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BLATTNER-KOSTANT-STERNBERG PAIRING AND FOURIER TRANSFORM ON SYMMETRIC SPACES

Wojciech Lisiecki

Abstract We show that Fourier transform on a symmetric space $X = G/K$ with G complex semisimple coincides with the operator given by geometric quantization that intertwines the quantizing Hilbert spaces associated with the vertical polarization and some other G-invariant polarization of T^*X.

0. Introduction

Let X be a Riemannian symmetric space of the noncompact type, that is, a coset space $X = G/K$, where G is a connected semisimple Lie group with finite center and K a maximal compact subgroup. Then there is a natural unitary representation of G on $L^2(X,dx)$ (dx being a G-invariant measure on X). Utilizing deep results of Harish-Chandra, Helgason showed that this representation decomposes into a direct integral of representations belonging to the spherical principal series (see [H] and [Wa]). This decomposition is obtained by means of a suitable Fourier transform, which is a natural generalization of the Fourier transform on $\mathbb{R}^n$. This transform maps a compactly supported smooth function f on X to a function $\tilde{f}$ on $B \times \mathfrak{a}^*_+$, where B is the real flag manifold, and $\mathfrak{a}^*_+$ is a dual Weyl chamber, given by

$$\text{(0.F)} \qquad \tilde{f}(b, \lambda) = \int_X f(x)e^{\langle -i\lambda + \rho , A(x,b)\rangle}dx , \qquad b \in B,\ \lambda \in \mathfrak{a}^*_+$$

(see 1.A below for all unexplained notations used in this introduction). Helgason showed that $f \longmapsto \tilde{f}$ extends to a unitary isomorphism of $L^2(X,dx)$ onto $L^2(B \times \mathfrak{a}^*_+, db|c(\lambda)|^{-2}d\lambda)$, where db is a K-invariant measure on B normalized such that the total measure is 1, $d\lambda$ is a suitably normalized Lebesgue measure on $\mathfrak{a}^*_+$ and $c(\lambda)$ is the so so called Harish-Chandra c-function.

The aim of the present paper is to obtain the Fourier transform $f \longmapsto \tilde{f}$ by means of geometric quantization. From the point of view of that theory the representation of G on $L^2(X,dx)$ "quantizes" the natural Hamiltonian action of G on the cotangent bundle T^*X. More precisely, $L^2(X,dx)$ is naturally isomorphic to the quantizing Hilbert space associated with the vertical polarization $\tau : T^*X \longrightarrow X$. By analogy with the Fourier transform on $\mathbb{R}^n$, $f \longmapsto \tilde{f}$ should be the operator which intertwines $L^2(X,dx)$ with the quantizing Hilbert space associated with another G-invariant real polarization whose space of leaves should be $B \times \mathfrak{a}^*_+$. A construction of this polarization is suggested by looking at the symplectic analog of the direct integral decomposition of $L^2(X,dx)$. To be more precise, the momentum mapping $J: T^*X \longrightarrow \mathfrak{g}^*$ induces a 1-1 correspondence between maximal dimensional G-orbits in T^*X and regular hyperbolic coadjoint orbits in $\mathfrak{g}^*$, which correspond, via geometric quantization, to representations of the spherical principal series. These representations are constructed using G-invariant real polarizations. We can fix on each of the orbits such polarization so that it "depends smoothly on the orbit". Taking inverse images under J of the leaves of so fixed polarizations we obtain a G-invariant real polarization π of $(T^*X)'$ (the union of the maximal dimensional orbits), which has the desired properties. We carry out the construction of π in §3, having analyzed, in §2, the orbit structure of T^*X. Moreover, we show that (τ,π): $(T^*X)' \longrightarrow X \times B \times \mathfrak{a}^*_+$ is a diffeomorphism. In §4 we show that π has a generating function S of the form $S(x,b,\lambda) = \langle \lambda, A(x,b) \rangle$.

Given a pair of polarizations, we can construct the so called Blattner-Kostant-Sternberg pairing, which in some cases leads to a unitary operator intertwining the quantizing Hilbert spaces associated with these polarizations. It turns out that applying this pairing construction to (τ,π) gives correct result only for complex G. §§5,6 and 7 are devoted to the computation of the BKS-pairing under this additional assumption on G. More precisely, in §5 we compute the Liouville form on $X \times B \times \mathfrak{a}^*_+$, in §6 we describe the quantizing Hilbert spaces associated with τ and π, and finally in §7 we obtain an explicit formula for the BKS pairing and conclude that the corresponding intertwining operator coincides with the Fourier transform $f \longmapsto \tilde{f}$.

We only sketch the proofs of main results; detailed proofs will appear elsewhere.

1. Preliminaries

1.A. Notation

The following standard notation concerning semisimple Lie groups will be used throughout the paper (with the exception of subsections 1.C and 1.D).

G denotes a (noncompact) connected semisimple Lie group with finite center. In §§5,6 and 7 we assume additionally that G is complex. The identity of G is denoted by e.

$\mathfrak{g}$ denotes the Lie algebra of G.

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a fixed Cartan decomposition of $\mathfrak{g}$.

$\mathfrak{a}$ is a fixed maximal Abelian subspace of $\mathfrak{p}$, $l = \dim \mathfrak{a}$.

$\mathfrak{m}$ = centralizer of $\mathfrak{a}$ in $\mathfrak{k}$

R = set of restricted roots of $(\mathfrak{g}, \mathfrak{a})$; for $\alpha \in R$, $\mathfrak{g}_\alpha$ is the corresponding root space, and $m_\alpha = \dim \mathfrak{g}_\alpha$ ($\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$ is the root space decomposition of $\mathfrak{g}$).

W is the Weyl group of R; $|W|$ denotes its order.

$\mathfrak{a}^*_+$ is a fixed Weyl chamber in the dual $\mathfrak{a}^*$ of $\mathfrak{a}$.

R_+ = subset of positive roots corresponding to $\mathfrak{a}^*_+$

$$\rho = \frac{1}{2}\sum_{\alpha \in R_+} m_\alpha \alpha$$

$$\mathfrak{n} = \sum_{\alpha \in R_+} \mathfrak{g}_\alpha \; , \; m = \dim \mathfrak{n} \; \left(= \sum_{\alpha \in R_+} m_\alpha \right)$$

$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is the Iwasawa decomposition of $\mathfrak{g}$.

K is the analytic subgroup of G with Lie algebra $\mathfrak{k}$ (a maximal compact subgroup of G); θ is the Cartan involution of G (fixing the elements of K).

A = $\exp \mathfrak{a}$, $\log: A \longrightarrow \mathfrak{a}$ is the inverse of $\exp: \mathfrak{a} \longrightarrow A$.

N = $\exp \mathfrak{n}$

G = KAN is the Iwasawa decomposition of G.

$H: G \longrightarrow \mathfrak{a}$ is the map given by $H(kan) = \log(a)$.

M = centralizer of A in K

MAN is a minimal parabolic subgroup of G (its Lie algebra equals $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$).

X = G/K (Riemannian symmetric space of the noncompact type), o = eK (the "origin" of X).

B = G/MAN = K/M (real flag manifold), b_0 = eMAN = eM.
Note that $\dim X = m + l$, $\dim B = m$.

$(x,b) \longmapsto A(x,b)$ is a $\mathfrak{a}$-valued function on $X \times B$ defined by the formula $A(x,b) = -H(g^{-1}k)$, where $x = g\cdot o$, $g \in G$, $b = k\cdot b_0$, $k \in K$.

1.B. Hyperbolic coadjoint orbits

The dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ is a G-module with respect to the coadjoint action of G given by

$$\langle Ad^*(g)f, \xi \rangle = \langle f, Ad(g^{-1})\xi \rangle,$$

where $g \in G$, $f \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$, and Ad denotes the adjoint representation of G in $\mathfrak{g}$.

(1.1) For each $f \in \mathfrak{g}^*$, we denote by B_f the skew symmetric bilinear form on $\mathfrak{g} \times \mathfrak{g}$ defined by $B_f(\xi, \eta) = -\langle f, [\xi, \eta]\rangle$. It gives rise to a G-invariant symplectic form $\omega_{\mathcal{O}}$ on the orbit $\mathcal{O}$ through f, which will be called the Kirillov form of $\mathcal{O}$.

(1.2) The Killing form of $\mathfrak{g}$ induces a G-equivariant isomorphism $\mathfrak{g}^* \longrightarrow \mathfrak{g}$, $f \mapsto f^\#$. An element $f \in \mathfrak{g}^*$ is called hyperbolic if $f^\#$ is so (that is, $ad(f^\#)$ is semisimple and has all real eigenvalues). We write $\mathfrak{g}^*_h$ for the set of hyperbolic elements. A coadjoint orbit is called hyperbolic if one (and hence any) of its elements is hyperbolic.

(1.2.1) Each hyperbolic orbit is a closed submanifold of $\mathfrak{g}^*$ (see [V], Part I, §1).

Let $\mathfrak{k}^\perp$ be the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$. Then we have the following.

(1.2.2) $\mathcal{O} \subset \mathfrak{g}^*_h$ iff $\mathcal{O} \cap \mathfrak{k}^\perp \neq \emptyset$,

and there is a bijection of orbit spaces

(1.2.3) $\mathfrak{g}^*_h/G \xrightarrow{\sim} \mathfrak{k}^\perp/K$, $\mathcal{O} \longmapsto \mathcal{O} \cap \mathfrak{k}^\perp$.

(1.3) Due to the root space decomposition of $\mathfrak{g}$ we have a natural imbedding $\mathfrak{a}^* \longrightarrow \mathfrak{k}^\perp$. Let $Cl(\mathfrak{a}^*_+)$ denote the closure of $\mathfrak{a}^*_+$ in $\mathfrak{a}^*$ and, for each $\lambda \in Cl(\mathfrak{a}^*_+)$, put $\mathcal{O}_\lambda = Ad^*(G)\lambda$. Then the mapping

(1.3.1) $Cl(\mathfrak{a}^*_+) \longrightarrow \mathfrak{g}^*_h/G$, $\lambda \longmapsto \mathcal{O}_\lambda$,

is a bijection. The orbits $\mathcal{O}_\lambda$ with $\lambda \in \mathfrak{a}^*_+$ will be called regular. The union of the regular orbits will be denoted by $(\mathfrak{g}^*_h)'$. The stabilizer of each $\lambda \in \mathfrak{a}^*_+$ equals MA, so that each $\mathcal{O}_\lambda \subset (\mathfrak{g}^*_h)'$ is G-isomorphic to G/MA. Moreover, each $\mathcal{O}_\lambda \subset (\mathfrak{g}^*_h)'$, being semisimple, has a G-invariant tubular neighborhood in $\mathfrak{g}^*$ ([V], Part I, §1). It follows that $(\mathfrak{g}^*_h)'$ is a submanifold of $\mathfrak{g}^*$ (of codimension $\dim \mathfrak{m}$) and the orbit space $(\mathfrak{g}^*_h)'/G$ has a natural manifold structure. Since $\mathfrak{a}^*_+$ intersects each orbit in $(\mathfrak{g}^*_h)'$ at a single point and transversely, the restriction of (1.3.1) to $\mathfrak{a}^*_+$ induces a diffeomorphism

(1.3.2) $$\mathfrak{a}^*_+ \xrightarrow{\sim} (\mathfrak{g}^*_h)'/G,$$

and the map

(1.3.3) $G/MA \times \mathfrak{a}^*_+ \longrightarrow (\mathfrak{g}^*_h)'$, $(gMA, \lambda) \mapsto Ad^*(g)\lambda$,

is a G-equivariant diffeomorphism ($\mathfrak{a}^*_+$ being considered as a trivial G-space).

The Kirillov form of $\mathcal{O}_\lambda \subset (\mathfrak{g}_n^*)'$ will be denoted by ω_λ rather than $\omega_{\mathcal{O}_\lambda}$.

1.C. An outline of geometric quantization

Let (P, ω) be a symplectic manifold.

(1.4) A prequantization of (P, ω) is a triple $(L, \langle , \rangle, \nabla)$, where L is a complex line bundle over P, $\langle , \rangle$ is a Hermitian inner product on L and ∇ is a metric connection on L whose curvature form is $-i\omega$. (P, ω) admits a prequantization iff the deRham cohomology class of ω is integral. If this is the case, the isomorphism classes of prequantizations of (P, ω) are in 1-1 correspondence with the characters of the fundamental group of P. See [Ko] for details.

(1.5) Given a Hamiltonian action (see [AM]) of a connected Lie group G on (P, ω), there is a natural infinitesimal action of the Lie algebra of G on $(L, \langle , \rangle, \nabla)$ via infinitesimal automorphisms ([Ko], Th. 4.5.1). By a prequantization of the action of G on (P, ω) we mean its lift to an action on L inducing this infinitesimal action.

(1.6) By a (real) polarization of (P, ω) we mean in this paper a Lagrangian fibration $\tau : P \longrightarrow X$ (i.e. $\tau : P \longrightarrow X$ is a fiber bundle whose fibers (or leaves) are Lagrangian submanifolds of (P, ω)). Given a prequantization L and a polarization τ , the restriction $L|_{\tau^{-1}(x)}$ is a flat bundle for any $x \in X$. We say the leaf $\tau^{-1}(x)$ is quantizable if the holonomy group of $L|_{\tau^{-1}(x)}$ is trivial. To any quantizable leaf $\tau^{-1}(x)$ there is naturally associated a complex line L_x^τ consisting of covariant constant sections of $L|_{\tau^{-1}(x)}$. We will be assuming that all leaves of τ are quantizable. Then the disjoint union

$$L^\tau = \bigsqcup_{x \in X} L_x^\tau$$

has a natural structure of a Hermitian line bundle over X. The pull-back $\tau^* L^\tau$ is canonically isomorphic to L, and for any section s of of L^τ its pull-back $\tau^* s$ is a covariant constant along τ section of L, i.e.,

$$\nabla \tau^* s|_{\operatorname{Ker} T\tau} = 0.$$

Conversely, any covariant constant along τ section of L is of the form $\tau^* s$ for a unique section s of L^τ.

(1.7) Let $D^{\frac{1}{2}}(X)$ be the bundle of complex half-densities on X and let $C_0^{\infty}(L^{\tau}\otimes D^{\frac{1}{2}}(X))$ denote the space of compactly supported smooth sections of $L^{\tau}\otimes D^{\frac{1}{2}}(X)$. For $s_i\otimes\delta_i\in C_0^{\infty}(L^{\tau}\otimes D^{\frac{1}{2}}(X))$, $i = 1,2$, $\langle s_1,s_2\rangle\delta_1\otimes\bar{\delta}_2$ is a compactly supported smooth density on X, so the following formula makes sense

$$\langle s_1\otimes\delta_1,\ s_2\otimes\delta_2\rangle_{\tau} = \int_X \langle s_1,s_2\rangle\,\delta_1\otimes\bar{\delta}_2\ .$$

Since sections of the form $s\otimes\delta$ generate $C_0^{\infty}(L^{\tau}\otimes D^{\frac{1}{2}}(X))$, this formula defines a Hermitian inner product on $C_0^{\infty}(L^{\tau}\otimes D^{\frac{1}{2}}(X))$. The resulting pre-Hilbert space will be denoted by H_0^{τ}. The completion H^{τ} of H_0^{τ} is the <u>quantizing Hilbert space</u> associated with τ and L. The details of the above constructions can be found in [Bl], [GS] and [We]. We remark that in many cases half-densities should be replaced by half-forms, but for our purposes the "half-density quantization" described above is sufficient.

(1.8) A Hamiltonian action of a Lie group G on (P,ω) which preserves τ and prequantizes to an action on $(L,\langle,\rangle,\nabla)$ gives rise to a unitary representation of G on H^{τ}.

1.D. <u>BKS pairing</u>

Remaining in the setting of 1.C assume additionally that $\pi: P \longrightarrow Y$ is another polarization of (P,ω) which is <u>strongly transverse</u> to τ in the sense that the mapping $P \longrightarrow X\times Y$, $p\longmapsto(\tau(p),\pi(p))$, is a diffeomorphism. Let Φ be the inverse of (τ,π). It is convenient to work on $X\times Y$ rather than P. Thus we replace ω, L, τ, π by $\Phi^*\omega$, Φ^*L, p_X, p_Y respectively, the latter two being the Cartesian projections.

(1.9) Assume that X and Y admit volume elements μ_X and μ_Y respectively. Let $|\mu_X|^{\frac{1}{2}}$ and $|\mu_Y|^{\frac{1}{2}}$ be the corresponding half-densities (see [Bl], §3). By a pairing of these we mean the unique function $\langle|\mu_X|^{\frac{1}{2}},|\mu_Y|^{\frac{1}{2}}\rangle$ on $X\times Y$ such that

$$(1.9.1)\qquad (2\pi)^d d!\ p_X^*\mu_X\wedge p_Y^*\mu_Y = (\langle|\mu_X|^{\frac{1}{2}},|\mu_Y|^{\frac{1}{2}}\rangle)^2\,\Phi^*\omega^d,$$

where $2d = \dim P$, and where we assume that μ_X and μ_Y have been chosen such that the corresponding product orientation of $X\times Y$ coincides with that induced by $\Phi^*\omega^d$. Now the <u>BKS pairing</u> (named so for Blattner, Kostant and Sternberg) of $s\otimes|\mu_X|^{\frac{1}{2}}\in H_0^{\tau}$ and $t\otimes|\mu_Y|^{\frac{1}{2}}\in H_0^{\pi}$ is given by

(1.9.2) $\langle s \otimes |\mu_X|^{\frac{1}{2}}, t \otimes |\mu_Y|^{\frac{1}{2}} \rangle_{\pi\tau} =$

$$= ((2\pi)^d d!)^{-1} \int_{X \times Y} \langle p_X^* s, p_Y^* t \rangle \langle |\mu_X|^{\frac{1}{2}}, |\mu_Y|^{\frac{1}{2}} \rangle |\Phi^* \omega^d|$$

$$= \int_{X \times Y} \langle p_X^* s, p_Y^* t \rangle (\langle |\mu_X|^{\frac{1}{2}}, |\mu_Y|^{\frac{1}{2}} \rangle)^{-1} |p_X^* \mu_X \wedge p_Y^* \mu_Y|,$$

where we write $|\mu|$ for the density corresponding to a volume element μ. This formula defines a sesquilinear form on $H_0^\tau \times H_0^\pi$, which we will call the <u>BKS pairing between</u> H_0^τ <u>and</u> H_0^π. See [Bl] and [GS] for a definition of this pairing in more general situation.

(1.10) We say τ and π are <u>unitarily related</u> if there is a unitary isomorphism $U_{\pi\tau} : H^\tau \longrightarrow H^\pi$ such that $\langle U_{\pi\tau} h, k \rangle_\pi = \langle h, k \rangle_{\pi\tau}$ for any $h \in H_0^\tau$ and any $k \in H_0^\pi$. The problem of characterizing pairs of unitarily related polarizations remains open.

(1.11) If we are in the situation of (1.8), and π is also G-invariant, the BKS pairing is G-invariant. Thus if τ and π are unitarily related, $U_{\pi\tau}$ is a (unitary) intertwining operator for the representations of G on H^τ and H^π.

2. <u>Orbit structure of T^*X</u>

(2.1) Let T^*X be the cotangent bundle to X, θ_X the canonical one-form on T^*X and $\omega_X = d\theta_X$ the canonical symplectic structure. The action of G on X lifts to an action by vector bundle automorphisms on T^*X. This lifted action preserves θ_X hence it is Hamiltonian, with momentum mapping $J: T^*X \longrightarrow \mathfrak{g}^*$ being the composition $T^*X \longrightarrow \mathfrak{g}^* \times X \longrightarrow \mathfrak{g}^*$ of the vector bundle morphism dual to the infinitesimal action of $\mathfrak{g}$ on X and the Cartesian projection onto the first factor. In particular, $J|_{T_o^*X}$ is the natural isomorphism $T_o^*X \xrightarrow{\sim} \mathfrak{h}^\perp$. Since J is G-invariant, its image $J(T^*X)$ is a G-invariant subset of $\mathfrak{g}^*$. It is clear from the above that a coadjoint orbit is contained in $J(T^*X)$ iff it has a nonempty intersection with $\mathfrak{h}^\perp$. Together with (1.2.2) and (1.2.3) this yields the following.

(2.2) <u>Proposition</u>. (i) $J(T^*X) = \mathfrak{g}_h^*$.

(ii) J induces a bijection of orbit spaces $T^*X/G \xrightarrow{\sim} \mathfrak{g}_h^*/G$. Hence G-orbits in T^*X are of the form $J^{-1}(\mathcal{O})$, where $\mathcal{O}$ is a coadjoint orbit in $\mathfrak{g}_h^*$.

From (ii) above and (1.2.1) we get

(2.3) <u>Proposition</u>. Each G-orbit in T^*X is a closed coisotropic submanifold.

(2.4) Let us put

$$(T^*X)' = J^{-1}((\mathfrak{g}_h^*)')$$

(see (1.3) for the definition of $(\mathfrak{g}_h^*)'$). This is a G-invariant connected open and dense subset of T^*X. It inherits the structure of a Hamiltonian G-space and we shall continue to write J for its mommentum mapping, as well as for the induced mapping $(T^*X)' \longrightarrow (\mathfrak{g}_h^*)'$. All G-orbits in $(T^*X)'$ have the same type G/M and they are the maximal dimensional orbits in T^*X.

Noting that $(J|_{T_o^*X})^{-1}(\mathfrak{a}_+^*)$ intersects each orbit in $(T^*X)'$ at a single point and transversely we can easily prove the following.

(2.5) Proposition. (i) $J: (T^*X)' \longrightarrow (\mathfrak{g}_h^*)'$ is a G-equivariant fibration.

(ii) The orbit space $(T^*X)'/G$ has a natural manifold structure and the map $(T^*X)'/G \longrightarrow (\mathfrak{g}_h^*)'/G$ induced by J is a diffeomorphism.

In what follows, we shall identify both $(T^*X)'/G$ and $(\mathfrak{g}_h^*)'/G$ with $\mathfrak{a}_+^*$ (cf. (1.3.2)) and we shall write $\widetilde{\mathcal{O}}_\lambda$ for the G-orbit corresponding to $\lambda \in \mathfrak{a}_+^*$, that is, $\widetilde{\mathcal{O}}_\lambda = J^{-1}(\mathcal{O}_\lambda)$.

3. Horizontal polarization

(3.1) For each $\lambda \in \mathfrak{a}_+^*$, the map

(3.1.1) $$\mathcal{O}_\lambda \longrightarrow B, \ Ad^*(g)\lambda \longmapsto g\cdot b_0,$$

is a G-invariant real polarization of $\mathcal{O}_\lambda$ (cf. [OW]). Since $\mathcal{O}_\lambda$ is closed in $\mathfrak{g}^*$ (1.2.1), this polarization satisfies Pukanszky condition, i.e., each of its leaves Λ_b is an affine subspace of $\mathfrak{g}^*$, in particular

(3.1.2) $$\Lambda_{b_0} = \lambda + (\mathfrak{m} + \mathfrak{a} + \mathfrak{n})^\perp$$

(see [Be], Chap. IV, §3)

(3.2) The maps $\mathcal{O}_\lambda \longrightarrow B$ can be pieced together to give a smooth G-equivariant fibration

$$(\mathfrak{g}_h^*)' \longrightarrow B \times \mathfrak{a}_+^*.$$

More precisely, this fibration is defined as the map corresponding to $G/MA \times \mathfrak{a}_+^* \longrightarrow G/MAN \times \mathfrak{a}_+^*$, $(gMA, \lambda) \longmapsto (gMAN, \lambda)$ under the isomorphism (1.3.3). Define

$$\pi : (T^*X)' \longrightarrow B$$

as the composition $(T^*X)' \longrightarrow (\mathfrak{g}_h^*)' \longrightarrow B \times \mathfrak{a}_+^*$. This is a G-equivariant fibration. The fiber $\widetilde{\Lambda}_b$ over (b, λ) is

$$\widetilde{\Lambda}_b = \pi^{-1}(b, \lambda) = J^{-1}(\Lambda_b).$$

Since each $\widetilde{\mathcal{O}}_\lambda$ is coisotropic and since $J: \widetilde{\mathcal{O}}_\lambda \longrightarrow \mathcal{O}_\lambda$ is its symplectic reduction, the fibers $\widetilde{\Lambda}_b$ are Lagrangian submanifolds of $(T^*X)'$. This proves part of the following.

(3.3) Proposition. $\pi : (T^*X)' \longrightarrow B \times \mathfrak{a}_+^*$ is a G-invariant real polarization of $(T^*X)'$ with the following properties:

(a) for each $p \in (T^*X)'$, the leaf of π through p is contained in the G-orbit through p,

(b) π is strongly transverse to the vertical polarization $\tau : (T^*X)' \longrightarrow X$ (cf. 1.D).

Property (a) follows directly from the definition of π. As for (b), since both polarizations are G-invariant and since the restriction of J to T_o^*X is an isomorphism onto $\mathfrak{k}^\perp$, it suffices to note that, in virtue of (3.1.2) and Iwasawa decomposition of $\mathfrak{g}$, $\Lambda_{b_0} \cap \mathfrak{k}^\perp = \{\lambda\}$ and $T_\lambda \Lambda_{b_0} \cap \mathfrak{k}^\perp = \{0\}$.

π will be called the horizontal polarization of $(T^*X)'$.

(3.4) Remark. It can be shown that $(T^*X)'$ has exactly $|W|$ G-invariant real polarizations satisfying (a) of (3.3). They are constructed in the same way as π was, but with (3.1.1) replaced by any other of the $|W|$ G-invariant real polarizations of $\mathcal{O}_\lambda$. Hence they satisfy also (b). All the following statements concerning π hold equally well for any of these polarizations.

4. Generating function of the horizontal polarization

(4.1) It follows from (3.3) (b) that each leaf $\widetilde{\Lambda}_b$ of π projects diffeomorphically onto X. Therefore there is a unique closed 1-form $\widetilde{\lambda}_b$ on X such that $\widetilde{\Lambda}_b = \widetilde{\lambda}_b(X)$ (we consider $\widetilde{\lambda}_b$ as a mapping $X \longrightarrow T^*X$). Since each closed 1-form on X is exact, there exists a function $S_{b,\lambda} : X \longrightarrow \mathbb{R}$ such that $\widetilde{\lambda}_b = dS_{b,\lambda}$. It is clear that these $S_{b,\lambda}$ can be chosen such that the function $S : X \times B \times \mathfrak{a}_+^* \longrightarrow \mathbb{R}$ given by $S(x,b,\lambda) = S_{b,\lambda}(x)$ is smooth. Such S is called a generating function of π (cf. [Wo], 4.6). It is determined by π up to the addition of an arbitrary function of (b,λ). In what follows, S will denote the unique generating function of π which vanishes on $\{o\} \times B \times \mathfrak{a}_+^*$.

(4.2) Theorem. S is given by

$$S(x,b,\lambda) = \langle \lambda, A(x,b) \rangle,$$

where, for $x = g \cdot o$, $g \in G$, and $b = k \cdot b_0$, $k \in K$, $A(x,b) = -H(g^{-1}k)$.

We sketch the proof. It is clear that

$$S(x,b,\lambda) = \int_o^x \widetilde{\lambda}_b \ ,$$

where the integral is along any path from o to x. Fix x and take $b = b_0$. The group AN acts transitively on X and leaves $\tilde{\lambda}_{b_0}$ invariant. From this one can easily deduce that $\tilde{\lambda}_{b_0}$ vanishes on each orbit of N. Since the action of AN on X is also free, there is a unique $a \in A$ such that $A\cdot o \cap N\cdot x = \{a\cdot o\}$. Take a path from o to x consisting of two pieces: $[0, 1] \longrightarrow A\cdot o$, $t \longmapsto (\exp(t\log(a)))\cdot o$, from o to $a\cdot o$ and an arbitrary path from $a\cdot o$ to x in $N\cdot x$. The integral of $\tilde{\lambda}_{b_0}$ over this path reduces to the integral over the first piece, which is easily seen to be equal $\langle\lambda, -H(g^{-1})\rangle$. Now to conclude the proof, it suffices to note that S is K-invariant.

(4.3) From G-invariance of π we obtain the following transformation rule of A under the action of G

$$A(g\cdot x, g\cdot b) = A(x,b) - A(g^{-1}\cdot o, b).$$

(4.4) Let $\Phi: X \times B \times \mathfrak{a}^*_+ \longrightarrow (T^*X)'$ be the inverse of (τ,π) (cf. (3.3) (b)). It is clear that

$$\Phi(x,b,\lambda) = \tilde{\lambda}_b(x) = dS_{b,\lambda}(x).$$

We can use Φ to transfer the structure of a Hamiltonian G-space to $X \times B \times \mathfrak{a}^*_+$. The pull-backs of the canonical forms θ_X and ω_X can be expressed in terms of derivatives of S, which will prove useful later on. Write Y for $B \times \mathfrak{a}^*_+$. Then the exterior derivative on $X \times Y$ decomposes as $d = d_X + d_Y$, where d_X (resp. d_Y) is the exterior derivative in the direction of X (resp. Y). Now it follows directly from the definitions of θ_X, ω_X and Φ that

$$\Phi^*\theta_X = d_X S \quad \text{and} \quad \Phi^*\omega_X = dd_X S.$$

When transferred to $X \times Y$, the polarizations τ and π become the Cartesian projections p_X and p_Y, respectively.

5. Liouville form on $X \times B \times \mathfrak{a}^*_+$

A decisive step in finding the BKS pairing consists in a computation of the Liouville form on $X \times B \times \mathfrak{a}^*_+$. We will do it now under the additional assumption that G is complex. In the first subsection, however, we work still without this assumption.

(5.1) Let $(e_1,\ldots,e_l)$ be a basis in $\mathfrak{a}$ and let $(e^1,\ldots,e^l)$ be the dual basis in $\mathfrak{a}^*$. The imbedding $\mathfrak{a}^* \longrightarrow \mathfrak{k}^\perp$ allows us to treat the e^i as elements of $\mathfrak{g}^*$. If $A^i(x,b)$ (resp. λ_i) are the coordinates of $A(x,b)$ (resp. λ) with respect to those bases, the formula for S (cf. (4.2)) reads

$$S(x,b,\lambda) = \sum_{i=1}^{l} \lambda_i A^i(x,b).$$

Hence the canonical forms on $X \times B \times \mathfrak{a}_+^*$ are given by (cf. (4.4))

$$d_X S = \sum_{i=1}^{l} \lambda_i d_X A^i,$$

$$dd_X S = \sum_{i=1}^{l} d\lambda_i \wedge d_X A^i + \sum_{i=1}^{l} \lambda_i dd_X A^i.$$

It is easy to see that, for each G-orbit $X \times B \times \{\lambda\} = \Phi^{-1}(\widetilde{\mathcal{O}}_\lambda)$,

(5.1.1) $$\left(\sum_{i=1}^{l} \lambda_i dd_X A^i\right)\Big|_{X \times B \times \{\lambda\}} = dd_X S\big|_{X \times B \times \{\lambda\}} = \widetilde{\omega}_\lambda,$$

where $\widetilde{\omega}_\lambda$ is the pull-back of the Kirillov form ω_λ by the mapping $X \times B \times \{\lambda\} \longrightarrow \mathcal{O}_\lambda$ induced by the momentum mapping. It follows that the rank of $\sum_{i=1}^{l} \lambda_i dd_X A^i$ equals $2m$ ($=\dim \mathcal{O}_\lambda$). Thus the Liouville form

(5.1.2) $$(dd_X S)^{m+l} = \binom{m+l}{l}\left(\sum_{i=1}^{l} d\lambda_i \wedge d_X A^i\right)^l \wedge \widetilde{\omega}_\lambda^m$$

$$= (-1)^{l(l+1)/2}\, l!\binom{m+l}{l}(d_X A^1 \wedge \ldots \wedge d_X A^l) \wedge \widetilde{\omega}_\lambda^m \wedge (d\lambda_1 \wedge \ldots \wedge d\lambda_l)$$

(with a slight abuse of notation). Put

(5.1.3) $$\delta_\lambda = (-1)^{l(l+1)/2}\, l!\binom{m+l}{l}(d_X A^1 \wedge \ldots \wedge d_X A^l) \wedge \widetilde{\omega}_\lambda^m.$$

This is a G-invariant $(2m+l)$-form, which may be considered as a form on $X \times B$ depending on the parameter λ. Let $r_{(o,b_0)}: G \longrightarrow X \times B$, $g \longmapsto (g\cdot o, g\cdot b_0)$ be the orbital mapping at (o,b_0). A simple calculation yields

(5.1.4) $$(r^*_{(o,b_0)}\, \delta_\lambda)_e = (-1)^{l(l+1)/2}\, l!\binom{m+l}{l}(e^1 \wedge \ldots \wedge e^l) \wedge B_\lambda^m.$$

(5.2) From now on we assume that G is (the underlying real group of) a complex (connected semisimple) Lie group. Under this assumption $\mathfrak{h} = \mathfrak{m} + \mathfrak{a}$ is a Cartan subalgebra of the complex Lie algebra $\mathfrak{g}$ and the restiction map $\mathfrak{h}^* \longrightarrow \mathfrak{a}^*$ estabishes a bijection of the set of roots of $(\mathfrak{g}, \mathfrak{h})$ onto R. Put $n = |R_+|$, so that $m = 2n$. Let $(X_\alpha)_{\alpha \in R}$ be a Chevalley system of $(\mathfrak{g}, \mathfrak{h})$ (see [Bo1], Chap. VIII, §3) and let $H_\alpha = -[X_\alpha, X_{-\alpha}]$. The vectors $u_\alpha = X_\alpha + X_{-\alpha}$, $v_\alpha = i(X_\alpha - X_{-\alpha})$, $\alpha \in R_+$, together with $\mathfrak{m}$ span a compact real form of $\mathfrak{g}$ (cf. [Bo2], Chap. IX, §3). We assume, as we may, that this coincides with $\mathfrak{k}$. Then $H_\alpha \in \mathfrak{a}$ and $\mathfrak{p} = i\mathfrak{k}$. The vectors u_α, v_α and $s_\alpha = iu_\alpha$, $t_\alpha = iv_\alpha$, $\alpha \in R_+$, form a basis of the orthogonal complement (with respect to the Killing form) of $\mathfrak{h}$. Let u^α, v^α, s^α, t^α, $\alpha \in R_+$, form the dual basis. Extend these to functions on the whole $\mathfrak{g}$ putting 0 on $\mathfrak{h}$. A straightforward calculation using the commutation relations satisfied by the X_α yields

$$B_\lambda = 2\sum_{\alpha \in R_+} \langle \lambda, H_\alpha \rangle (v^\alpha \wedge s^\alpha - u^\alpha \wedge t^\alpha).$$

It follows that

$$(5.2.1)\qquad B_\lambda^{2n} = 2^{2n}(2n)!\prod_{\alpha\in R_+}\langle\lambda, H_\alpha\rangle(s^{\alpha_1}\wedge t^{\alpha_1}\wedge\ldots\wedge s^{\alpha_n}\wedge t^{\alpha_n})\wedge \wedge(v^{\alpha_1}\wedge u^{\alpha_1}\wedge\ldots\wedge v^{\alpha_n}\wedge u^{\alpha_n}),$$

where we have chosen some ordering of the positive roots.

(5.3) It is not hard to see that there exist a unique G-invariant volume element μ on X and a unique K-invariant volume element ν on B such that

$$(5.3.1)\quad (r_o^*\mu)_e = (-1)^{l(l+1)/2}c_X e^1\wedge\ldots\wedge e^l\wedge s^{\alpha_1}\wedge t^{\alpha_1}\wedge\ldots\wedge s^{\alpha_n}\wedge t^{\alpha_n},$$

$$(5.3.2)\quad ((r_{b_0}^K)^*\nu)_e = c_B v^{\alpha_1}\wedge u^{\alpha_1}\wedge\ldots\wedge v^{\alpha_n}\wedge u^{\alpha_n},$$

where $r_o: G\longrightarrow X$ and $r_{b_0}^K: K\longrightarrow B$ denote the orbital mappings at o and b_0 respectively, and c_X and c_B are some positive real constants, which will be determined below. It is a standard result that ν transforms under the action of G according to

$$(5.3.3)\qquad g_B^*\nu = e^{\langle 2\rho, A(g^{-1}\cdot o,\cdot)\rangle}\nu\quad \forall g\in G,$$

where g_B denotes the diffeomorphism of B corresponding to g. Using this, the transformation rule of A (see (4.3)) and formulae (5.1.4) and (5.2.1) we get

$$(5.3.4)\quad \delta_\lambda = (2n+l)!2^{2n}(c_Xc_B)^{-1}\Big(\prod_{\alpha\in R_+}\langle\lambda, H_\alpha\rangle\Big)^2 e^{\langle 2\rho, A\rangle}p_X^*\mu\wedge p_B^*\nu,$$

where p_X and p_B stand for the Cartesian projections of $X\times B$ onto X and B respectively.

(5.4) In order to determine the constants c_X and c_B we must choose a normalization of invariant measures on G and some of its subgroups. We adopt the normalization used by Helgason (see [H], pp. 5-6). That is, the Haar measures on K and M are normalized such that the total measure is 1. This implies that

$$(5.4.1)\qquad \int_B\nu = 1.$$

The Haar measures on N and $\bar N = \theta(N)$ are normalized such that

$$(5.4.2)\qquad \theta(dn) = d\bar n,\quad \int_{\bar N} e^{\langle -2\rho, H(\bar n)\rangle}d\bar n = 1.$$

The Haar measure on A is the one corresponding under the exponential mapping to the Euclidean Lebesgue measure on $\mathfrak{a}$ (the Euclidean structure on $\mathfrak{a}$ being that induced by the Killing form) multiplied by the factor $(2\pi)^{-l/2}$. The Haar measure dg on G is normalized such that

$$(5.4.3)\qquad \int_G f(g)dg = \int_{K\times A\times N} f(kan)e^{\langle 2\rho, \log(a)\rangle}dk\, da\, dn .$$

These conditions determine a G-invariant measure on X.

Noting that B is K-isomorphic to the coadjoint K-orbit $\mathcal{O}_{-i\rho/2}$ in $\mathfrak{h}^*$ and that

$$\int_{\mathcal{O}_{-i\rho/2}} (\omega_{-i\rho/2})^n = 1,$$

which is a very special case of the Kirillov's character formula (see [Ki1], §3), we can show that in order to have (5.4.1) we should take

$$c_B = (2\pi)^{-n} \prod_{\alpha \in R_+} \langle \rho, H_\alpha \rangle. \tag{5.4.4}$$

In order to have also (5.4.2) we must assume that the basis $(e_1, \ldots, e_l)$ is orthonormal (with respect to the Euclidean structure induced by the Killing form) and take

$$c_X = (2\pi)^{-(n+l/2)} 2^{2n} \prod_{\alpha \in R_+} \langle \rho, H_\alpha \rangle. \tag{5.4.5}$$

The normalized volume elements μ and ν and the normalized Haar measure on M determine a Haar measure on G fof which (5.4.3) holds.

(5.5) It follows from the explicit formula for the Harish-Chandra c-function (see for instance [Wa], p. 326) that, in the case of complex G,

$$|c(\lambda)|^{-2} = \left(\prod_{\alpha \in R_+} \frac{\langle \lambda, H_\alpha \rangle}{\langle \rho, H_\alpha \rangle} \right)^2$$

(note that since H_α is the co-root associated with α, $\langle \lambda, H_\alpha \rangle = 2(\lambda, \alpha)(\alpha, \alpha)^{-1}$, where bracket denotes the scalar product on $\mathfrak{a}^*$ induced by that on $\mathfrak{a}$). It is clear from (5.3.4), (5.4.4) and (5.4.5) that $|c(\lambda)|^{-2}$ will appear as a multiplicative factor in the final expression for the Liouville form. It is convenient to include this factor in the definition of a volume element on $\mathfrak{a}_+^*$. More precisely, we take this volume element as

$$(2\pi)^{-l/2} |c(\lambda)|^{-2} d\lambda_1 \wedge \ldots \wedge d\lambda_l. \tag{5.5.1}$$

The following proposition, which is a direct consequence of (5.1.2), (5.1.3) and (5.3.4), summarizes the foregoing discussion.

(5.6) Proposition. Let μ and ν be the volume elements on X and B determined by (5.3.1), (5.4.5) and (5.3.2), (5.4.4) respectively and let the volume element on $\mathfrak{a}_+^*$ be as in (5.5.1). Then the Liouville form on $X \times B \times \mathfrak{a}_+^*$ is given by

$$(dd_X S)^{2n+l} = (2\pi)^{2n+l} (2n+l)! e^{\langle 2\rho, A \rangle} p_X^* \mu \wedge p_B^* \nu \wedge$$
$$\wedge (2\pi)^{-l/2} |c(\lambda)|^{-2} d\lambda_1 \wedge \ldots \wedge d\lambda_l,$$

where p_X and p_B denote now the Cartesian projections of $X \times B \times \mathfrak{a}^*_+$ onto X and B respectively.

6. Quantizing Hilbert spaces associated with τ and π

Results of this section hold without the assumption that G is complex. (As a matter of fact, we use below the volume elements on X, B and $\mathfrak{a}^*_+$ defined in the preceding section, but it is easy to see that such volume elements exist in the case of arbitrary G.)

(6.1) A natural prequantization of T^*X is the trivial line bundle $L = T^*X \times \mathbb{C}$ with the obvious inner product $\langle , \rangle$ and with a connection ∇ given by

$$\nabla F = dF - iF\theta_X,$$

where we identify sections of L with functions on T^*X. Since X is simply connected, any other prequantization of T^*X is isomorphic to this one (cf. (1.4)). It follows from the G-invariance of θ_X that if we let G act trivially on $\mathbb{C}$, we get an action of G on L which prequantizes its action on T^*X. Restricting $(L, \langle,\rangle, \nabla)$ to $(T^*X)'$ and pulling back by Φ we obtain a prequantization of $X \times B \times \mathfrak{a}^*_+$, which we will denote by the same symbol. In the remainder of this section we will work on $X \times B \times \mathfrak{a}^*_+$ rather than $(T^*X)'$ (cf. (4.4)).

(6.2) Since $d_X S$ vanishes on the leaves of p_X (which now plays the role of τ), the covariant constant along p_X sections of L can be naturally identified with functions on X (in other words, L^τ is naturally isomorphic to $X \times \mathbb{C}$). Take the G-invariant volume element μ on X defined in §5. Let dx be the corresponding G-invariant measure and $|\mu|^{\frac{1}{2}}$ the corresponding G-invariant half-density. Then we have a G-equivariant isomorphism

(6.2.1) $$C_0^\infty(X) \longrightarrow C_0^\infty(L^\tau \otimes D^{\frac{1}{2}}(X)), \; f \longmapsto f \otimes |\mu|^{\frac{1}{2}},$$

which extends to a G-invariant unitary isomorphism

$$L^2(X, dx) \xrightarrow{\sim} H^\tau .$$

Note that quantization of the whole T^*X would have given the same H^τ.

(6.3) $F \in C^\infty(L)$ is covariant constant along p_Y iff

$$d_X F - iF d_X S = 0.$$

It is obvious that e^{iS} satisfies this equation. Let s be the section of L corresponding to e^{iS} (so that $p_Y^* s = e^{iS}$). Take the volume elements ν and $(2\pi)^{-l/2}|c(\lambda)|^{-2} d\lambda_1 \wedge \ldots \wedge d\lambda_l$ on B and $\mathfrak{a}^*_+$ respectively as in §5. These give rise to a K-invariant measure on $B \times \mathfrak{a}^*_+$, which we will denote by $db|c(\lambda)|^{-2}d\lambda$, and a nowhere vanishing K-invariant half-density $|\nu \wedge (2\pi)^{-l/2}|c(\lambda)|^{-2} d\lambda_1 \wedge \ldots \wedge d\lambda_l|^{\frac{1}{2}}$. As s is a nowhere vanishing K-invariant section of L^π, we get a K-equivariant isomorphism

$$(6.3.1)\qquad C_0^\infty(B\times\mathfrak{a}_+^*) \longrightarrow C_0^\infty(L^\pi \otimes D^{\frac12}(B\times\mathfrak{a}_+^*)),$$

$$\varphi \longmapsto \varphi \otimes |\nu\wedge(2\pi)^{-l/2}|c(\lambda)|^{-2}d\lambda_1\wedge\ldots\wedge d\lambda_l|^{\frac12},$$

which extends to a K-equivariant unitary isomorphism

$$L^2(B\times\mathfrak{a}_+^*, db\,|c(\lambda)|^{-2}d\lambda) \xrightarrow{\sim} H^\pi.$$

7. BKS pairing between H_0^τ and H_0^π

In this section we assume that G is complex. We fix the volume elements on X, B and $\mathfrak{a}_+^*$ as in §5 and we write $dx\,db\,|c(\lambda)|^{-2}d\lambda$ for the corresponding product measure on $X\times B\times\mathfrak{a}_+^*$.

(7.1) Take the half-densities on X and $B\times\mathfrak{a}_+^*$ induced by the volume elements we fixed above. It follows from (5.6) that the pairing of these half-densities (see (1.9.1)) is given by

$$\langle|\mu|^{\frac12}, |\nu\wedge(2\pi)^{-l/2}|c(\lambda)|^{-2}d\lambda_1\wedge\ldots\wedge d\lambda_l|^{\frac12}\rangle = e^{\langle\rho, A\rangle}.$$

Now if $f\in C_0^\infty(X)$, $\varphi\in C_0^\infty(B\times\mathfrak{a}_+^*)$ and p_X^*f, $p_Y^*\varphi s$ are the corresponding sections of L (see (6.2) and (6.3)), then

$$\langle p_X^*f,\ p_Y^*\varphi s\rangle = f\,\bar\varphi e^{-iS} = f\,\bar\varphi e^{\langle -i\lambda, A\rangle}.$$

We are ready to compute the BKS pairing (1.9) between H_0^τ and H_0^π. Due to the isomorphisms (6.2.1) and (6.3.1) we may view this pairing as a sesquilinear form on $C_0^\infty(X)\times C_0^\infty(B\times\mathfrak{a}_+^*)$. As a direct consequence of the two above formulae we obtain our main result, namely

(7.2) Theorem. The BKS pairing of $f\in C_0^\infty(X)$ and $\varphi\in C_0^\infty(B\times\mathfrak{a}_+^*)$ is given by

$$\langle f,\varphi\rangle_{\pi\tau} = \int_{X\times B\times\mathfrak{a}_+^*} f(x)\,\overline{\varphi}(b,\lambda)e^{\langle -i\lambda+\rho, A(x,b)\rangle}dx\,db\,|c(\lambda)|^{-2}d\lambda\,.$$

Noting that $\langle f,\varphi\rangle_{\pi\tau} = \langle\tilde f,\varphi\rangle_\pi$, where $\tilde f$ denotes the Fourier transform of f (see (0.F)) and $\langle\,,\rangle_\pi$ stands for the inner product on $C_0^\infty(B\times\mathfrak{a}_+^*)$ (transferred from H_0^π by means of the isomorphism (6.3.1)), and using a theorem of Helgason ([H], Th. 5.8 of Chap. III), which asserts that $f\mapsto\tilde f$ extends to a unitary isomorphism of $L^2(X,dx)$ onto $L^2(B\times\mathfrak{a}_+^*, db\,|c(\lambda)|^{-2}d\lambda)$, we get the following.

(7.3) Corollary. τ and π are unitarily related (1.10) and the intertwining isomorphism $U_{\pi\tau}$ coincides with the Fourier transform $f\mapsto\tilde f$.

(7.4) Remarks. (a) The reason why, in the case of complex G, the BKS pairing leads to the Fourier transform is that the function of λ which appears in the formula for the Liouville form on $X\times B\times\mathfrak{a}_+^*$ coincides with $|c(\lambda)|^{-2}$ (see §5). Now it is clear from (5.1.2) and (5.1.1) that, in the case of arbitrary G, this function is a polynomial. On the other hand, $|c(\lambda)|^{-2}$ is a polynomial iff $\mathfrak{g}$ has but one conjugacy class of Cartan subalgebras (see [Wa], p. 327). A case by case inspection shows that complex groups are the only ones among the groups with this property for which the corresponding polynomial coincides with $|c(\lambda)|^{-2}$. This seems to be related to the fact that Kirillov's formula for the Plancherel measure of G leads to a correct result only when G is complex (cf. [Ki2], 15.6).

(b) The horizontal polarization gives rise to another G-invariant real polarization of $(T^*X)'$, whose space of leaves coincides with the space of horocycles in X (when transferred to $X \times B \times \mathfrak{a}^*_+$, it sends (x,b,λ) to the horocycle determined by (x,b)). It is reasonable to think that, at least for complex G, the BKS pairing corresponding to this polarization and the vertical one should lead to the Radon transform on X (in the sense of [H]).

A deeper analysis of these questions should help to understand why the geometric quantization scheme works well only in the complex case. We shall deal with these matters in a later article.

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