \overset{\circ}{f} \in \mathbb{C}^{N \times 1} ,$$

admits for each $y \in \mathbb{R}^n \setminus \{0\}$ the unique solution $\overset{\circ}{f} = 0$.

Hereby $\overset{\circ}{f}(iy)$ is the matrix obtained from $\overset{\circ}{f}(D)$ by replacing

$$\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \text{ by } (iy_1)^{\alpha_1} , \quad 1 = 1, \dots, n.$$

Putting $f^+(-D) = [f_{kj}^+(-D)]$ whereby

$$f_{kj}^+(-D) = \sum_{|\alpha| \leq r_{jk}} (-1)^{|\alpha|} \bar{c}_{jk\alpha} \partial_{x_1^{\alpha_1} \dots x_n^{\alpha_n}}^{|\alpha|}$$

we then have that

$$L(D) = f^+(-D) f(D)$$

is a strongly elliptic operator of order $2r$ (see [4]). Moreover, if for $\Omega \subset \mathbb{R}^n$ open and $J \in \mathbb{N}$, $L_{2,J}(\Omega)$ stands for the space of $\mathbb{C}^{J \times 1}$ -valued L_2 -functions in Ω , then we put $H_1 = L_{2,N}(\Omega)$, $H_2 = L_{2,M}(\Omega)$ and $V = \overset{\circ}{W}_{2,N}^r(\Omega)$.

In the sequel we assume that $r = 1$, i.e. $L(D)$ is a second order strongly elliptic operator, that Ω is bounded and of the class C^1 and that the Dirichlet problem for the operator $L(D)$ is well-posed in $\mathcal{N} \subset L_{2,N}(\Omega)$.

Taking $\text{dom}(f) = W_{2,N}(\Omega)$, we thus obtain that $L = f^* f$ is a positive definite self-adjoint operator with $\text{dom}(L) = \mathcal{N}$ (see also [2]). Moreover, as the embedding of $\overset{\circ}{W}_{2,N}^r(\Omega)$ into $L_{2,N}(\Omega)$ is compact, L is

an operator having a pure point spectrum, whence the results from section 1 may be applied. Hence we have a.o. that $f^{-1} : L_{2,M}(\Omega) \rightarrow L_{2,N}(\Omega)$ is a bounded operator such that for each $g \in L_{2,M}(\Omega)$

$$f^{-1}(g) = \sum_{j \in \mathbb{N}} \frac{1}{\lambda_j} \langle g, f f_j \rangle f_j \quad (*)$$

$(f_j)_{j \in \mathbb{N}}$ being an orthonormal basis of $L_{2,N}(\Omega)$ consisting of eigenfunctions of L with corresponding eigenvalues λ_j .

3. The gradient operator in \mathbb{R}^3

3.1 Take Ω to be the unit ball B in \mathbb{R}^3 , $H_1 = L_2(B)$, $H_2 = L_{2,3}(B)$

and $f(D) = \text{grad} = \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{bmatrix}$, with $\text{dom}[f(D)] = V = \dot{W}_2^4(B)$. Then

$$f^+(-D) = - \begin{bmatrix} \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \end{bmatrix} = -\text{div} \quad \text{and} \quad L(D) = f^+(-D) \cdot f(D) = (-\Delta).$$

An orthogonal basis of $L_2(B)$ consisting of eigenfunctions of $(-\Delta)$ is given (using spherical co-ordinates) by :

$$u_{1,m,k} = e^{i \sin \varphi} P_1^m(\cos \theta) r^{-1/2} J_{1+\frac{1}{2}}(\mu_k^{(1+\frac{1}{2})} r),$$

$$l = 0, 1, 2, \dots; \quad m = 0, 1, \dots, l; \quad k = 1, 2, \dots,$$

the corresponding eigenvalues being given by

$$\lambda_{1,m,k} = \left[\mu_k^{(1+\frac{1}{2})} \right]^2,$$

where $\mu_k^{(1+\frac{1}{2})}$, $k = 1, 2, \dots$, represent the positive zeros of the Bessel function $J_{1+\frac{1}{2}}$.

Putting

$$\begin{aligned} a_{1,m,k}^2 &= \langle u_{1,m,k}, u_{1,m,k} \rangle \\ &= \frac{2\pi}{2l+1} \frac{(l+m)!}{(l-m)!} \left[J_{1+\frac{1}{2}}(\mu_k^{(1+\frac{1}{2})}) \right]^2 \end{aligned}$$

and

$$b_{1,m,k} = \langle g, \text{grad} u_{1,m,k} \rangle, \quad g \in L_{2,3}(B),$$

the unique solution f in V of the system

$$\text{grad} f = g, \quad g \in L_{2,3}(B)$$

is given, accordingly to formula (*) of the previous section, by

$$f = f^{-1}(g) = \sum_{l=0}^{\infty} \sum_{n=0}^l \sum_{k=1}^{\infty} \frac{1}{a_{l,m,k}^2} \frac{1}{\lambda_{l,m,k}} b_{l,m,k} u_{l,m,k}.$$

Given an arbitrary $g \in L_{2,3}(B)$ the computation of the solution f by the above formula is practically unfeasible. Therefore we developed REDUCE programs, which take over the by hand calculations; we used the version 3.2 of REDUCE [5] implemented on a VAX 750 computer. For a brief introduction on the nature of REDUCE see also [1].

3.2 The correctitude of the REDUCE programs had first to be checked on a case where the computation of the solution was possible by hand.

Therefore we focussed on the special case where $g \in L_{2,3}(B)$ is spherically symmetric, i.e. has the specific form $g = g(r)e_r$. In this case the constants $b_{l,m,k}$ are easily seen to be zero unless $l = m = 0$, which reduces the form the solution takes to

$$f = f^{-1}(g) = \frac{\sqrt{2}}{4\pi} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} b_{0,0,k} \frac{\sin(k\pi r)}{k\pi r}$$

In this way the (known) potential of the unit ball B with homogeneous electrical charge may easily be computed. The electrostatic field is radial and of magnitude proportional to the distance to the origin, say $g = re_r$. This yields for the potential vanishing on the sphere ∂B :

$$v = -f = \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \frac{\sin(k\pi r)}{k\pi r}$$

where the series is uniformly convergent in $[0,1]$.

From the Fourier series of the function $(r^3 - r)$ it follows at once that the obtained series converges to the function $(1-r^2)/2$, of course the expected potential.

Our REDUCE program calculated exactly the terms of the above series; we show the first seven terms :

```
term(0,0,1) := (- 6*SIN(R*PI))/(R*PI**3)$
term(0,0,2) := (3*SIN(2*R*PI))/(4*R*PI**3)$
term(0,0,3) := (- 2*SIN(3*R*PI))/(9*R*PI**3)$
term(0,0,4) := (3*SIN(4*R*PI))/(32*R*PI**3)$
term(0,0,5) := (- 6*SIN(5*R*PI))/(125*R*PI**3)$
term(0,0,6) := SIN(6*R*PI)/(36*R*PI**3)$
term(0,0,7) := (- 6*SIN(7*R*PI))/(343*R*PI**3)$
```

3.3 Next by the same REDUCE program we solved the system

$$\text{grad } f = g, \quad f \in V$$

for $g = 2r \cos\theta e_r + (1-r^2)/r \sin\theta e_\theta$, the solution of which is seen to be $f = (r^2-1)\cos\theta$. We computed the first seven terms of the series solution and found, after having introduced the zeros of $J_{3/2}$ with their numerical values :

```
term(1,0,1) := (COS(TH)*(- 0.0875874*SIN(4.493409*R) +
                0.3935659*COS( 4.493409*R)*R))/R**2$
term(1,0,2) := (COS(TH)*(- 0.0059394*SIN(7.725252*R) +
                0.0458836*COS( 7.725252*R)*R))/R**2$
term(1,0,3) := (COS(TH)*(- 0.0040778*SIN(10.90412*R) +
                0.0444648*COS( 10.90412*R)*R))/R**2$
term(1,0,4) := (COS(TH)*(- 0.0011643*SIN(14.06619*R) +
                0.0163777*COS( 14.06619*R)*R))/R**2$
term(1,0,5) := (COS(TH)*(- 0.0009338*SIN(17.22075*R) +
                0.0160806*COS( 17.22075*R)*R))/R**2$
term(1,0,6) := (COS(TH)*(- 0.0004089*SIN(20.3713*R) +
                0.0083294*COS( 20.3713*R)*R))/R**2$
term(1,0,7) := (COS(TH)*(- 0.0003496*SIN(23.51945*R) +
                0.008222*COS( 23.51945*R)*R))/R**2$
```

The value on the sphere ∂B of this partial sum turned out to be 0. with an error less than 10^{-6} .

3.4 We conclude this section by showing an excerpt of the REDUCE program to give a flavour of what it looks like. The complete programs can be obtained on simple request.

```
comment : This program computes the bessel-functions of order
        n+(1/2);
operator J;
J(1/2):=(2/(PI*z))**(1/2)*SIN(z);
J(3/2):=(2/(PI*z))**(1/2)*(SIN(z)/z - COS(z));
for i:=N1 step 2 until N2 do
<< begin scalar u;
    u:=i/2;
    J(u):=2*(u-1)*J(u-1)/z - J(u-2);
    end >>;
;end;
```

```
comment : this program computes the inner product of the
function G = GR e(r) + GT e(th) + GF e(fi)
with grad u(1,m,k);
```

```
procedure b(1,m);
  begin scalar e1,e2,e3,e4,e5,e6,e7,e8,e9;
    e1:= GR*dru(1,m)+(1/r)*GT*dthu(1,m)+(GF*dfiu(1,m))/(r*SIN(TH));
    e2:=INT(e1,FI);
    e3:=sub(FI=2*PI,e2) - sub(FI=0,e2);
    e4:=SIN(TH)*e3;
    e5:=INT(e4,TH);
    e6:=sub(TH=PI,e5) - sub(TH=0,e5);
    e7:=r**2*e6;
    e8:=INT(e7,r);
    e9:=sub(r=1,e8) - hosp(e8);
    return e9
  end;
;end;
```

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