ON MUTATIONAL DEFORMATION RETRACTS

José M. R. Sanjurjo

Let X be a closed subset of a metrizable space X' considered as a closed subset of an ANR($\mathcal{M}$)-space P. The family $\underline{U}(X',P)$ of all open neighborhoods of X' in P is called the complete neighborhood system of X' in P. By a mutational deformation retraction of X' to X we mean a mutational retraction (see [5]) $\underline{r}:\underline{U}(X',P) \longrightarrow \underline{U}(X,P)$ such that for every $U' \in \underline{U}(X',P)$ and for every $r \in \underline{r}$ with range U' there exists $V' \in \underline{U}(X',P)$ contained in U' and in the domain of r such that $r_{|V'} \simeq i$ (the identity) in U'. If the homotopy can be chosen stationary on X we say that $\underline{r}$ is a stationary mutational deformation retraction. A mutational retraction $\underline{r}:\underline{U}(X',P) \longrightarrow \underline{U}(X,P)$ is said to be regular if for every $U' \in \underline{U}(X',P)$ and for every $r,r' \in \underline{r}$ with range U' there exists $V' \in \underline{U}(X',P)$ such that $r_{|V'} \simeq r'_{|V'}$ (rel X) in U'. The notion of regular mutational retraction is a generalization of Dydak's notion of regular fundamental retraction [2] in which a more restrictive condition is imposed on homotopies.

The problem whether every W-shape deformation retract is stationary has been raised by K. Borsuk in his book [1] (p. 190, Problem 4.15) and, up to the author's knowledge, is open even in the compact case. In the present note we give a partial answer to the analogous problem in Fox shape theory [4]. The reader is referred to [1],[3] and [6] for information about theory of shape.

Theorem 1. Let $\underline{r}:\underline{U}(X',P) \longrightarrow \underline{U}(X,P)$ be a deformation mutational retraction. Then $\underline{r}$ is stationary if and only if $\underline{r}$ is regular.

Proof. The part "only if" is trivial, we are going to prove the converse. Let $U' \in \underline{U}(X',P)$ and consider $r \in \underline{r}$ with range U' and domain $U'_0 \in \underline{U}(X',P)$. Since $\underline{r}$ is a mutational deformation retraction there exists an open neighborhood $V' \in U'_0$ of X' in P such that

(1) $r_{|V'} \simeq i$ in U'.

Since $U' \in$ ANR it is easy to see, by using the homotopy extension theorem, that there exist a map $s:V' \longrightarrow U'$ and an open neighborhood

This paper is in final form and no version of it will be submitted elsewhere.

$U \subset V'$ of X in P such that

(2) $s(x)=x$ for every $x \in U$

(3) $r_{|V'} \simeq s_{|V'}$ (rel X) in U'.

Since $\underline{r}$ is regular, there exists $r' \in \underline{r}$ such that $r': W' \longrightarrow U$ where $W' \subset V'$ is an open neighborhood of X' in P and such that

(4) $r' \simeq r_{|W'}$ (rel X) in U'.

Let us now define a map $\phi: K = W' \times \{0\} \cup X \times I \cup W' \times \{1\} \longrightarrow V'$ by

$$\phi(x,0)=x,\ \phi(x,1)=r'(x) \text{ for every } x \in W'$$

and $$\phi(x,t)=x \text{ for } (x,t) \in X \times I.$$

Since $\phi(x,1) \in U$ it follows from (2) that

$$s\phi(x,0)=s(x),\ s\phi(x,1)=r'(x) \text{ for every } x \in W'$$

$$\text{and } s\phi(x,t)=x \text{ for } (x,t) \in X \times I.$$

It follows from (1) and (3) that $\phi \simeq s\phi$ in U'. Moreover $s\phi$ is homotopic in U' to the map $\psi: K \longrightarrow U'$ defined by

$$\psi(x,0)=\psi(x,1)=r'(x) \text{ for every } x \in W'$$

$$\text{and } \psi(x,t)=x \text{ for } (x,t) \in X \times I.$$

To see it consider a homotopy $\chi: W' \times I \longrightarrow U'$ such that $\chi(x,0)=s(x)$, $\chi(x,1)=r'(x)$ for $x \in W'$ and $\chi(x,t)=x$ for $(x,t) \in X \times I$. We define a map $F: K \times I \longrightarrow U'$ by

$$F((x,o),t')=\chi(x,t'),F((x,1),t')=r'(x) \text{ for } x \in W',\ t' \in I$$

$$\text{and } F((x,t),t')=x \text{ for } x \in X \text{ and } t,t' \in I.$$

Obviously, $F((x,t),0)=s\phi(x,t)$ and $F(x,t),1)=\psi(x,t)$ for $(x,t) \in K$. Hence $\psi \simeq s\phi \simeq \phi$. Since $U' \in$ ANR and ψ can be extended to $W' \times I$ (by the map $\hat{\psi}(x,t)=r'(x)$) then, in virtue of the homotopy extension theorem, ϕ can also be extended to a map $\hat{\phi}: W' \times I \longrightarrow U'$ which realizes a homotopy between i and r' stationary on X. Since $r_{|W'} \simeq r'$ (rel X) in U' we conclude that $r_{|W'} \simeq i$ (rel X) in U' and, consequently, $\underline{r}$ is stationary.

Corollary. Let X be an MANR [5] with compact components. If X is a mutational deformation retract of a metrizable space X' lying in $P \in \mathrm{ANR}(\mathcal{M})$, then X is a stationary mutational deformation retract of X'.

Proof. By Corollary 3.11 of [5] $X = \bigoplus\{X_i, i \in I\}$, where $\{X_i, i \in I\}$ is the family of all components of X. Since $X_i \in$ FANR for every $i \in I$ it follows from Dydak's Corollary 1, [2], that each X_i is a regular mutational retract of one of its neighborhoods in P. Hence, there exists a neighborhood W of X in P which can be represented as a topological sum $W = \bigoplus\{W_i, i \in I\}$, where W_i is a neighborhood of X_i in P and X_i is a regular mutational retract of W_i for $i \in I$. Consequently, there exists a regular mutational retraction $\underline{r}: \underline{U}(W,P) \longrightarrow \underline{U}(X,P)$. Since X is a mutational retract of X' there exists a map $s: X' \longrightarrow W$

such that $s(x)=x$ for every $x \in X$. Let $\underline{s}:\underline{U}(X',P) \longrightarrow \underline{U}(W,P)$ be a mutation generated by s. Then $\underline{r}'=\underline{r}\ \underline{s}$ is a regular mutational retraction and, since X is a mutational deformation retract of X', we can easily get from Theorem 1 that $\underline{r}'$ is a stationary mutational deformation retraction.

REFERENCES

1. BORSUK, K. Theory of Shape, Monografie Matematyczne 59, Polish Scientific Publishers, Warszawa, 1975.
2. DYDAK, J. A simple proof that pointed FANR spaces are regular fundamental retracts of ANR's, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys, 25 (1977) 55-62.
3. DYDAK, J. and SEGAL, J. Shape Theory: An introduction, Lecture Notes in Math. 688, Springer-Verlag, Berlin, 1978.
4. FOX, R.H. On Shape, Fund. Math. 74 (1972), 47-71.
5. GODLEWSKI, S. On components of MANR-spaces, Fund. Math. 114 (1981) 1-9.
6. MARDEŠIĆ, S. and SEGAL, J. Shape Theory, North Holland, Amsterdam, 1982.

Departamento de Geometría y Topología
Facultad de Matemáticas
Universidad Complutense
28040 Madrid. Spain.