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ON MUTATIONAL DEFORMATION RETRACTS

José M. R. Sanjurjo

Let  $X$  be a closed subset of a metrizable space  $X'$  considered as a closed subset of an ANR( $\mathcal{A}$ )-space  $P$ . The family  $\underline{U}(X', P)$  of all open neighborhoods of  $X'$  in  $P$  is called the complete neighborhood system of  $X'$  in  $P$ . By a mutational deformation retraction of  $X'$  to  $X$  we mean a mutational retraction (see [5])  $\underline{r}: \underline{U}(X', P) \rightarrow \underline{U}(X, P)$  such that for every  $U' \in \underline{U}(X', P)$  and for every  $r \in \underline{r}$  with range  $U'$  there exists  $V' \in \underline{U}(X', P)$  contained in  $U'$  and in the domain of  $r$  such that  $r|_{V'} \simeq i$  (the identity) in  $U'$ . If the homotopy can be chosen stationary on  $X$  we say that  $\underline{r}$  is a stationary mutational deformation retraction. A mutational retraction  $\underline{r}: \underline{U}(X', P) \rightarrow \underline{U}(X, P)$  is said to be regular if for every  $U' \in \underline{U}(X', P)$  and for every  $r, r' \in \underline{r}$  with range  $U'$  there exists  $V' \in \underline{U}(X', P)$  such that  $r|_{V'} \simeq r'|_{V'}$  (rel  $X$ ) in  $U'$ . The notion of regular mutational retraction is a generalization of Dydak's notion of regular fundamental retraction [2] in which a more restrictive condition is imposed on homotopies.

The problem whether every  $W$ -shape deformation retract is stationary has been raised by K. Borsuk in his book [1] (p. 190, Problem 4.15) and, up to the author's knowledge, is open even in the compact case. In the present note we give a partial answer to the analogous problem in Fox shape theory [4]. The reader is referred to [1], [3] and [6] for information about theory of shape.

Theorem 1. Let  $\underline{r}: \underline{U}(X', P) \rightarrow \underline{U}(X, P)$  be a deformation mutational retraction. Then  $\underline{r}$  is stationary if and only if  $\underline{r}$  is regular.

Proof. The part "only if" is trivial, we are going to prove the converse. Let  $U' \in \underline{U}(X', P)$  and consider  $r \in \underline{r}$  with range  $U'$  and domain  $U'_0 \in \underline{U}(X', P)$ . Since  $\underline{r}$  is a mutational deformation retraction there exists an open neighborhood  $V' \in U'_0$  of  $X'$  in  $P$  such that

$$(1) \quad r|_{V'} \simeq i \text{ in } U'.$$

Since  $U' \in \text{ANR}$  it is easy to see, by using the homotopy extension theorem, that there exist a map  $s: V' \rightarrow U'$  and an open neighborhood

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$U \subset V'$  of  $X$  in  $P$  such that

(2)  $s(x)=x$  for every  $x \in U$

(3)  $r|_{V'} \approx s|_{V'}$  (rel  $X$ ) in  $U'$ .

Since  $\underline{r}$  is regular, there exists  $r' \in \underline{r}$  such that  $r':W' \rightarrow U$  where  $W' \subset V'$  is an open neighborhood of  $X'$  in  $P$  and such that

(4)  $r' \approx r|_{W'}$  (rel  $X$ ) in  $U'$ .

Let us now define a map  $\phi:K=W' \times \{0\} \cup X \times I \cup W' \times \{1\} \rightarrow V'$  by

$$\phi(x,0)=x, \phi(x,1)=r'(x) \text{ for every } x \in W'$$

and  $\phi(x,t)=x$  for  $(x,t) \in X \times I$ .

Since  $\phi(x,1) \in U$  it follows from (2) that

$$s\phi(x,0)=s(x), s\phi(x,1)=r'(x) \text{ for every } x \in W'$$

$$\text{and } s\phi(x,t)=x \text{ for } (x,t) \in X \times I.$$

It follows from (1) and (3) that  $\phi \approx s\phi$  in  $U'$ . Moreover  $s\phi$  is homotopic in  $U'$  to the map  $\psi:K \rightarrow U'$  defined by

$$\psi(x,0)=\psi(x,1)=r'(x) \text{ for every } x \in W'$$

$$\text{and } \psi(x,t)=x \text{ for } (x,t) \in X \times I.$$

To see it consider a homotopy  $\chi:W' \times I \rightarrow U'$  such that  $\chi(x,0)=s(x)$ ,  $\chi(x,1)=r'(x)$  for  $x \in W'$  and  $\chi(x,t)=x$  for  $(x,t) \in X \times I$ . We define a map  $F:K \times I \rightarrow U'$  by

$$F((x,0),t')=\chi(x,t'), F((x,1),t')=r'(x) \text{ for } x \in W', t' \in I$$

$$\text{and } F((x,t),t')=x \text{ for } x \in X \text{ and } t, t' \in I.$$

Obviously,  $F((x,t),0)=s\phi(x,t)$  and  $F(x,t,1)=\psi(x,t)$  for  $(x,t) \in K$ .

hence  $\psi \approx s\phi \approx \phi$ . Since  $U' \in \text{ANR}$  and  $\psi$  can be extended to  $W' \times I$  (by the map  $\hat{\psi}(x,t)=r'(x)$ ) then, in virtue of the homotopy extension theorem,  $\phi$  can also be extended to a map  $\hat{\phi}:W' \times I \rightarrow U'$  which realizes a homotopy between  $i$  and  $r'$  stationary on  $X$ . Since  $r|_{W'} \approx r'$  (rel  $X$ ) in  $U'$  we conclude that  $r|_{W'} \approx i$  (rel  $X$ ) in  $U'$  and, consequently,  $\underline{r}$  is stationary.

Corollary. Let  $X$  be an MANR [5] with compact components. If  $X$  is a mutational deformation retract of a metrizable space  $X'$  lying in  $P \in \text{ANR}(\mathcal{M})$ , then  $X$  is a stationary mutational deformation retract of  $X'$ .

Proof. By Corollary 3.11 of [5]  $X = \bigoplus \{X_i, i \in I\}$ , where  $\{X_i, i \in I\}$  is the family of all components of  $X$ . Since  $X_i \in \text{FANR}$  for every  $i \in I$  it follows from Dydak's Corollary 1, [2], that each  $X_i$  is a regular mutational retract of one of its neighborhoods in  $P$ . Hence, there exists a neighborhood  $W$  of  $X$  in  $P$  which can be represented as a topological sum  $W = \bigoplus \{W_i, i \in I\}$ , where  $W_i$  is a neighborhood of  $X_i$  in  $P$  and  $X_i$  is a regular mutational retract of  $W_i$  for  $i \in I$ . Consequently, there exists a regular mutational retraction  $\underline{r}:U(W,P) \rightarrow U(X,P)$ . Since  $X$  is a mutational retract of  $X'$  there exists a map  $s:X' \rightarrow W$

such that  $s(x)=x$  for every  $x \in X$ . Let  $\underline{s}: \underline{U}(X', P) \longrightarrow \underline{U}(W, P)$  be a mutation generated by  $s$ . Then  $\underline{r}' = \underline{r} \underline{s}$  is a regular mutational retraction and, since  $X$  is a mutational deformation retract of  $X'$ , we can easily get from Theorem 1 that  $\underline{r}'$  is a stationary mutational deformation retraction.

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