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ON THE FORMALITY OF PRODUCTS AND WEDGES

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Roughly speaking, we prove that the intrinsical formality of the cartesian product or a wedge of (finitely many) topological spaces implies the intrinsical formality of each factor. Our proofs are based on a deformation theory for bigraded algebras which is a generalization of that defined by Y. Félix in [F2]. We deduce anew also some results of Y. Félix and D. Tanré characterizing the intrinsical formality in terms of an associated cohomology theory. An outline of the corresponding results in the dual situation (for coformality) is given in Appendix.

0. Introduction and results. In this note we restrict our attention to the category of simply connected spaces having the rational cohomology of finite type. Consider in our category the following relation: There exists a continuous map $F : X \longrightarrow Y$ inducting an isomorphism $F^*: H^*(Y;Q) \longrightarrow H^*(X;Q)$ in rational cohomology. This relation induces in our category an equivalence relation, called the rational homotopy equivalence and the corresponding classes are called the rational homotopy types. Finally, a topological space X is called intrinsically formal, if each space whose rational cohomology algebra is isomorphic to $H^*(X;Q)$ belongs to the same rational homotopy type as X. In other words, there is precisely one rational homotopy type with the rational cohomology isomorphic to $H^*(X;Q)$ (see [T1]).

In fact, as is usual in algebraic topology, all spaces are supposed to have a base point. Recall that the wedge of two spaces A and B is the topological space $A \vee B$ obtained from the disjoint union $A \sqcup B$ by the identification of the base points. This definition clearly generalizes to an arbitrary number of factor. Our main result then reads:

Theorem 1. Let $X_1,\ldots,X_n$ be simply connected spaces having the rational cohomology of finite type. If the space $X_1 \times \ldots \times X_n$ or $X_1 \vee \ldots \vee X_n$ is intrinsically formal, then each $X_1,\ldots,X_n$ is intrinsically formal, too.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

The dual statement is valid, too (see Appendix). Our proof is based on the usual methods of rational homotopy theory (minimal models ...). In the following paragraph we describe a deformation theory for the set of rational homotopy types, motivated by the computation of Y. Félix [F1,F2], see also [SS]. In the second paragraph we discuss the case of free products, which is the algebraic analogy of the cartesian product and wedge. In the third paragraph is our theory applied to the proof of Theorem 1. For the basic definitions, notation and results we refer the reader to [T2].

1. <u>Deformation theory for bigraded algebras</u>. All objects (vector spaces, algebras ...) are considered over the field of rationals $\mathbb{Q}$.

1.1. Let $A^*_* = \bigoplus_{i,j\in\mathbb{Z}} A^i_j$ be a bigraded algebra (i.e. $A^i_j . A^p_q \subset A^{i+p}_{j+q}$) which is either graded commutative or graded Lie with respect to the upper grading. Suppose that the following condition is satisfied:

(<u>bound</u>) The set $\{i \in \mathbb{Z};\ A^n_i \neq \{0\}\}$ is finite for each $n \in \mathbb{Z}$.

We denote by $Der^i(A)$ the set of all derivations of degree i of the algebra A and let $Der^i_j(A) = \{\theta \in Der^i(A);\ \theta(A^*_k) \subset A^*_{k+j}$ for each $j \in \mathbb{Z}\}$. Let $d \in Der^1_1(A)$ be a derivation satisfying $[d,d] = 0$ (recall that the commutator is taken in graded sense, hence $[d,d] = d^2$). We define the bigraded vector space $E^*_* = E^*_*(A,d)$ by:

$$E^k_j = Der^k_{j+k}(A) \text{ for } k > 0,\ j \geq 0 \text{ or } k=0 \text{ and } j > 0,$$
$$E^0_0 = \{\theta \in Der^0_0(A);\ [d,\theta] = 0\} \text{ and}$$
$$E^k_j = \{0\} \text{ otherwise.}$$

Clearly E^*_* with the (graded) commutator as the product is a bigraded Lie algebra. The linear map D_d defined by $D_d(\theta) = [d,\theta]$ is a differential on the bigraded vector space E^*_*. Because it is homogeneous (of bidegree (1,0)) the homology $H(E^*_*, D_d)$ inherits the natural structure of a bigraded vector space as well.

1.2. For $g \in Aut(A)$, let $F_0(g)$ be the linear endomorphism of A^*_* defined by $F_0(g)(y) = P_i(g(y))$ for $y \in A^*_i$; here $P_i\colon A^*_* \to A^*_i$ denotes the projection. Then the set

$$G = G(A,d) = \{g \in Aut(A);\ g(A^i_j) \subset A^i_{\geq j} \text{ and } F_0(g) \circ d = d \circ F_0(g)\}$$

can be shown to be a subgroup of Aut(A), this need not be true without the condition (bound). The natural (left) action of the group G on E^*_* is defined by $g(\Delta) = g \circ \Delta \circ g^{-1}$.

1.3. Under the notation above we denote by $M_d = M_d(A)$ the set

$$M_d = \{d+m_1+m_2+\ldots \in \prod_{j\geq 0} E^1_j;\ [d+m_1+m_2+\ldots, d+m_1+m_2+\ldots] = 0\}.$$

The equation $[d+m_1+m_2+\dots,d+m_1+m_2+\dots] = 0$ plays here the role of a deformation equation in the sense of [NR]. Because of (bound), each element of $\prod_{j\geq 0}\mathrm{Der}^i_j$ defines a derivation, hence the elements of M_d are again derivations of the algebra A.

1.4. The action of the group G clearly restrict to an action on the set M_d and we can denote by $\#(M_d/G)$ the number of elements of the orbit space M_d/G. In our theory is valid the following "rigidity" theorem (see [NR],[G],[T1],[F1],[F2],[SS],...)

Proposition 2. If $H^1_{\geq 1}(E,D_d) = 0$, then $\#(M_d/G) = 1$. On the other hand, if $H^2_{\geq 2}(E,D_d) = 0$ and $\#(M_d/G) = 1$, then $H^1_{\geq 1}(E,D_d) = 0$.

1.5. Example. Let H^* be a graded commutative algebra of finite type with $H^0 \cong Q$ and $H^1 = \{0\}$. The Lie differential graded algebra $(\mathbb{L}(W),\partial_2)$ is defined (up to isomorphism) by $\mathcal{L}_*(H^*,d=0)) \cong$ $= (\mathbb{L}(W),\partial_2)$ (see [T2] for notation). Define the bigraded space Z^*_* by

$$Z^i_1 = W^{-i},\ Z^i_j = \{0\} \text{ for } i,j \in \mathbb{Z},\ j \neq 1.$$

Then ∂_2 defines on $\mathbb{L}(Z)$ a differential $d \in \mathrm{Der}^1_1(\mathbb{L}(Z))$ and the algebra $E^*_*(\mathbb{L}(Z),d)$ will be called the Quillen deformation algebra for H^*. In this case, the points of the set M_d/G are in one-to-one correspondence with rational homotopy types with the cohomology isomorphic to H^* (see [LS]). It is possible to show that $H^i_j(E(\mathbb{L}(Z),d),D_d)$ is isomorphic with $\mathrm{Harr}^{i+j+1,i}(H,H)$, where Harr denotes the Harrison cohomology in the notation of [T2]. Proposition 2 in this case gives:

Theorem (D. Tanré). If $\mathrm{Harr}^{>2,1}(H,H) = 0$ then H is intrinsically formal. If $\mathrm{Harr}^{>4,2}(H,H) = 0$ then H is intrinsically formal if and only if $\mathrm{Harr}^{>2,1}(H,H) = 0$.

1.6. Example. Being H^* as above, let $(\Lambda X,d_{-1})$ be the Halperin-Stasheff bigraded model of the algebra H^* (see [HS]), $X = \bigoplus_{i,j\geq 0} X^i_j$. define the bigraded space Z^*_* by $Z^i_j = X^i_{-j}$ and denote by d the differential induced by d_{-1} on $\Lambda(Z)$. The corresponding algebra $E^*_*(\Lambda(Z),d)$ will be called the HS deformation algebra for H . Again M_d/G is the set of all rational homotopy types with fixed cohomology H . The object $H^*_*(E(\Lambda(Z),d),D_d)$ is precisely the "filtered cohomology" introduced by Y. Félix in [F1,F2] and Proposition 2 in this case gives:

Theorem (Y. Félix). If $H^1_{\geq 1}(H) = 0$ then H is intrinsically formal. If $H^2_{\geq 2}(H) = 0$ then H is intrinsically formal if and only if $H^1_{\geq 1}(H) = 0$. Here we write for brevity $H^*_*(H) = H^*_*(E(\Lambda(Z),d),D_d)$.

1.7. Suppose $i \geq 1$ and let $\theta \in E^0_i = \mathrm{Der}^0_i(A)$. Because of the condition (bound) the sum

$$\exp(\theta)(x) = \sum_{n\geq 0}(1/n!)\,\theta^n(x)$$

is finite for each $\theta \in A$ and defines an element $\exp(\theta) \in G$.

Proposition 3. Let $E^*_* = E(A,d)$ and $G = G(A,d)$ be as in 1.1 and 1.2. Suppose that $\bar{E}^*_*$ is a bigraded Lie subalgebra of E and $d \in \bar{E}^1_0 \subset E^1_0$. Let $\bar{G}$ be a subgroup of the group G satisfying:

(i) $\bar{G}(\bar{E}^*_*) \subset \bar{E}^*_*$,

(ii) the element $\exp(\bar{\theta})$ belongs to the subgroup $\bar{G}$ for each $\bar{\theta} \in \bar{E}^0_i$, $i \geq 1$.

Finally, denote by $\bar{M}_d$ the $\bar{G}$-space $\bar{M}_d = M_d \cap \prod_{j \geq 0} \bar{E}^1_j$.

If the inclusion $\bar{E} \hookrightarrow E$ induces a monomorphism $H^1_{\geq 1}(\bar{E},D_d) \rightarrowtail H^1_{\geq 1}(E,D_d)$ then $\#(M_d/G) = 1$ implies that $\#(\bar{M}_d/\bar{G}) = 1$.

Proof of the proposition. Suppose that $\#(\bar{M}_d/\bar{G}) > 1$. Then there exists an element $\Delta = d + \bar{m}_s + \bar{m}_{s+1} + \ldots$ (here $\bar{m}_i \in \bar{E}^1_i$ are the homogeneous components, $i \geq s$) with $\Delta \in \bar{M}_d$, which is not $\bar{G}$-equivalent to d. Counting the degrees in the equation $[d + \bar{m}_s + \bar{m}_{s+1} + \ldots, d + \bar{m}_s + \bar{m}_{s+1} + \ldots] = 0$ we see that $[d,\bar{m}_s] = D_d(\bar{m}_s) = 0$, hence $\bar{m}_s$ determines an element $[\bar{m}_s]$ in $H^1_s(\bar{E};D_d)$.

If $[\bar{m}_s] = 0$, there exists $\bar{\theta} \in \bar{E}^0_s \subset Der^0_s(A)$ with $[d,\bar{\theta}] = \bar{m}_s$. By our assumptions the element $\bar{g} = \exp(\bar{\theta})$ belongs to the subgroup $\bar{G}$ and

$$\bar{g}\Delta = \bar{g}(d + \bar{m}_s + \bar{m}_{s+1} + \ldots) = (id + \bar{\theta} + \bar{\theta}^2/2! + \ldots)(d + m_s + \ldots)(id - \bar{\theta} + \bar{\theta}^2/2! - \ldots)$$
$$= d + (\bar{m}_s - [d,\bar{\theta}]) + (\text{terms of lower degree} \geq s+1),$$

hence Δ is $\bar{G}$-equivalent with a point of the form $d + \bar{m}'_{s+1} + \bar{m}'_{s+2} + \ldots$. Repeating this process sufficiently many times we can show, using (bound) and the assumptions above, that Δ is $\bar{G}$-equivalent with a point of the form $d + \bar{m}_k + \bar{m}_{k+1} + \ldots \in \bar{M}_d$ with $[\bar{m}_k] \neq 0$ in $H^1_k(\bar{E},D_d)$.

Our point $d + \bar{m}_k + \bar{m}_{k+1} + \ldots$ is also an element of the set M_d. Because $\#(M_d/G) = 1$, there exists $g \in G$ with $gd = d + \bar{m}_k + \bar{m}_{k+1} + \ldots$ Using the similar arguments as above we can find $g' \in G$ of the form $g' = id + g'_k + g'_{k+1} + \ldots$, where g'_i are, for $i \geq k$, linear endomorphisms of E^*_* of bidegree $(0,i)$, such that $g'd = gd$. The endomorphism g'_k is clearly a derivation and the equation $g'd = d + \bar{m}_k + \bar{m}_{k+1} + \ldots$ gives $[d,g'_k] = D_d(g'_k) = \bar{m}_k$, hence $[\bar{m}_k] = 0$ in $H^1_k(E,D_d)$. But this is a contradiction, because $H^1_k(\bar{E},D_d)$ is mapped monomorphically into $H^1_k(E,D_d)$ by the assumption.

2. Free products. In this paragraph we show, how to apply Prop. 3 in the case when A is a free product of subalgebras.

2.1. For a graded vector space Z^* we denote by $F^*(Z)$ either the free graded commutative or the free graded Lie algebra on Z^*. If Z^* has another "lower" grading, $Z^i_* = \bigoplus_{j \in \mathbb{Z}} Z^i_j$ for each $i \in \mathbb{Z}$, then there is the natural induced "lower" grading on $F(Z)$; suppose that it satisfies the condition (bound) of 1.1. Note that there is one-to-one correspondence between elements of Der(F(Z)) and linear maps from Z to F(Z).

2.2. Suppose that there are bigraded spaces X^*_* and Y^*_* such that $Z^*_* \cong X^*_* \oplus Y^*_*$. Then we can consider in the clear sense F(X) and F(Y) as bigraded subalgebras of F(Z). Define $\oplus : Der^*_*(F(X)) \oplus Der^*_*(F(Y)) \longrightarrow \longrightarrow Der^*_*(F(Z))$ by $\oplus(\Omega,\theta) = \Omega\oplus\theta$, where the derivation $\Omega\oplus\theta$ is defined by $(\Omega\oplus\theta)|F(X) = \Omega$, $(\Omega\oplus\theta)|F(Y) = \theta$. The map $\times: Aut(F(X)) \times Aut(F(Y)) \longrightarrow Aut(F(Z))$ can be defined similarly.

2.3. If $d' \in Der^1_1(F(X))$ and $d'' \in Der^1_1(F(Y))$ are derivations satisfying $d'^2 = d''^2 = 0$, then clearly $d = d' \oplus d''$ is also a derivation satisfying $d^2 = 0$. The subspace $\overline{E}^*_*$ of the algebra $E^*_* = E(F(Z),d)$ consisting of all derivations θ with $\theta(X) \subset F(X)$ and $\theta(Y) \subset F(Y)$ clearly forms a bigraded subalgebra. lso $\overline{G} = \{g \in G(F(Z),d);\ g(X) \subset F(X)$ and $g(Y) \subset F(Y)\}$ is a subalgebra of $G = G(F(Z),d)$ (see 1.1 and 1.2). The couple $(\overline{E},\overline{G})$ clearly satisfies (i) and (ii) of Proposition 3.

2.4. Let us denote
$E' = E(F(X),d')$, $G' = G(F(X),d')$, $E'' = E(F(Y),d'')$ and $G'' = G(F(Y),d'')$. Then there are isomorphisms $\overline{E}^*_* \cong E'^*_* \oplus E''^*_*$ and $\overline{G} \cong G' \times G''$ and it is not hard to verify that $M_{d'}/G' \times M_{d''}/G'' \cong \overline{M}d/\overline{G}$ (we use the notation of Proposition 3), hence $\#(M_{d'}/G').\#(M_{d''}/G'') = \#(\overline{M}_d/\overline{G})$.

Proposition 4. There exists a linear map $J: E^*_* \longrightarrow \overline{E}^*_*$ of differential spaces, homogeneous of degree (0,0), such that $J \circ \iota = id_{\overline{E}}$, where $\iota: \overline{E} \hookrightarrow E$ is the inclusion.

We note that J is not, in general, a homomorphism of Lie algebras. Combining the statement above and Proposition 3 we easily obtain:

Corrolary 5. The map $H^*_*(\overline{E},D_d) \longrightarrow H^*_*(E,D_d)$ induced by the inclusion is a monomorphism, hence $\#(M_d/G) = 1$ implies that

$$\#(M_{d'}/G') = \#(M_{d''}/G'') = 1.$$

Proof of the proposition. Let $P_1: F(Z) \longrightarrow F(X)$ and $P_2: F(Z) \longrightarrow F(Y)$ be the canonical projections. Given $\theta \in Der(F(Z))$, we can define the linear endomorphism $J_1(\theta)$ of F(X) by $J_1(\theta)(x) = P_1(\theta(x))$. Similarly, the linear endomorphism $J_2(\theta)$ of F(Y) is defined by $J_2(\theta)(y) = = P_2(\theta(y))$.

At first, we show that $J_1(\theta)$ is a derivation of F(X). Let us denote by $F^+(X)$ the augmentation ideal of F(X) (= subspace of all elements of positive lenght) and let I be the ideal generated by $F^+(X)$ in F(Z). Now, for each $a \in F(Z)$, $\theta(a)$ can be decomposed uniquely in the form $P_1(\theta(a)) + \theta_+(a)$ with $\theta_+(a) \in I$. For a,b F(X) compute $J_1(\theta)(a.b)$. By the definition, $J_1(\theta)(a.b) = P_1(\theta(a.b)) =$
$= P_1(\theta(a).b \pm a.\theta(b)) = P_1((P_1(\theta(a)) + \theta_+(a)).b \pm a.(P_1(\theta(b)) + + \theta_+(b))) = P_1(J_1(\theta)(a).b \pm a.J_1(\theta)(b) + (\text{an element of } I) = J_1(\theta)(a)b$

$\pm$ a.$J_1(\theta)$(b). The endomorphism $J_2(\theta)$ is a <u>derivation</u> by the same argument. Hence the map $J(\theta) = J_1(\theta) \oplus J_2(\theta)$ is a well-defined linear homomorphism $J:E \longrightarrow \bar{E}$ of bidegree (0,0) and it clearly satisfies $J\circ \iota = id$.

It remains to show that J <u>commutes</u> with the <u>differentials</u>. In other words, we show that for each $\theta \in Der(F(Z))$, $D_d(J(\theta)) = [d,J(\theta)] = J([d,\theta]) = J(D_d(\theta))$. For $x \in F(X)$ we have $J([d,\theta])(x) = P_1(d\circ\theta(x) \pm \theta\circ d(x)) = P_1((d' \oplus d'')(P_1(\theta(x)) + \theta_+(x)) \pm (P_1(\theta(d'(x)) + \theta_+(d'(x))) = P_1(d'(P_1(\theta(x)) + \text{an element of } I) \pm (P_1(\theta(d'(x)) + \theta_+(d'(x))) = [d,J(\theta)](x)$. The same equation is clearly valid also on F(Y) and therefore is valid on F(Z) as well.

3. <u>Proof of Theorem 1.</u>

3.1. <u>The case of wedges</u>. Suppose that the algebra H^* is a "wedge" $H'^* \vee H''^*$ of algebras H'^* and H''^* (i.e. $H^0 \cong Q$, $H^{\geq 1} \cong H'^{\geq 1} + H''^{\geq 1}$ with the product defined by the clear way). Denote by $E = E(\mathbb{L}(Z),d)$, $E = E(\mathbb{L}(X),d')$ and $E = E(\mathbb{L}(Y),d'')$ the Quillen deformation algebras for H^*, H'^* and H''^* respectively (see 1.5). Because the Quillen model of the algebra (H,d=0) is of the form $(\mathbb{L}(s^{-1}(H'^* \oplus H''^*)), \partial' \oplus \partial'')$, we can suppose that $Z^*_* \cong X^*_* \oplus Y^*_*$ and $d = d' \oplus d''$. This situation was studied in the previous paragraph and Corrolary 5 gives:

<u>Proposition 6</u>. If the algebra $H^* = H'^* \vee H''^*$ is intrinsically formal, then both H'^* and H''^* are intrinsically formal, too.

<u>Remark</u>. As kindly pointed me Daniel Tanré, Proposition 6 follows in the light of his theorem (see 1.5) in the case when the Harrison cohomology $Harr^{\geq 4,2}(H,H)$ is zero from the additivity of this cohomology.

Now the statement of Theorem 1 for wedges follows from the clear generelization of Proposition 6 to the case of n factors.

3.2. <u>The case of products</u> can be discussed similarly as the previous one. If $H^* \cong H'^* \otimes H''^*$ and $\eta': (\Lambda U, d'_{-1}) \longrightarrow (H'^*, d=0)$ and $\eta'': (\Lambda V, d''_{-1}) \longrightarrow (H''^*, d=o)$ are the Halperin-Stasheff bigraded models of H'^* and H''^* respectively, then clearly $\eta' \otimes \eta'': (\Lambda(U+V), d'_{-1} \oplus d''_{-1}) \longrightarrow (H'^* \otimes H''^*, d=o)$ is the Halperin-Stasheff model for H^*. Now, if $E = E(\Lambda Z,d)$ is the HS-deformation algebra for H^* (see 1.6) then again $(\Lambda Z,d)$ is of the form $(\Lambda(X+Y), d' \oplus d'')$ and we conclude similarly as in 3.1

<u>Proposition 7</u>. If $H^* = H'^* \otimes H''^*$ is intrinsically formal, then both H'^* and H''^* are intrinsically formal, too.

Theorem 1 in the case of products again follows immediately from the clear generalization of the previous proposition to the case of n factors.

3.3. For a 1-connected graded algebra H^* of finite type denote by $\#(H)$ the number of (simply connected) rational homotopy types with the cohomology isomorphic to H^*. Our main result in this terminology reads:

If $\#(H'^* \vee H''^*) = 1$ (or $\#(H'^* \otimes H''^*) = 1$), then $\#(H'^*).\#(H''^*) = 1$.

It is possible to construct a graded differential space whose acyclycity in dimension one implies the inequality $\#(H'^* \vee H''^*) \leq \#(H'^*).\#(H''^*)$ (similarly for tensor product). The construction, based on the theory developed in the present paper and some results of [SS] is described in [Ma]. Inequalities of this kind can be obtained even for pullbacks and attaching of cells.

Appendix. Here we briefly show how our theory applies to the study of the set of rational homotopy types with a given homotopy Lie algebra. The symbol Π_* denotes a (positively) graded Lie algebra of finite type.

A.1. The bigraded algebra $E = E(\wedge Z,d)$, where $(\wedge Z,d) = C^*(\Pi_*,\partial=0)$ (see [T1] for the notation), describes the rational homotopy types with homotopy Lie algebra isomorphic to Π_*. The associated cohomology was introduced and studied by Y. Félix in [F1, Annexe 2]. Because the Sullivan model $(\wedge Z,d)$ behaves well under products, we can deduce:

Theorem 8. Let $X_1,\dots,X_n$ be simply connected spaces having the rational cohomology of finite type. If $X_1 \times \dots \times X_n$ is intrinsically coformal, then each $X_1,\dots,X_n$ is intrinsically coformal, too.

A.2. Let $(\mathbb{L}(X),\partial_{-1})$ be the Halperin-Stasheff bigraded Lie model of $(\Pi_*,\partial=0)$. If we denote by Z_*^* the bigraded space $Z_*^* = X_{-*}^{-*}$ and by d the differential induced on $\mathbb{L}(Z)$ by ∂_{-1} then again $E(\mathbb{L}(Z),d)$ describes the set of all rational homotopy types with given homotopy Lie algebra. Because the Halperin-Stasheff bigraded Lie model behaves well under wedges, we can formulate statement, similar to Theorem 8, also in the case of wedge.

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