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CONFORMALLY INVARIANT WAVE EQUATIONS IN 3+2-DE SITTER LINEAR GRAVITY

J.P. Gazeau and M. Hans

Massless fields theories on 3+2 de Sitter space ("Anti-de Sitter space") share with their flat-space counterparts common characteristics like gauge invariance and conformal invariance.
In the present work we examine these aspects in the lowest-spin cases s = 0,1, and 2 with a particular emphasis on the de Sitter gravity (s = 2). Conformally invariant wave equations are given. Finally we show that only two values of the gauge-fixing parameter have a conformal origin.

Introduction.

It has been known for a long time that conformal invariance is intimately connected with massless field theories [1]. Field theories being usually built up around wave equations, the latter should present in the massless case specific behaviour under conformal transformations.

Basic relativity principles already impose invariance properties on wave equations : e.g. they are invariant under Poincaré-transformations if space-time is flat. They are instead invariant under the Anti-de Sitter group $SO_0(3,2)$ (resp. de Sitter group $SO_0(4,1)$) if space time is curved with negative (resp. positive) constant curvature ρ. Dealing with both latter relativities of maximal symmetry is most interesting at several levels : in general relativity, the Einstein cosmological constant Λ is proportional to ρ and working with constant-curvature space-times gives solid foundations and brings in rich information for further developments concerning field theories on more general pseudo-Riemanian manifolds.

We shall restrict ourselves to $SO_0(3,2)$ in the present work.

"This paper is in final form and no version of it will be submitted for publication elsewhere"

This group possesses a set of very attractive features. For instance, some of its unitary irreducible representations can be associated with physical elementary systems. In particular, massless elementary systems are defined in this way and related representations show features familiar to the flat-space situation : gauge invariance of the wave equations and indecomposability of the representation. Furthermore, conformal invariance of wave equations strongly depends on the choice of a certain parameter, the "gauge-fixing" coefficient.

Massless - de Sitterian - conformal trilogy has been examined in the recent past by Binegar, Fronsdal and Heidenreich (see [2] , [3] and references therein). Several points however require clarification, particularly the passage conformal six-cone - (3+2) de Sitter formalism, the explicit form of the conformally invariant ("CI") wave equations, and finally the "conformal" origin of two and only two values of the de Sitter gauge-fixing parameter : $c = 1$ and $c = 2/(2s+1)$ (s is the spin).

In order to simplify the presentation, we have restricted our approach to the lowest integral spin cases : $s = 0,1$ and 2.

The six-cone formalism is presented in Section I. De Sitter coordinates and projection techniques are described in Section II. Afterwards, we give in Section III a summary concerning the de Sitter group and its representations necessary for our purposes. The rest of the paper is devoted to the specific cases $s = 0,1$, and 2.

1. 4+2-conformal action through the Dirac's six-cone formalism [1], [2].

The Dirac's six cone is the 5-dimensional surface $u^2 = \delta^{AB} u_A u_B$ in $\mathbb{R}^6$. The metric is given by :

$$\delta_{AB} = \mathrm{diag}(1, -1, -1, -1, -1, 1), \tag{1.1}$$

$A, B = 0,1,2,3,4,5.$

An operator $\hat{A}$ which acts on scalar fields f over $\mathbb{R}^6$ is said intrinsic if

$$\hat{A}\, u^2 f = u^2 \hat{A}' f \qquad \text{for any } f$$

For instance, are intrinsic

- the 15 generators $M_{AB} = i(u_A\partial_B - u_B\partial_A)$ of the conformal group $SO_0(4,2)$,
- the conformal-degree operator $\hat{N}_4 \equiv u^A \partial_A$
- the intrinsic gradient
$$\nabla_A \equiv u_A \partial_B \partial^B - 2\partial_A(\hat{N}_4 + 1)$$
- conditionnally the powers of the d'Alembertian $\partial_A\partial^A$; $(\partial_A\partial^A)^p$ acts intrinsically on f if $\hat{N}_4 f = (p-2) f$, i.e. if the conformal degree of f is $p-2$.

Therefore, we can start from the following conformally invariant ("CI") system on the cone :

$$\begin{cases} (\partial_A\partial^A)^p \psi = 0, \\ \hat{N}_4 \psi = (p-2)\psi, \end{cases} \tag{1.2}$$

where ψ is a tensor field of a certain rank and a certain symmetry pictured by some Young tableau, e.g. $\psi \equiv \psi_{\tiny\begin{array}{|c|c|c|c|}\hline & & & \\ \hline \\ \cline{1-1}\end{array}}$.

Other conformally invariant conditions can be added to (1.2) in order to restrict its solution space, e.g. for symmetric tensors,

- transversality
$$u^A \psi_{ABC\ldots} = 0, \tag{1.3}$$
- divergencelessness
$$\nabla^A \psi_{ABC\ldots} = 0, \tag{1.4}$$
- tracelessness
$$(\mathrm{Tr}\,\psi)_{BC\ldots} \equiv \psi^A{}_{ABC\ldots} = 0. \tag{1.5}$$

and so on.

Conformal invariance of (1.2) - (1.5) is understood as invariance under infinitesimal transformations

$$\delta\psi \equiv \varepsilon^{AB} L_{AB}\, \psi, \quad L_{AB} \equiv M_{AB} + S_{AB}; \tag{1.6}$$

S_{AB} acts unically on indices of ψ .

Invariant subspaces of solutions to (1.2) are easily put in evidence, using constant vectors or tensors z, or gradients ∇, and suitable symmetrizers Σ .

$$\tilde{\psi} = \Sigma\, z\, \varphi \quad , \quad \tilde{\psi} = \Sigma\, \nabla \varphi \ .$$

2. <u>(3+2)-de Sitter coordinates for the projective six-cone and reduction of the tensor fields.</u>

The (3+2)-de Sitter coordinates for the cone $u^2 = 0$ are a set of five numbers

$$\{ y^\alpha ,\ y^2 \equiv \delta^{\alpha\beta} y_\alpha y_\beta = 1/\rho \} \times \{ y_4 \} \ .$$

$\alpha, \beta, \ldots.$ run over $\{0,1,2,3,5\}$. $\delta_{\alpha\beta}$ designates the (3+2) metric of $\mathbb{R}^5$: $\delta_{\alpha\beta}$ = diag (1,-1,-1,-1,1). ρ is some positive constant, the curvature of the de Sitter space. The y^α's are related to the "u" variables by

$$y^\alpha = \frac{u^\alpha}{\sqrt{\rho}\, u_4} \quad , \quad y_4 = \sqrt{\rho}\, u_4 \tag{2.1}$$

y_4 becomes superfluous when we deal with the projective cone. We keep the notation y for the $\{y^\alpha\}$ solely.

The various intrinsic operators introduced in the first section now read :

- the conformal-degree operator
$$\hat{N}_4 = y_4\, \partial/\partial y_4 \ ,$$
- the ten $SO_0(3,2)$ generators
$$M_{\alpha\beta} = i\, (y_\alpha \partial_\beta - y_\beta \partial_\alpha) ,$$
- the five purely conformal generators

$$M_{4\alpha} = i\, \rho^{-1/2} \left(\bar{\partial}_\alpha + \rho\, y_\alpha \hat{N}_4 \right) , \tag{2.2}$$

where $\bar{\partial}_\alpha = \partial_\alpha - \rho\, y_\alpha y^\beta \partial_\beta \equiv T \partial_\alpha$ stands for transverse derivative on de Sitter space. We designate by T the transverse projector which makes transverse a tensor field $k = (k_{\alpha_1 \alpha_2 \ldots \alpha_i \ldots})$ over $\mathbb{R}^5$

$$y^{\alpha_i} \, (T k)_{\alpha_1 \alpha_2 \ldots \alpha_i \ldots} = 0 \qquad \text{for any i.}$$

- the powers of the conformal d'Alembertian <u>when</u> acting on fields of conformal degree p-2

$$(\partial^A \partial_A)^p = (-\rho)^p\, y_4^{-2p} \prod_{j=1}^{p} \left(Q_0 - (j+1)(j-2) \right) \tag{2.3}$$

where $Q_0 \equiv 1/2\, M^{\alpha\beta} M_{\alpha\beta}$ is a second order-Casimir operator representative for SO(3,2)

- the conformal gradient

$$\nabla_\alpha = - y_4^{-1} \left\{ y_\alpha \rho \left(Q_0 + \hat{N}_4 (\hat{N}_4 - 1) \right) + 2 \rho \bar{\partial}_\alpha (\hat{N}_4 + 1) \right\} \tag{2.4}$$

As we shall see, the tensor fields on de Sitter space which are of real importance are transverse and totally symmetrical in their indices.
To obtain such fields through projection, say $k = (k_{\alpha_1 \alpha_2 \cdots \alpha_s})$ of rank s, the most economical way starts from a s-rank symmetric tensor field $\psi = (\psi_{A_1 A_2 \cdots A_s})$. The latter may be issued from a tensor of higher rank and of mixed symmetry : the case s = 2 is very illustrating [3] of this situation.

Let us define from ψ (s+1)(s+2)/2 tensor fields using contraction procedures and transverse projection :

$$K^{p}_{p-q} \equiv T\Big(\underbrace{y \cdot (y \cdot (\cdots (y \cdot}_{q \text{ times}} \psi_{\underbrace{44\cdots4}_{s-p}}) \cdots))\Big) \tag{2.5}$$

$$0 \leqslant p \leqslant s \quad , \quad 0 \leqslant q \leqslant p$$

Only the first one is of rank s. We shall introduce the purely-de Sitter field

$$k = y_4^{\,2-p} K^{s}_{s} \tag{2.6}$$

The remaining ones, $\left\{ K^{p}_{p-q} \ , \ p < s \text{ or } p = s \text{ and } q > 0 \right\}$, form a set of $\frac{s(s+3)}{2}$ auxiliary fields.

The main points that we would now like to elucidate are the following :

(i) Starting from the "minimal" conformally invariant system (1.2) what is the equation $\mathcal{O} k = 0$, obeyed by k alone ?

(ii) Given the invariance of (1.2) under the conformal transformation (1.6), what comes about the transformational properties of k or/and of the operator $\mathcal{O}$?

(iii) How can the answers to (i) and (ii) be modified if we enlarge the system (1.2) by adding other conformally invariant conditions, say $u \cdot \psi = 0$, $\nabla \cdot \psi = 0$, etc..., which restrict the solution space ?

(iv) What is the $SO_0(3,2)$-content of the equation $\mathcal{O} k = 0$? What is its physical content too ?

Before giving (partial) answers to these questions, the last one need more precision. That will be the aim of the next section.

3. A Survey of 3+2 - de Sitter - Group Representations [4].

The 3+2 de Sitter geometry for four-dimensional space-time is one of the three geometrics of maximal symmetry (besides 4+1 de Sitter and Poincaré) [5] . De Sitter universe can be visualized as the hyperboloïd in $\mathbb{R}^5$: $y^2 = 1/\varrho$. Its kinematical group is $SO_0(3,2)$. By extension of the Wigner ideas, massive elementary systems are defined as associated to unitary irreducible representations (URREP) of this group. Massless elementary systems are rather associated to nondecomposable representations where gauge invariance plays a crucial role. Unitarity and irreducibility are restored on quotient spaces and give significance to the physical degrees of freedom of the system.

Representation spaces are conveniently realised as set of symmetric, transverse, homogeneous, s-rank tensor fields on some open set U including the hyperboloïd (only integral spins will be considered here)

$$y \in U \longrightarrow k(y) = (k_{\alpha_1\alpha_2\cdots\alpha_s}(y)),$$

$$y\cdot k = 0 \ , \quad (\hat{N} - N)\,k = 0 \ , \quad \hat{N} \equiv y\cdot\partial \ .$$

The ten basic elements $X_{\alpha\beta}$ of the Lie algebra so(3,2) are represented by

$$X_{\alpha\beta} \longrightarrow L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}\ , \tag{3.1}$$

where $M_{\alpha\beta} = i(y_\alpha\partial_\beta - y_\beta\partial_\alpha)$. The spin part acts as follows :

$$S_{\alpha\beta}\, k_{\alpha_1\alpha_2\cdots\alpha_s} = i\sum_i \left(\delta_{\alpha\alpha_i} k_{\alpha_1\cdots\overset{\beta}{\cancel{\alpha_i}}\cdots\alpha_s} - \delta_{\alpha_i\beta} k_{\alpha_1\cdots\overset{\alpha}{\cancel{\alpha_i}}\cdots\alpha_s}\right)$$

Solely the K-finite, minimal-weight representations $D(E_0,s)$ can be of physical interest. E_0 is the lowest eigenvalue of the "energy" L_{50} whereas s is the angular momentum (spin) of the state of lowest energy.

$$\begin{array}{l} D(E_0,s) \text{ is an URREP if} \\ \quad s = 0 \text{ and } E_0 > 1/2 \\ \quad s = 1 \text{ and } E_0 > 1 \\ \quad s > 1 \text{ and } E_0 > s+1 \end{array} \tag{3.2}$$

The limit cases of unitarity D(1/2,0), D(1,1/2) (singletons) and D(s+1,s) for s > 0 are nondecomposable.

Note that the finite irreducible representations (FIRREP) correspond to the values $E_0 = -s-\sigma$, for positive integers σ.

The usual way for obtaining carrier states passes through solving invariant wave equations. They involve the second-order Casimir operator

$$Q = \frac{1}{2} L_{\alpha\beta} L^{\alpha\beta} . \tag{3.3}$$

For instance, the $D(E_0,s)$ -states lie among the solutions of the SO(3,2)-invariant system (the index s means an action on tensors of rank s)

$$\begin{cases} (Q_s - \langle Q_s^{E_0} \rangle) k = 0 , \\ \partial \cdot k = 0 . \end{cases} \tag{3.4}$$

where $\langle Q_s^{E_0} \rangle \equiv E_0 (E_0 - 3) + s(s+1)$.

They are distinguished from $D(3-E_0,s)$-states by different behaviour at spatial infinity $r = (y_1^2 + y_2^2 + y_3^2)^{1/2} \longrightarrow +\infty$. Equations (3.4) are not suitable for massless fields. They have to be modified in order to account for some gauge invariance

$$(Q_s - \langle Q_s^{s+1} \rangle) k + c D_s \partial_s \cdot k = 0, \tag{3.5a}$$

$$Tr\, k = 0 \text{ if } c \neq 1 , \; Tr(Tr\, k) = 0 \text{ if } c = 1 . \tag{3.5b}$$

Tr is the trace operator

$$(Tr\, k)_{\alpha_1\alpha_2 \ldots \alpha_{s-2}} = \delta^{\alpha_{s-1}\alpha_s} k_{\alpha_1\alpha_2 \ldots \alpha_{s-2}\alpha_{s-1}\alpha_s} .$$

D_s is a symmetric transverse gradient operator :

$$D_s \equiv \rho^{-1} T \Sigma \partial . \tag{3.6}$$

$\partial_s \cdot$ is a double-traceless transverse divergence :

$$\partial_s \cdot \equiv T \partial \cdot - \rho/2\, D_{s-1} Tr . \tag{3.7}$$

D_s acts on tensors of rank s-1 whereas $\partial_s \cdot$ acts on tensors of rank s. Both achieve a factorization of Q_s :

$$\partial_s \cdot D_s \zeta = - (Q_{s-1} - \langle Q_s^{s+1} \rangle) \zeta , \tag{3.8}$$

if $Tr\, \zeta = 0$.

c is the gauge-fixing parameter. If c = 1, (3.5) is fully gauge-invariant, i.e. is identically satisfied by gauge fields :

$$\tilde{k} = D_s \xi \quad , \quad \mathrm{Tr}\, \xi = 0 \quad , \tag{3.9}$$

while applying $\partial_s \cdot$ to the left side of $(3.5)_a$ gives zero. On the other hand, the choice [6] , [7],

$$c = 2/(2s+1) \tag{3.10}$$

restricts the space of solutions to the minimal content of any massless invariant theory : such a space is made up of traceless, double-divergenceless tensors and carries the direct sum of two nondecomposable representations or Gupta-Bleuler triplets :

$$\begin{array}{ccccc} D(s+2,s-1) & \longrightarrow & D(s+1,s) & \longrightarrow & D(s+2,s-1) \\ & & \oplus & & \\ D(2-s,s) & \longrightarrow & D(s+1,s) & \longrightarrow & D(2-s,s) \\ & \searrow & \oplus & \nearrow & \\ & & D(1-s,s-1) & & \end{array} \tag{3.11}$$

The arrow in A $\longrightarrow$ B means an extension of the representation B by A. In other words, a group element g will be represented by a nondecomposable block matrix :

$$g \longrightarrow \begin{pmatrix} B(g) & Z(g)A(g) \\ 0 & A(g) \end{pmatrix},$$

Z(g) being some nontrivial 1-cocycle of extension [8] :

$$Z(gg') = Z(g) + B(g)Z(g')A(g^{-1})$$

Carrier states of (3.11) propagate on the de Sitter light cone , i.e. singularities of two-point functions are solely meromorphic in $\rho\, y.y' = 1$ [7],[9] . Any other choice of c introduces logarithmic singularities which implies reverberation inside the light cone. However, physical states remain unaffected by any choice of c: they carry D(s+1,s), the URREP parts of the triplets, once cleaned up from the D(s+2,s-1) and D(2-s,s)-gauge fields. Physical states really propagate on the light cone.

Conformal invariance and light-cone propagation are intimately linked. It results that the very special value (3.10) taken on by c should denote a conformal lineage. This has been proved for general s [10] and will just appear as a by-product in the particular cases s = 0,1,2 we are going to consider in the next.

4. Conformally Invariant Wave Equations in 3+2 de Sitter Space : spin-0 and 1 cases.

a) Scalar Fields s = 0.

The simplest CI system is obtained from (1.2) with p = 1

$$\partial^2 \psi = 0 \quad , \quad \hat{N}_4 \psi = -\psi \ . \tag{4.1}$$

The scalar de Sitter field is defined by (2.6)

$$k = y_4 \psi \ . \tag{4.2}$$

It obeys the C.I. equation issued from (2.3)

$$(Q_0 + 2)\, k = 0 \ . \tag{4.3}$$

The solutions of (4.3) carry the scalar massless representation of $SO_0(3,2)$, direct sum of two URREPS.

$$D(1,0) \oplus D(2,0) \ . \tag{4.4}$$

Eq. (4.3) is invariant under the purely-conformal transformation

$$\delta k = -i \rho^{-1/2} (z \cdot \bar{\partial} - \rho\, y \cdot z)\, k \tag{4.5}$$

which mixes up D(1,0)-states and D(2,0)-states.
The symbol z stands for the infinitesimal five-vector $z_\alpha = \varepsilon_{4\alpha}$.

b) Conformal features of the de Sitter Q.E.D. : s = 1.

We first adopt the procedure described in Ref. [11]. Let ψ be of rank one. Assume it be solution of

$$(\partial_A \partial^A)\, \psi = 0 \quad , \quad \hat{N}_4 \psi = -\psi \ . \tag{4.6}$$

The purely-de Sitter field $k_\alpha \equiv y_4 T \psi_\alpha \, (= y_4 K_1^{'})$ and the auxiliary field $\psi_4 \, (= K_0^{\circ})$ obey the following system, once eliminated the other auxiliary field $y \cdot \psi \ (= K_0^{'})$.

$$Q_1 k + \frac{2}{3} D_1 k = \frac{1}{6} D_1 Q_0 \partial . k \ , \tag{4.7}$$

$$(Q_0 + 2)\, \psi_4 = 0 \ . \tag{4.8}$$

Subsequently, taking the divergence of (4.7) leads to

$$Q_0 (Q_0 + 2)\, \partial . k = 0 \ . \tag{4.9}$$

General solutions to (4.7) can be found out from solutions to (4.6). Consider the combinations

$$\psi_A = Z_A\, \varphi_1 + \nabla_A\, \varphi_2 \tag{4.10}$$

with $\partial_A \partial^A \varphi_1 = 0$ and $(\partial_A \partial^A)^2 \varphi_2 = 0$.

Z_A is a constant six vector. Trivially, each term of (4.10) form a conformally invariant space of solutions to (4.6). They induce solutions to (4.7) :

$$k_\alpha = \theta_\alpha \cdot Z\, \varphi_1 + D_{1\alpha}\, \varphi_2 \tag{4.11}$$

with

$$(Q_0 + 2)\varphi_1 = 0 \;,\; Q_0 (Q_0 + 2)\varphi_2 = 0\;. \tag{4.12}$$

Z is now a constant five-vector.

$\theta_{\alpha\beta} \equiv \delta_{\alpha\beta} - \rho\, y_\alpha y_\beta$ projects on transverse tensors.

A (not CI) subspace of gauge solution $D_1 \varphi_2$ is therefore made explicit, directly issued from $\nabla_A \varphi_2$. Even (4.11) is not CI, since we must take into account the auxiliary fields whenever purely conformal transformations $k \longrightarrow k + \delta k$, $\psi_4 \longrightarrow \psi_4 + \delta\psi_4$, are carried out. Explicitly

$$\begin{aligned}\delta k_\alpha = & -\rho^{-1/2}\left[T\, z . \bar{\partial}\, k_\alpha - \rho\, y \cdot z\, k_\alpha \right] + \\ & - \rho^{-1/2}\, \theta_\alpha \cdot Z \left[y \cdot \psi - \rho^{-1/2}\, \psi_4 \right] .\end{aligned}$$

Besides, an invariant bilinear form on the space of the two-component fields $\binom{k}{\psi_4}$ can be defined [12] .

At this stage, we would like to give up auxiliary fields in order to deal with k alone. Two possibilities exist. Either we wish to keep up conformal invariance or we aim to the minimal structure (3.11).

(i) CI wave equations on de Sitter space.

The first purpose is achieved by adding to (4.6) the CI condition :

$$u^A \psi_A = y_4 \left(y \cdot \psi - \rho^{-1/2}\, \psi_4 \right) = 0\;. \tag{4.13}$$

We are then left with the system,

$$Q_1 k + D_1\, \partial \cdot k = 0, \tag{4.14}$$

$$(Q_0 + 2)\, \partial \cdot k = 0\,. \tag{4.15}$$

which corresponds to the gauge fixing c = 1 in $(3.5)_a$. However this is not the fully gauge-invariant case since condition (4.15) restricts the gauge-field space and preserves the null-cone propagation of the solutions. As expected, (4.14)-(4.15) is invariant under

$k \to k + \delta k$ with

$$\delta k = -\rho^{-1/2}[\, T z\cdot\bar{\partial} k - \rho\, y\cdot z\, k\,]\,. \tag{4.16}$$

(ii) Minimal structure.

It is obtained from (4.7) by imposing

$$Q_0\, \partial\cdot k = 0, \tag{4.17}$$

which is compatible with (4.9) although it obviously breaks the conformal invariance. Eq. (3.5) then follows and the gauge is fixed to be c = 2/3.

As a final remark, we mention that imposing further to (4.13) the other CI condition

$$\nabla^A \psi_A = 0 \,.$$

does not bring in anything new.

5. About Some Conformal Aspects of Linear de Sitter Gravity.

We now deal with the case s = 2. ψ is a symmetric two-rank tensor field which obeys (1.2). Because we hope to attain a de Sitter field for which the lowest energy E_o is s+1 = 3, the power p has to take on a value equal to or larger than 2. It is clear from (2.3) where the terms (j+1) (j-2) run over possible values of $E_o(E_o-3)$. It is reasonable to fix our choice on the value p = 2.

The auxiliary fields K_1^1 , K_0^1 , K_0^0 are unnecessary to write down the de Sitter equations. We will adopt the notation for the remaining ones :

$$k = K_2^2 \quad , \quad K = K_1^2 \quad , \quad A = K_0^2 \,.$$

They obey the system

$$\begin{aligned}(Q_2 - 6)(Q_2 - 4)k - 4 D_2 \partial^T\!\!\cdot k + 4 Q_2 \Theta k' + \\ -4\rho Q_2 D_2 K - 16\Theta\partial\cdot K + 4\rho^2 D_2 D_1 A - 4\rho(Q_2+6)\Theta A = 0,\end{aligned} \tag{5.1}$$

$$\begin{aligned}Q_1(Q_1+14)K - 12 D_1 \partial^T\!\!\cdot k - 4\rho(Q_1+8) D_1 A + \\ + 4\rho^{-1} Q_1 \partial^T\!\!\cdot k + 8 D_1 k' = 0\end{aligned} \tag{5.2}$$

$$(Q_0^2 + 26 Q_0 + 96)A + 8\rho^{-1}(Q_0+8)\partial\cdot K - 4\rho^{-1}(Q_0+6)k' + 8\rho^{-2}\partial.\partial^T\!\!\cdot k = 0. \tag{5.3}$$

" $\partial^T_{\cdot}$ " stands for " $T\,\partial\cdot$ " . Θ was introduced in (4.12). k' is shortened notation for $Tr\, k$.

Without entering into details about deriving these formulas, we just mention the intertwining properties of Q_s, D_s, $\Sigma\,\theta$:

$$Q_s D_s = D_s Q_{s-1}\ ,\ Q_s \Sigma\,\theta = \Sigma\,\theta\, Q_{s-2}\ ,\ D_s \Sigma\,\theta = \Sigma\,\theta\, D_{s-2}\ .$$

As for spin one, it is possible to get a (not CI) equation involving k alone. The (actually valid for any spin) procedure consists to carry out trace and divergences of (5.1) and divergence of (5.2). A sufficient number of equations is then available in order to eliminate the one-rank tensor K and the scalars $\partial\cdot K$ and A. The result is somewhat complicated :

$$(Q_2-6)(Q_2-4)k - \frac{1}{20}(Q_2^2 - 22Q_2 + 80)D_2\partial^T\cdot k +$$
$$-\frac{1}{15}(Q_2-6)D_2D_1\partial\cdot\partial^T\cdot k - \frac{\rho^{-1}}{3}(Q_2-6)\theta\,\partial\cdot\partial^T\cdot k +$$
$$-\frac{\rho}{360}(Q_2^3 + 7Q_2^2 + 70Q_2 - 360)D_2D_1 k' +$$
$$-\frac{1}{72}(Q_2^3 + 16Q_2^2 - 236Q_2 + 432)\theta k' = 0\ . \tag{5.4}$$

It follows the one-rank and scalar equations

$$Q_1(Q_1-2)(Q_1-6)\partial^T\cdot k - (12Q_1-8)D_1\partial\cdot\partial^T\cdot k +$$
$$-\rho\, Q_1(Q_1+2)(Q_1+20)D_1 k' = 0\ , \tag{5.5}$$

$$Q_0(Q_0+2)[2\partial\cdot\partial^T\cdot k + \rho(3Q_0+20)k'] = 0, \tag{5.6}$$

$$(Q_0-10)(Q_0-4)Q_0(Q_0+2)k' = 0. \tag{5.7}$$

As we already mentioned Eq. (5.4) is not invariant under purely conformal transformations. The latter mix up components k, $T(y\cdot\psi) \equiv K_1^2$, and $T\psi_4 \equiv K_1^1$:

$$\delta k = -\rho^{-1/2}[T z\cdot\bar\partial k - \Sigma\,\theta\cdot z(K_1^2 - \rho^{-1/2}K_1^1)] \tag{5.8}$$

The discussion is now similar to that one in Sec. 4.b.

(i) <u>CI wave equations on de Sitter Space</u>.

By imposing the CI condition $u^A\psi_A = K_1^2 - \rho^{-1/2}K_1^1 = 0$, we obtain a CI system made up of one two-rank wave equation, one one-rank, and one scalar condition. Note that we have reintroduced the symbol $\partial_2\cdot$ defined by (3.7)

$$(Q_2-4)[(Q_2-6)k+D_2\partial_2\cdot k]+$$

$$+\frac{1}{3}[D_2D_1-\rho^{-1}\theta(Q_0-6)][\partial.\partial^T.k+\rho(Q_0-3)k']=0, \quad (5.9)$$

$$Q_1(Q_1-2)\partial^T.k+\frac{4}{3}(Q_1-6)D_1\partial.\partial^T.k=0, \quad (5.10)$$

$$Q_0(Q_0+2)k'=0. \quad (5.11)$$

(5.10) entails the scalar constraint $Q_0(Q_0+2)\partial.\partial^T.k=0$.
(5.9)-(5.11) is invariant under $k\longrightarrow k+\delta k$,

$$\delta k=-\rho^{-1/2}\,T\,Z.\bar{\partial}k. \quad (5.12)$$

A very interesting feature of (5.9) holds in the fact that it is completely factorizable, i.e. it can be written ;

$$\mathcal{O}_6\mathcal{O}_4k=\mathcal{O}_4\mathcal{O}_6k=0, \quad (5.13)$$

which is reminiscent of $(\partial^A\partial_A)(\partial^A\partial_A)\psi=0$!
Precisely $\mathcal{O}_4$ is this operator we would have obtained if we had started from $\partial^A\partial_A\psi=0$ instead

$$\mathcal{O}_4k\equiv(Q_2-4)k-\frac{1}{12}(Q_2-8)D_2\partial^T.k+$$

$$-\frac{1}{12}D_2D_1\partial.\partial^T.k-\frac{\rho^{-1}}{3}\theta\partial.\partial^T.k+$$

$$-\frac{\rho}{24}(Q_2+6)D_2D_1k'-\frac{1}{3}(Q_2-3)\theta k'=0. \quad (5.14)$$

The partner factor $\mathcal{O}_6$ is given by

$$\mathcal{O}_6k\equiv(Q_2-6)k-\frac{1}{10}(Q_2-10)D_2\partial^T.k+$$

$$-\frac{3}{20}D_2D_1\partial.\partial^T.k-\frac{\rho^{-1}}{2}\theta\partial.\partial^T.k+$$

$$-\frac{\rho}{40}(Q_2+10)D_2D_1k'-\frac{1}{4}(Q_2-6)\theta k'. \quad (5.15)$$

Contrarily to the "s=1" situation the system (5.9)-(5.11) is still far from $(3.5)_a$ with c = 1. However we can supplement $u^A\psi_A=0$ with two other CI conditions.

$$\delta^{AB}\psi_{AB}=0 \qquad \text{(tracelessness)}. \quad (5.16)$$

$$\nabla^A \psi_A = 0 \qquad \text{(divergencelessness)} . \tag{5.17}$$

(5.16) does not bring out anything new. Condition (5.17) (which together with $u^A \psi_A = 0$ actually imply (5.16)) amounts to the following one

$$\partial^T . k = -5 \rho K ,$$

which in turn entails :

$$Q_1 \partial^T . k - \frac{1}{6} (Q_1 - 4) D_1 \partial . \partial^T . k = 0 , \tag{5.18}$$

a formula to be compared to (4.7).

Besides, the de Sitter tracelessness k' = 0 is clearly invariant under $k \longrightarrow k - \rho^{-1/2} T Z . \bar{\partial} k$. To sum up, the best we can reach is the following CI system :

$$(Q_2 - 4) [(Q_2 - 6) k + D_2 \partial^T . k] +$$

$$+ \frac{1}{3} [D_2 D_1 - \rho^{-1} \theta (Q_0 - 6)] \partial . \partial^T . k = 0 , \tag{5.19}$$

$$Q_1 \partial^T . k - 1/6 \; (Q_1 - 4) D_1 \partial . \partial^T . k = 0 , \tag{5.20}$$

$$k' = 0 . \tag{5.21}$$

Let us now consider the two-rank tensors k which are solutions to $(3.5)_a$ with the gauge fixing c = 1 . They form a $SO_0(3,2)$-(but not conformally) invariant subspace of solutions to (5.19) if furthermore they are traceless and obey the (not C.I.) conditions :

$$Q_1 \partial^T . k = 0 , \tag{5.22}$$

$$\partial . \partial^T . k = 0 . \tag{5.23}$$

This means that $\partial^T . k$ describes a physical spin-one massless field carrying D(2,1) (modulo D(3,0) or D(1,1)-gauge fields).
A second $SO_0(3,2)$-invariant subspace of solutions is made up of solutions to the following system, straightly issued from $\mathcal{O}_4 k = 0$:

$$(Q_2 - 4) k + \frac{2}{3} D_2 \partial^T . k - \frac{\rho^{-1}}{3} \theta \partial . \partial^T . k = 0 , \tag{5.24}$$

$$Q_1 \partial^T . k + D_1 \partial . \partial^T . k = 0 , \tag{5.25}$$

$$(Q_0 + 2) \partial . \partial^T . k = 0 , \tag{5.26}$$

$$k' = 0 . \tag{5.27}$$

(Compare the constraints (5.25) and (5.26) with (4.14) and (4.15)).

This system is not conformally invariant under (5.12) but instead under a transformation similar to (4.16). Its solutions have no physical meaning and are given the name of ghosts. Those which are divergenceless are associated to the nonunitary infinite representations D(2,2) and D(-1,2).

The role of Transformation (5.12) amounts to mix up both subspaces of solutions. Therefore conformal invariance in 3+2-de Sitter linear gravity keeps up with inevitable appearance of ghosts.

ii) Minimal structure.

Let us now try to recover the minimal structure (3.11). It is then necessary to impose the following conditions on solutions to the system (5.4)-(5.7) :

$$\partial \cdot \partial^T \cdot k = 0, \tag{5.28}$$

$$k' = 0, \tag{5.29}$$

$$(Q_1 - 6)\,\partial \cdot k = 0 . \tag{5.30}$$

They are compatible with (5.5)-(5.7) and considerably lighten (5.4) :

$$(Q_2 - 6)(Q_2 - 4)\,k + \frac{4}{5}\, D_2\, \partial^T \cdot k = 0 . \tag{5.31}$$

Dropping the operator (Q_2-4) by using (5.30) yields Eq. $(3.5)_a$ with the "good" gauge-fixing $c = \frac{2}{5}$.

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