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ON MUTATIONAL DEFORMATION RETRACTS

José M. R. Sanjurjo

Let X be a closed subset of a metrizable space X' considered as a closed subset of an ANR(\mathcal{A})-space P . The family $\underline{U}(X', P)$ of all open neighborhoods of X' in P is called the complete neighborhood system of X' in P . By a mutational deformation retraction of X' to X we mean a mutational retraction (see [5]) $\underline{r}: \underline{U}(X', P) \rightarrow \underline{U}(X, P)$ such that for every $U' \in \underline{U}(X', P)$ and for every $r \in \underline{r}$ with range U' there exists $V' \in \underline{U}(X', P)$ contained in U' and in the domain of r such that $r|_{V'} \simeq i$ (the identity) in U' . If the homotopy can be chosen stationary on X we say that \underline{r} is a stationary mutational deformation retraction. A mutational retraction $\underline{r}: \underline{U}(X', P) \rightarrow \underline{U}(X, P)$ is said to be regular if for every $U' \in \underline{U}(X', P)$ and for every $r, r' \in \underline{r}$ with range U' there exists $V' \in \underline{U}(X', P)$ such that $r|_{V'} \simeq r'|_{V'}$ (rel X) in U' . The notion of regular mutational retraction is a generalization of Dydak's notion of regular fundamental retraction [2] in which a more restrictive condition is imposed on homotopies.

The problem whether every W -shape deformation retract is stationary has been raised by K. Borsuk in his book [1] (p. 190, Problem 4.15) and, up to the author's knowledge, is open even in the compact case. In the present note we give a partial answer to the analogous problem in Fox shape theory [4]. The reader is referred to [1], [3] and [6] for information about theory of shape.

Theorem 1. Let $\underline{r}: \underline{U}(X', P) \rightarrow \underline{U}(X, P)$ be a deformation mutational retraction. Then \underline{r} is stationary if and only if \underline{r} is regular.

Proof. The part "only if" is trivial, we are going to prove the converse. Let $U' \in \underline{U}(X', P)$ and consider $r \in \underline{r}$ with range U' and domain $U'_0 \in \underline{U}(X', P)$. Since \underline{r} is a mutational deformation retraction there exists an open neighborhood $V' \in U'_0$ of X' in P such that

$$(1) \quad r|_{V'} \simeq i \text{ in } U'.$$

Since $U' \in \text{ANR}$ it is easy to see, by using the homotopy extension theorem, that there exist a map $s: V' \rightarrow U'$ and an open neighborhood

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$U \subset V'$ of X in P such that

(2) $s(x)=x$ for every $x \in U$

(3) $r|_{V'} \approx s|_{V'}$ (rel X) in U' .

Since \underline{r} is regular, there exists $r' \in \underline{r}$ such that $r':W' \rightarrow U$ where $W' \subset V'$ is an open neighborhood of X' in P and such that

(4) $r' \approx r|_{W'}$ (rel X) in U' .

Let us now define a map $\phi:K=W' \times \{0\} \cup X \times I \cup W' \times \{1\} \rightarrow V'$ by

$$\phi(x,0)=x, \phi(x,1)=r'(x) \text{ for every } x \in W'$$

and $\phi(x,t)=x$ for $(x,t) \in X \times I$.

Since $\phi(x,1) \in U$ it follows from (2) that

$$s\phi(x,0)=s(x), s\phi(x,1)=r'(x) \text{ for every } x \in W'$$

$$\text{and } s\phi(x,t)=x \text{ for } (x,t) \in X \times I.$$

It follows from (1) and (3) that $\phi \approx s\phi$ in U' . Moreover $s\phi$ is homotopic in U' to the map $\psi:K \rightarrow U'$ defined by

$$\psi(x,0)=\psi(x,1)=r'(x) \text{ for every } x \in W'$$

$$\text{and } \psi(x,t)=x \text{ for } (x,t) \in X \times I.$$

To see it consider a homotopy $\chi:W' \times I \rightarrow U'$ such that $\chi(x,0)=s(x)$, $\chi(x,1)=r'(x)$ for $x \in W'$ and $\chi(x,t)=x$ for $(x,t) \in X \times I$. We define a map $F:K \times I \rightarrow U'$ by

$$F((x,0),t')=\chi(x,t'), F((x,1),t')=r'(x) \text{ for } x \in W', t' \in I$$

$$\text{and } F((x,t),t')=x \text{ for } x \in X \text{ and } t, t' \in I.$$

Obviously, $F((x,t),0)=s\phi(x,t)$ and $F(x,t,1)=\psi(x,t)$ for $(x,t) \in K$.

hence $\psi \approx s\phi \approx \phi$. Since $U' \in \text{ANR}$ and ψ can be extended to $W' \times I$ (by the map $\hat{\psi}(x,t)=r'(x)$) then, in virtue of the homotopy extension theorem, ϕ can also be extended to a map $\hat{\phi}:W' \times I \rightarrow U'$ which realizes a homotopy between i and r' stationary on X . Since $r|_{W'} \approx r'$ (rel X) in U' we conclude that $r|_{W'} \approx i$ (rel X) in U' and, consequently, \underline{r} is stationary.

Corollary. Let X be an MANR [5] with compact components. If X is a mutational deformation retract of a metrizable space X' lying in $P \in \text{ANR}(\mathcal{M})$, then X is a stationary mutational deformation retract of X' .

Proof. By Corollary 3.11 of [5] $X = \bigoplus \{X_i, i \in I\}$, where $\{X_i, i \in I\}$ is the family of all components of X . Since $X_i \in \text{FANR}$ for every $i \in I$ it follows from Dydak's Corollary 1, [2], that each X_i is a regular mutational retract of one of its neighborhoods in P . Hence, there exists a neighborhood W of X in P which can be represented as a topological sum $W = \bigoplus \{W_i, i \in I\}$, where W_i is a neighborhood of X_i in P and X_i is a regular mutational retract of W_i for $i \in I$. Consequently, there exists a regular mutational retraction $\underline{r}:U(W,P) \rightarrow U(X,P)$. Since X is a mutational retract of X' there exists a map $s:X' \rightarrow W$

such that $s(x)=x$ for every $x \in X$. Let $\underline{s}: \underline{U}(X', P) \longrightarrow \underline{U}(W, P)$ be a mutation generated by s . Then $\underline{r}' = \underline{r} \underline{s}$ is a regular mutational retraction and, since X is a mutational deformation retract of X' , we can easily get from Theorem 1 that \underline{r}' is a stationary mutational deformation retraction.

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