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In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1989. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 21. pp. [295]–323.

Persistent URL: `http://dml.cz/dmlcz/701450`

SPINGROUPS AND SPHERICAL MEANS III

F. Sommen (*)

Abstract. In this paper we study the mean values of functions defined on the unit sphere S^{m-1} over spheres of a given dimension inside S^{m-1}. These mean value operators satisfy certain first order systems of differential equations with values in a Clifford algebra, which are generalizations of the Darboux equation. We construct explicit solutions to these equations and apply the solution to the Radon transforms on the unit sphere, leading to an improvement of the inversion formula obtained by S. Helgason in [3]. We also show that the Radon transform over geodesie spheres of dimension $p-1$ may be expressed in terms of the Radon transform over spheres of codimension p inside S^{m-1}.

Introduction. This paper belongs to a series of papers about applications of representations of $Spin(m)$ to problems concerning spherical means and Radon transforms. However, this paper can be read independently from the previous papers [13], [14] in this series.

In the first paper [13] we considered spherical means of functions defined in Euclidean space over spheres of codimension one. This paper was much inspired by the work of F. John [6] and lead to a refinement of the classical Darboux equation. The idea goes as follows. Let f be defined in $\mathbb{R}^m$. Then its spherical mean

$$Pf(\vec{x},r) = \frac{1}{\omega_m} \int_{S^{m-1}} f(\vec{x} + r\vec{\omega})\, d\vec{\omega}$$

satisfies the Euler-Poisson-Darboux equation

$$\left[\Delta_x - \left(\frac{\partial^2}{\partial r^2} + \frac{m}{r}\frac{\partial}{\partial r}\right)\right] Pf = 0 .$$

(*) Research Associate supported by the National Fund for Scientific Research, Belgium

By introducing the oriented spherical mean

$$Qf(\vec{x},r) = \frac{1}{\omega_m}\int_{S^{m-1}} \vec{\omega} f(\vec{x} + r\vec{\omega})\, d\vec{\omega} ,$$

this equation may be replaced by the "Darboux system"

$$D_x Pf(\vec{x},r) = \left(\frac{\partial}{\partial r} + \frac{m-1}{r}\right) Qf(\vec{x},r) , \quad D_x Qf(\vec{x},r) = -\frac{\partial}{\partial r} Pf(\vec{x},r) ,$$

where $D_x = \sum_{j=1}^{m} e_j \frac{\partial}{\partial x_j}$ is the Dirac operator and the function f takes values in the complex Clifford algebra $\mathbb{C}_m$. Hence, Clifford algebra valued functions show to be quite useful for the theory of spherical means and Radon transform (see also [15]). For function theoretical properties of quaternion and Clifford algebra valued functions we refer to [2], [4], [8], [10], [18], [19].

In our second paper [14], we generalized the Darboux system to various types of spherical means of functions in $\mathbb{R}^m$ over spheres of higher codimension. Especially the case of spherical means of codimension 2 seems interesting because of its links with complex analysis. Indeed, consider the spherical mean operator

$$Mf(\vec{x},\vec{y}) = \int_{S^{m-1}} (1 + i\vec{\nu}\,\vec{\omega})\,(\vec{\nu}\cdot\vec{\omega}) f(\vec{x} + r\vec{\omega})\, d\vec{\omega} , \quad \vec{y} = r\vec{\nu} \;; \quad \text{then}$$

$$(D_x + iD_y) Mf(\vec{x},\vec{y}) = 0 \text{ where } \frac{1}{2}(D_x + iD_y) = \sum_{j=1}^{m} e_i \frac{\partial}{\partial \bar{z}_j} \text{ and } z_j = x_j + iy_j .$$

In the present paper we deal with spherical means of functions defined on the unit sphere S^{m-1} in $\mathbb{R}^m$. We establish and solve the spherical analogue of the Darboux equations. These equations are not merely a transformation of the ones in Euclidean space because of the cuvature of the unit sphere. Also the methods of solving them are quite typical for the unit sphere, because we make extensive use of spherical monogenics (see [12], [17], [19]). Spherical monogenics are restrictions to the unit sphere of homogeneous nullsolutions of the Dirac operator and thus lead to a refinement of spherical harmonics.

Another motivation for the study of spherical means of functions on S^{m-1} is the fact that the Radon transform on S^{m-1} in the sence of S. Helgason (see [3]) is includes in the spherical mean operator, since it suffices to restrict that operator to geodesic spheres. In this way, the solutions of the Darboux equations lead to new inversion formulae for the Radon transform as well as to a relation between the Radon transform evaluated over spheres of dimension $p-1$ and that over spheres of codimension p , called the dual Radon transform.

Our paper is organized as follows.
In a first section we recall the mains definitions and notations concerning Clifford algebra valued functions, representations of Spin(m) and spherical monogenics, needed for this paper. We also give a brief introduction to spherical monogenics on the Lie sphere $LS^{m-1} = \{e^{i\theta}\vec{\omega} : \theta \in [0,\pi[, \vec{\omega} \in S^{m-1}\}$, which may be interpreted as "parameter set" for the set of oriented spheres of codimension one inside S^{m-1} (see also [1], [7]). In our forthcoming paper [17] we made an extensive study of spherical monogenics on LS^{m-1} and already started the theory of spherical means on S^{m-1}.
In section 2 we recall our results on the spherical mean

$$Pf(e^{i\theta},\vec{v}) = \frac{1}{\omega_{m-1}} \int_{\bar{S}(\vec{v})} f(\cos\theta\vec{v} + \sin\theta\vec{\mu})\, d\vec{\mu} ,$$

where $\vec{v} \in S^{m-1}$ and $\bar{S}(\vec{v})$ is the unit sphere orthogonal to $\vec{v}$, obtained in our paper [17]. We also generalize this spherical mean operator by considering spaces of spherical monogenics on the spheres $\bar{S}(\vec{v})$.
In section 3 we consider higher codimension spherical means of the form

$$Pf(e^{i\theta},\nu,\vec{v}) = \frac{1}{\omega_{m-p}} \int_{\bar{S}(\nu)} f(\cos\theta\vec{v} + \sin\theta\vec{\mu})\, d\vec{\mu} ,$$

where ν is a unit p-blade, $\vec{v} \in S^{m-1}$ is parallel to ν and $\bar{S}(\nu)$ is the unit sphere orthogonal to ν. We derive the Darboux equations for these operators $Pf(e^{i\theta},\nu,\vec{v})$, which turn out to be too complicated to solve directly. Instead of this, we introduce the bispherical means

$$Pf(e^{i\theta},\nu) = \frac{1}{\omega_p\omega_{m-p}} \int_{S(\nu)\times\bar{S}(\nu)} f(\cos\theta\vec{v} + \sin\theta\vec{\mu})\, d\vec{\mu}d\vec{v} ,$$

where $S(\nu)$ is the unit sphere parallel to ν. For the bispherical means it is possible to establish and solve the Darboux equations explicitly. The solutions may be expressed in terms of Jacobi polynomials.
Finally we apply the results of section 2 to obtains an inversion formula for the Radon transform

$$B_+f(\vec{v}) = Pf(i,\vec{v}) .$$

We also apply section 3 to the Radon type transforms

$$B_+f(\nu) = Pf(1,\nu) , \quad \bar{B}_+f(\nu) = Pf(i,\nu) ,$$

which are dual to one another and we establish a direct relation between them.

Acknowledgement. This paper was written during the author's stay at the RWTH, Aachen, which was sponsored by an Alexander von Humboldt fellowship.

1. Basic tools

Let $\{e_1,\ldots e_m\}$ be an orthonormal basis of $\mathbb{R}^m$. Then the complex Clifford algebra $\mathbb{C}_m$ is the set of elements $a = \sum_{A\subseteq M} a_A e_A$, $M = \{1,\ldots,m\}$, $a_A \in \mathbb{C}$, where $e_\emptyset = 1$ and $e_A = e_{\alpha_1} \ldots e_{\alpha_h}$ for $A = \{\alpha_1,\ldots,\alpha_h\}$ with $\alpha_1 < \ldots < \alpha_h$. The product in $\mathbb{C}_m$ is governed by the rules $e_i e_j + e_j e_i = -2\delta_{ij}$. By $\mathbb{C}_{m,k}$ we denote the subspace of k-vectors, i.e. Clifford numbers of the form $a = \sum_{|A|=k} a_A e_A$. Hence $\mathbb{C}^m$ is imbedded in $\mathbb{C}_m$ as the set of 1-vectors $\mathbb{C}_{m,1}$. By $[a]_k$ we denote the natural projection of $a \in \mathbb{C}_m$ onto $\mathbb{C}_{m,k}$.
An involution in $\mathbb{C}_m$ is given by

$$\bar{a} = \sum_A \bar{a}_A \bar{e}_A \ , \ \bar{e}_A = \bar{e}_{\alpha_h} \ldots \bar{e}_{\alpha_1} \ , \ \bar{e}_j = -e_j \ ; \ j = 1,\ldots,m$$

and a Hermitian inner product in $\mathbb{C}_m$ is given by $(a,b) = [\bar{a}\,b]_0 = [b\,\bar{a}]_0$. Notice that the product of two vectors is given by $\vec{v}\vec{w} = -\vec{v}\cdot\vec{w} + \vec{v}\wedge\vec{w}$, where $\vec{v}\cdot\vec{w} = -[\vec{v}\vec{w}]_0 = \sum_{j=1}^{m} v_j w_j$ is the standard bilinear form in $\mathbb{C}^m$.

By $\tilde{\mathbb{C}}_{m,k}$ we denote the set of pure k-vectors or k-blades $y = \vec{y}_1 \ldots \vec{y}_k$, $\vec{y}_j \in \mathbb{C}^m$ and $\vec{y}_j\vec{y}_k = -\vec{y}_k\vec{y}_j$.
The real Clifford algebra, space of real k-vectors and set of real k-blades are respectively denoted by $\mathbb{R}_m$, $\mathbb{R}_{m,k}$ and $\tilde{\mathbb{R}}_{m,k}$. Let $S_{m,k}$ be the unit sphere in $\mathbb{R}_{m,k}$. Then $\tilde{G}_{m,k}(\mathbb{R}) = S_{m,k} \cap \tilde{\mathbb{R}}_{m,k}$ is the set of unit k-blades, which we'll denote by $\nu = \vec{\nu}_1 \ldots \vec{\nu}_k$. These k-blades may be thought of as representatives for oriented k-subspaces of $\mathbb{R}^m$ and so, $\tilde{G}_{m,k}(\mathbb{R})$ forms a double covering of the Grassmann manifold $G_{m,k}(\mathbb{R})$.
The spingroup is the set of Clifford members

$$\mathrm{Spin}(m) = \{\mathfrak{s} = \vec{\omega}_1 \ldots \vec{\omega}_{2k} : \vec{\omega}_j \in S^{m-1}\} .$$

Its Lie algebra is the space $\mathbb{R}_{m,2}$ of bivectors, provided with the commutator product $[a,b] = ab - ba$.
Let $L_2(S^{m-1})$ be the space of $\mathbb{C}_m$-valued L_2-functions on the unit sphere S^{m-1}, provided with the inner product

$$(f,g) = \int_{S^{m-1}} \bar{f}(\vec{\omega})g(\vec{\omega})\,d\vec{\omega} .$$

Then we consider the following unitary representations of Spin(m) on $L_2(S^{m-1})$:

$$H(\mathfrak{s})f(\vec{\omega}) = f(\bar{\mathfrak{s}}\vec{\omega}\mathfrak{s}) \ , \ L(\mathfrak{s})f(\vec{\omega}) = \mathfrak{s}f(\bar{\mathfrak{s}}\vec{\omega}\mathfrak{s}) .$$

The representation H in fact corresponds to the standard representation of $SO(m)$ on $L_2(S^{m-1})$ while L corresponds to the spin 1/2 representation. Let R be any representation of $Spin(m)$. Then the infinitesimal representation dR of $\mathbb{R}_{m,2}$ is given by

$$dR(e_{ij}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(R(1+\varepsilon e_{ij}) - 1) .$$

The Casimir operator of R is given by (see also [8], [13])

$$C(R) = \frac{1}{4} \sum_{i<j} dR(e_{ij})^2$$

Notice that

$$dH(e_{ij}) = -2L_{ij} , \ L_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} , \ dL(e_{ij}) = -2L_{ij} + e_{ij} ,$$

$$C(H) = \Delta_S \quad \text{and} \quad C(L) = \Delta_S + \Gamma - \frac{1}{4}\begin{pmatrix} m \\ s \end{pmatrix} ,$$

where $\Gamma = -\sum_{i<j} e_{ij}L_{ij}$ is called "spherical Dirac operator" or "momentum operator" and $\Delta_S = \Gamma(m-2-\Gamma)$ is the stanard Laplace-Beltrami operator on the unit sphere.

Let $D = \sum_{j=1}^{m} e_j \frac{\partial}{\partial x_j}$ be the Dirac operator in $\mathbb{R}^m$ and let $f \in C_1(\Omega), \Omega \subseteq \mathbb{R}^m$ open. Then f is called left monogenic in Ω if $Df = 0$. Notice that, as $D^2 = -\Delta$, monogenic functions are also harmonic. Hence are may expect a refinement of the theory of spherical harmonics, namely the theory of spherical monogenics, which forms the main tool used in this paper.

Spherical monogenics can be introduces as the homogeneous monogenic functions in $\mathbb{R}^m \setminus \{0\}$.

Let $f(\vec{x}) = |\vec{x}|^\lambda f(\vec{\omega})$, $\vec{\omega} \in S^{m-1}$, $\vec{\omega} = \vec{x} / |\vec{x}|$ be monogenic in $\mathbb{R}^m \setminus \{0\}$; then λ is bound to have the values $\lambda = k$ and $\lambda = -(k+m-1)$, $k \in \mathbb{N}$, in which case f is called an inner (resp. outer) spherical monogenic of degree k . Spherical monogenics are considered as being defined on S^{m-1} or in $\mathbb{R}^m \setminus \{0\}$, depending on the context. By $\mathcal{M}_{+,k}$ (resp. $\mathcal{M}_{-,k}$) we denote the spaces of inner (resp. outer) spherical monogenics of degree k .

Both spaces $\mathcal{M}_{+,k}$ and $\mathcal{M}_{-,k}$ are linked as follows. The inversion $I f(\vec{x}) = \frac{\vec{x}}{|\vec{x}|^m} f\left(\frac{\vec{x}}{|\vec{x}|^2}\right)$ transforms monogenic functions into monogenic functions and intertwines the homogenous functions of degree k with those of degree

$-(k+m-1)$. Hence $f(\vec{\omega}) \in \mathcal{M}_{+,k}$ iff $\vec{\omega}f(\vec{\omega}) \in \mathcal{M}_{-,k}$ and so it suffices to consider homogeneous monogenic polynomials. Let $\mathcal{H}_k$ be the space of spherical harmonics of degree k , then clearly $\mathcal{M}_{+,k} \subseteq \mathcal{H}_k$. On the other hand, we have that $D = \vec{\omega}(\frac{\partial}{\partial r} + \frac{1}{r}\Gamma_\omega)$, $x = r\vec{\omega}$, $\vec{\omega} \in S^{m-1}$ so that

$$f(\vec{\omega}) \in \mathcal{M}_{+,k} \Leftrightarrow \Gamma_\omega f(\vec{\omega}) = -kf(\vec{\omega})$$

$$f(\vec{\omega}) \in \mathcal{M}_{-,k} \Leftrightarrow \Gamma_\omega f(\vec{\omega}) = (k+m-1)f(\vec{\omega}) .$$

As $\Delta_S = \Gamma_\omega(m-2-\Gamma_\omega)$ and $f(\vec{\omega}) \in \mathcal{H}_k$ iff $\Delta_S f(\vec{\omega}) = -k(k+m-2)f(\vec{\omega})$, it follows that

$$\mathcal{H}_k = \mathcal{M}_{+,k} \oplus \mathcal{M}_{-,k-1}$$

Similarly, as $C(L) = \Gamma(m-1-\Gamma) - \frac{1}{4}\binom{m}{2}$, the eigenspaces $\mathcal{M}_k$ of $C(L)$ admits decompositions

$$\mathcal{M}_k = \mathcal{M}_{+,k} + \mathcal{M}_{-,k} .$$

Moreover as H and L are unitary representations on $L_2(S^{m-1})$, we have that $\mathcal{H}_k \perp \mathcal{H}_\ell$ and $\mathcal{M}_k \perp \mathcal{M}_\ell$ for $k \neq \ell$ and hence $\mathcal{M}_{\pm,k} \perp \mathcal{M}_{\pm,\ell}$ for $k \neq \ell$ and $\mathcal{M}_{+,k} \perp \mathcal{M}_{-,1}$ for all k and ℓ .

Let P_k, Q_k, S_k, Π_k be the orthogonal projection operators onto respectively $\mathcal{M}_{+,k}$, $\mathcal{M}_{-,k}$, $\mathcal{H}_k$ and $\mathcal{M}_k$, then we have that

$$P_k = \frac{k+m-2-\Gamma}{2k+m-2} S_k \ , \ Q_k = \frac{k+1+\Gamma}{2k+m} S_{k+1} .$$

These formulae lead to explicit integral representations for P_k and Q_k . Let $\theta = \vec{v}\cdot\vec{\omega}$ and $C_k^\lambda(\theta)$ the Gegenbauer polynomials, then we have that (see [13], [17])

$$P_k f(\vec{v}) = \frac{1}{\omega_m}\int_{S^{m-1}} C^+_{m,k}(\vec{v},\vec{\omega})f(\vec{\omega})\,d\vec{\omega} \ , \ Q_k f(\vec{v}) = \frac{1}{\omega_m}\int_{S^{m-1}} C^-_{m,k}(\vec{v},\vec{\omega})f(\vec{\omega})\,d\vec{\omega} ,$$

where

$$C^+_{m,k}(\vec{v},\vec{\omega}) = \frac{1}{m-2}\left[(k+m-2)\,C_k^{\frac{m}{2}-1}(\theta) + (m-2)\,\vec{v}\wedge\vec{\omega}\,C_{k-1}^{\frac{m}{2}}(\theta)\right] ,$$

$$C^-_{m,k}(\vec{v},\vec{\omega}) = \frac{1}{m-2}\left[(k+1)\,C_{k+1}^{\frac{m}{2}-1}(\theta) - (m-2)\,v\wedge\vec{\omega}\,C_k^{\frac{m}{2}}(\theta)\right] .$$

Notice that these functions are of the form $f(\vec{v}\cdot\vec{\omega}, \vec{v}\wedge\vec{\omega})$ where $f(z_1, z_2)$

is a homogeneous polynomial of two variables.
Spherical monogenics lead to a powerful treatment of homogeneous polynomials (see also [15], [17]).
Let $\mathcal{P}_k$ be the space of homogeneous polynomials. Then every $R_k \in \mathcal{P}_k$ has the "monogenic decomposition"

$$R_k(\vec{x}) = \sum_{\ell=0}^{k} \vec{x}^{\ell} P_{k-\ell}(\vec{x}) \ , \ P_{k-\ell} \in \mathcal{M}_{+,k-\ell} \, .$$

This decomposition is quite useful in the construction of spherical solutions of

$$\sum \Lambda_k(\vec{x}) D_x^k f = g \ , \ \Lambda_k(z) \text{a } \mathbb{C}\text{-valued polynomial.}$$

A typical examples is the "Hermite equation" (see [16])

$$(D^2 - \vec{x}\, D - \lambda)\, f = 0 \, .$$

Functions of the form $\vec{x}^{\ell} P_k(\vec{x})$, $P_k \in \mathcal{M}_{+,k}$ are called Clifford monomials. They are not determined by their restrictions to the unit sphere since $\vec{\omega}^2 = -1$, $\vec{\omega} \in S^{m-1}$. Hence they have to be treated on a "slightly bigger" compact manifold, namely the Lie sphere. The Lie sphere LS^{m-1} is the set of points $e^{i\theta}\vec{\omega} \in \mathbb{C}^m$ where $\theta \in [0,\pi[$, $\vec{\omega} \in S^{m-1}$. Hence, functions on the Lie sphere can be considered as functions $f(e^{i\theta},\vec{\omega})$ on $S^1 \times S^{m-1}$, satisfying $f(-e^{i\theta},\vec{\omega}) = f(e^{i\theta},-\vec{\omega})$. Functions which satisfy $f(-e^{i\theta},\vec{\omega}) = -f(e^{i\theta},-\vec{\omega})$ are quite similar to functions on the Lie sphere and will be called functions on the "anti-Lie sphere", in spite of the fact that there is no manifold which could play the role of "anti Lie sphere".
The Lie sphere is the Shilov boundary of the Lie ball

$$LB(0,1) = \{\vec{x} + i\vec{y} : |\vec{x}|^2 + |\vec{y}|^2 + 2(|\vec{x}|^2|\vec{y}|^2 - (\vec{x}\cdot\vec{y})^2)^{1/2} < 1\} \ ,$$

which is the optimal domain in $\mathbb{C}^m$ to which a normally convergent series $\sum_{k=0}^{\infty} R_k(\vec{x})$ of homogeneous polynomials in $B(0,1)$ may be holomorphically extended (see alse [11]).
A more geometrical interpretation of Lie balls and spheres, useful for the theory of spherical means, is the following (see also [1], [7]). Let $\vec{z} = \vec{x} + i\vec{y}$ and put $S_{\vec{x}}(\vec{y}) = \{\vec{u} \in \mathbb{R}^m : |\vec{u} - \vec{x}| = |\vec{y}| \ \&(\vec{u} - \vec{x}) \perp \vec{y}\}$; then

$$LB(0,1) = \{\vec{z} : S_{\vec{x}}(\vec{y}) \subseteq B(0,1)\} \ , \ LS^{m-1} = \{\vec{z} : S_{\vec{x}}(\vec{y}) \subseteq S^{m-1}\} \, .$$

Spherical harmonics on the Lie sphere were considered by M. Morimoto in [9] to study hyperfunctions on LS^{m-1}. They are functions of the form

$$f(e^{i\theta},\vec{\omega}) = e^{i\ell\theta}S_k(\vec{\omega}) \ , \ S_k \in \mathcal{H}_k \ , \ \ell \in \mathbb{Z} \ ,$$

where $\ell = k+2s$, $s \in \mathbb{Z}$, in view of the topological constraint $f(e^{i\theta},-\vec{\omega}) = f(-e^{i\theta},\vec{\omega})$. Spherical monogenics on the Lie sphere were considered in our paper [17]. The are of the form

$$f(e^{i\theta}\vec{\omega}) = (e^{i\theta}\vec{\omega})^{\ell} e^{ik} P_k(\vec{\omega}) \ , \ P_k \in \mathcal{M}_{+,k} \ , \ \ell \in \mathbb{Z}$$

and correspond to the restrictions to LS^{m-1} of complexified Clifford monomials $\vec{z}^{\ell}P_k(\vec{z})$. For the orthogonal decomposition of $L_2(LS^{m-1})$ in spherical monogenics and the corresponding decomposition of the Cauchy-Hua kernel, see our paper [17]. On the Lie sphere we also consider the operators

$$\mathbb{D} = \Gamma_\omega - i\frac{\partial}{\partial\theta} \ , \ \tilde{\mathbb{D}} = \Gamma_\omega + i\frac{\partial}{\partial\theta} -(m-2)$$

$$\mathcal{D} = \tilde{\mathbb{D}}\,\mathbb{D} = \frac{\partial^2}{\partial\theta^2} + i(m-2)\frac{\partial}{\partial\theta} - \Delta_S \ .$$

Notice that $\mathcal{D}$ is the restriction to LS^{m-1} of the complexified Laplacian, while $\mathbb{D}$ and $\tilde{\mathbb{D}}$ are Dirac type operators on LS^{m-1}. In order to characterize all hyperfunction solutions of $\mathbb{D}$ and $\tilde{\mathbb{D}}$ we first recall some results on boundary value theory (see also [12], [17]).

Let F be a hyperfunction on S^{m-1}, i. e. a functional on the space $\mathcal{A}(S^{m-1})$ of analytic functions on S^{m-1}. Then the Cauchy transform of F is given by

$$\hat{F}(\vec{x}) = -\frac{1}{\omega_m}\int_{S^{m-1}} \frac{1+\vec{x}\,\vec{\omega}}{|1+\vec{x}\vec{\omega}|^m} F(\vec{\omega})\, d\vec{\omega} ,$$

which is monogenic in $\mathbb{R}^m\setminus\{0\}$ and vanishes at infinity. F may then be represented as the boundary value

$$F(\vec{\omega}) = H_+F(\vec{\omega}) + H_-F(\vec{\omega}) \ , \ H_\pm F(\vec{\omega}) = \pm\lim_{\varepsilon\to 0}\hat{F}((1\pm\varepsilon)\vec{\omega}) \ .$$

The Hilbert-Riesz transform of F is given by

$$H_SF(\vec{\omega}) = H_+F(\vec{\omega}) - H_-F(\vec{\omega})$$

and satisfies $H_S^2 = 1$.

Global nullsolutions of $\mathbb{D}$ are constructed as follows. First notice that $\hat{F}(\vec{x})$, $\vec{x} \in B(0,1)$ extends to a holomorphic function $F_{-}(\vec{z})$ in the Lie ball, and so determines a hyperfunction $F_{-}(e^{i\theta}\vec{\omega})$ on LS^{m-1} , which is nullsolution of $\mathbb{D}$. For m odd all global nullsolutions of $\mathbb{D}$ are of this form. For m even, $\hat{F}(\vec{x})$, $\vec{x} \in \mathbb{R}^m \setminus B(0,1)$ admits the complex extension $F_{+}(\vec{z}) = \vec{z}/(\Sigma z_j^2)^{m/2} G(\vec{z}/\Sigma z_j^2)$, where G is complex monogenic in $LB(0,1)$ and so, $F_{+}(e^{i\theta}\vec{\omega})$ is a globally defined hyperfunction on LS^{m-1} which satisfies $\mathbb{D}f = 0$. It can easily be seen that there are no other global nullsolutions and that the nullsolutions of $\tilde{\mathbb{D}}$ are of the form $\vec{\omega} e^{i\theta} f(e^{i\theta}\vec{\omega})$, where $\mathbb{D}f = 0$. In the even dimensional case we use the notation

$$F(e^{i\theta}\vec{\omega}) = F_{+}(e^{i\theta}\vec{\omega}) - F_{-}(e^{i\theta}\vec{\omega})$$

and we call it the canonical complexification of the hyperfunction $F(\vec{\omega})$.

2. Spherical means of codimension one

In our papers [13], [14], we introduced a theory of spherical means of functions defined in Euclidean space, leading to a refinement of the classical Darboux equation. Indeed, let f be defined in some subset of $\mathbb{R}^{m-1}$; then the spherical mean

$$Pf(\vec{x},r) = \frac{1}{\omega_{m-1}} \int_{S^{m-2}} f(\vec{x}+r\vec{\omega})\, d\vec{\omega}$$

satisfies the classical Darboux equation (see [6])

$$\left(\Delta_x - \left(\frac{\partial^2}{\partial r^2} + \frac{m-1}{r}\frac{\partial}{\partial r}\right)\right) Pf(\vec{x}, r) = 0 .$$

By introducing the oriented spherical mean

$$Qf(\vec{x},r) = \frac{1}{\omega_{m-1}} \int_{S^{m-2}} \vec{\omega} f(\vec{x}+r\vec{\omega})\, d\vec{\omega} ,$$

the Darboux equation may be refined to the following system

$$D_x Pf(\vec{x},r) = \left(\frac{\partial}{\partial r} + \frac{m-2}{r}\right) Qf(\vec{x},r) , \quad D_x Qf(\vec{x},r) = -\frac{\partial}{\partial r} Pf(\vec{x},r) ,$$

called **Darboux system**.

To incorporate the Radon transform on projective space into the theory, it is better to study spherical means of functions on the unit sphere S^{m-1} . The Radon transform on real projective space may indeed be represented by taking integrals of a function on S^{m-1} over geodesic spheres. Part of this work was already done in our previous paper [17] of which we recall the main definitions and results.

Definition 1. Let $f(\vec{\omega})$ be a function on S^{m-1}. Then for $0 < \theta < \pi$ the inner and outer spherical means of f are respectively given by

$$Pf(e^{i\theta},\vec{v}) = \frac{1}{\omega_{m-1}\sin^{m-3}\theta}\int_{S^{m-1}} \delta(\vec{\omega}\cdot\vec{v} - \cos\theta)\, f(\vec{\omega})\, d\vec{\omega}\ ,$$

$$Qf(e^{i\theta},\vec{v}) = \frac{1}{\omega_{m-1}\sin^{m-2}\theta}\int_{S^{m-1}} \vec{v}\wedge\vec{\omega}\ \delta(\vec{\omega}\cdot\vec{v} - \cos\theta)\, f(\vec{\omega})\, d\vec{\omega}\ .$$

Notice that $Pf(e^{i(\pi-\theta)},\vec{v}) = Pf(e^{i\theta},-\vec{v})$ and that $Qf(e^{i(\pi-\theta)},\vec{v}) = -Qf(e^{i\theta},-\vec{v})$. Hence it is natural to extend Pf and Qf for all values of θ by putting

$$Pf(e^{-i\theta},\vec{v}) = Pf(e^{i\theta},-\vec{v}),\ Qf(e^{-i\theta},\vec{v}) = -Qf(e^{i\theta},-\vec{v})\ .$$

In this way, Pf and Qf are function on $S^1\times S^{m-1}$ satisfying the relation $g(-e^{i\theta},-\vec{v}) = g(e^{i\theta},\vec{v})$ and so they may be considered as functions over the Lie sphere LS^{m-1}, i. e. functions of the variable $e^{i\theta}\vec{\omega}$. It is clear that on the one hand

$$\lim_{\theta\to 0} Pf(e^{i\theta}\vec{\omega}) = f(\vec{\omega})\ ,$$

while in [17] we proved that

$$\lim_{\theta\to 0}\frac{1}{\sin\theta}\, Qf(e^{i\theta}\vec{\omega}) = -\frac{1}{m-1}\,\Gamma_\omega\, f(\vec{\omega})\ .$$

On the other hand, for $\theta = \frac{\pi}{2}$ we have that

$$B_+f(\vec{v}) = Pf(i\vec{v}) = \frac{1}{\omega_{m-1}}\int_{S^{m-1}} \delta(\vec{v}\cdot\vec{\omega})\, f(\vec{\omega})\, d\vec{\omega}\ ,$$

$$B_-f(\vec{v}) = Qf(i\vec{v}) = \frac{1}{\omega_{m-1}}\int_{S^{m-1}} \vec{v}\wedge\vec{\omega}\ \delta(\vec{v}\cdot\vec{\omega})\, f(\vec{\omega})\, d\vec{\omega}\ .$$

B_+ vanishes on odd functions and hence may be called Radon transform on real projective space $\mathbb{R}P_{m-1}$. B_- vanishes on even functions and is injective on the set of odd functions. Hence we call it the Radon transform on "anti-projective space".

This means that the identity operator, the operator Γ_ω and the Radon transforms B_+ and B_- are part of the spherical means Pf and Qf. To unify these integral transforms, we also introduce the total spherical mean of f

$$Mf(e^{i\theta}\vec{v}) = Pf(e^{i\theta}\vec{v}) + i\, Qf(e^{i\theta}\vec{v})$$

and the spherical Radon transform of f

$$Bf(\vec{v}) = (B_+ + iB_-)\, f(\vec{v}) = \frac{1}{\omega_{m-1}} \int_{S^{m-1}} (1+i\vec{v}\wedge\vec{\omega})\delta(\vec{v}\cdot\vec{\omega})\, f(\vec{\omega})\, d\vec{\omega}\,.$$

In [17] we proved that Pf and Qf satisfy the spherical Darboux system

$$\Gamma_v Pf(e^{i\theta}\vec{v}) = -\left(\frac{\partial}{\partial\theta} + (m-2)\cot\theta\right) Qf(e^{i\theta}\vec{v})\,,$$

$$(\Gamma_v - (m-2))\, Qf(e^{i\theta}\vec{v}) = \frac{\partial}{\partial\theta} Pf(e^{i\theta}\vec{v})\,,$$

which is a system of equations on the Lie sphere. Using the differential operator $\mathbb{D} = \Gamma_\omega - i\frac{\partial}{\partial\theta}$, this system may be rewritten as

$$\mathbb{D} Mf(e^{i\theta}\vec{v}) = (m-2)(i - \cot\theta)\, Qf(e^{i\theta}\vec{v})\,.$$

Conversely, when P and Q are solutions on LS^{m-1} of the above system, with the property $P(e^{i\theta}\vec{v}) = P(e^{-i\theta}\vec{v})$, then P and Q are the inner and outer spherical means of the function $f(\vec{\omega}) = P(\vec{\omega})$, $\vec{\omega} \in S^{m-1}$.
Next, consider the m-dimensional Legendre polynomials

$$P_{m,k}(t) = \frac{k!\,(m-3)!}{(k+m-3)!}\, C_k^{\frac{m}{2}-1}(t)\,,$$

then in [17] we proved the following

Theorem 1. Let $f = \sum_{k=0}^{\infty} (P_k f + Q_k f)$ be the expansion of f in spherical monogenics on S^{m-1}. Then Mf admits the expansion in spherical monogenics on LS^{m-1}:

$$Mf(e^{i\theta}\vec{v}) = \sum_{k=0}^{\infty} \Big[\Big(P_{m,k}(\cos\theta) + \frac{ik}{m-1}\sin\theta\, P_{m+2,k-1}(\cos\theta)\Big)\, P_k f(\vec{v})$$
$$+ \Big(P_{m,k+1}(\cos\theta) - \frac{i(k+m-1)}{m-1}\sin\theta\, P_{m+2,k}(\cos\theta)\Big)\, Q_k f(\vec{v})\Big]\,.$$

From this, it follows that for all values of θ, the total spherical mean $Mf(e^{i\theta}\vec{v})$ is injective on all function spaces on S^{m-1}. The classical spherical mean $Pf(e^{i\theta}\vec{v})$ is non-injective as soon as $\cos\theta$ is zero of a Gegenbauer polynomial $C_k^{\frac{m}{2}-1}(t)$. Hence, the introduction of Clifford numbers is really more than just a formality; it makes the spherical means so to speak "Ghost free" on the whole unit sphere.

The spherical means of codimension one may now be generalized as follows. Let $\vec{v}\in S^{m-1}$. Then we put $\bar{S}(\vec{v}) = \{\vec{\omega}\in S^{m-1} : \vec{v}\cdot\vec{\omega} = 0\}$.

By $\mathcal{M}_{+,k}(\vec{v})$, $\mathcal{M}_{-,k}(\vec{v})$, $\mathcal{M}_k(\vec{v})$, $\mathcal{H}_k(\vec{v})$ we denote respectively the spaces of inner and outer spherical monogenics, the total space of spherical monogenics and the space of spherical harmonics of degree k on the sphere $\bar{S}(\vec{v})$. As every function may uniquely be written into the form $f(\cos\theta\vec{v} + \sin\theta\vec{\mu})$, $\theta\in[0,\pi]$, where $\vec{\mu}$ varies inside $\bar{S}(\vec{v})$, we may introduce

Definition 2. The inner and outer spherical means of degree k of f are given by

$$P_{+,k}f(e^{i\theta},\vec{v})(\vec{\eta}) = P_k(f(\cos\theta\vec{v} + \sin\theta\vec{\mu}))(\vec{\eta})$$

$$= \frac{1}{\omega_{m-1}}\int_{\bar{S}(\vec{v})} C^{+}_{m-1,k}(\vec{\eta},\vec{\mu})f(\cos\theta\vec{v} + \sin\theta\vec{\mu})\,d\vec{\mu} ,$$

$$P_{-,k}f(e^{i\theta},\vec{v})(\vec{\eta}) = \vec{v}P_{+,k}(\vec{\mu}f(\cos\theta\vec{v} + \sin\theta\vec{\mu}))(\vec{\eta}) .$$

Notice that $Pf = P_{+,0}f$ and $Qf = P_{-,0}f$. Furthermore, for every value of θ, $P_{+,k}f(e^{i\theta},\vec{v})$ and $P_{-,k}f(e^{i\theta},\vec{v})$ are to be interpreted as sections of the vector bundle $\mathcal{M}_{+,k}(\vec{v})$ over S^{m-1}.

The projections of $f(\cos\theta\vec{v} + \sin\theta\vec{\mu})$, θ and $\vec{v}$ fixed, on the spaces $\mathcal{M}_k(\vec{v})$ and $\mathcal{H}_k(\vec{v})$ are respectively given by

$$\Pi_k f(e^{i\theta},\vec{v}) = P_{+,k}f(e^{i\theta},\vec{v}) + \vec{\eta}\vec{v}P_{-,k}f(e^{i\theta},\vec{v}) ,$$

$$S_k f(e^{i\theta},\vec{v}) = P_{+,k}f(e^{i\theta},\vec{v}) + \vec{\eta}\vec{v}P_{-,k-1}f(e^{i\theta},\vec{v}) .$$

We now prove a generalized version of the Darboux systems.

Theorem 2. The spherical means $P_{+,k}f$ and $P_{-,k}f$ satisfy the Darboux system

$$P_{+,k}(\Gamma_\omega f) = -kP_{+,k}f - \left(\frac{\partial}{\partial\theta} + (k+m-2)\cot\theta\right)P_{-,k}f ,$$

$$P_{-,k}(\Gamma_\omega f) = (k+m-2)P_{-,k}f + \left(\frac{\partial}{\partial\theta} - k\cot\theta\right)P_{+,k}f .$$

Proof. From the definition of Γ_ω , it easily follows that for $\vec{\omega} = \cos\theta\vec{\nu} + \sin\theta\vec{\mu}$, $\vec{\nu}$ fixed,

$$\Gamma_\omega = -\vec{\nu}\vec{\mu}\frac{\partial}{\partial\theta} + (1-\cot\theta\,\vec{\nu}\,\vec{\mu})\,\Gamma_\mu \ .$$

Furthermore one always has that

$$f(\cos\theta\vec{\nu} + \sin\theta\vec{\mu}) = \sum_{k=0}^{\infty} \Pi_k f(e^{i\theta},\vec{\nu})(\vec{\eta}) \ ,$$

from which it follows that

$$P_{+,k}(\Gamma_\mu f) = \Gamma_\eta P_{+,k}f = -\,k\,P_{+,k}f$$

$$P_{-,k}(\Gamma_\mu f) = \vec{\nu}\vec{\eta}\Gamma_\eta\vec{\eta}\vec{\nu}\ P_{-,k}f = (k+m-2)\ P_{-,k}f \ .$$

It is now sufficient to insert the expression for Γ_ω in the left hand side of the Darboux equations and see what happens. □

Notice that for $k = 0$ we obtain the supplementary relations $\Gamma_\nu Pf = P(\Gamma_\omega f)$ and $\Gamma_\nu Qf = Q(\Gamma_\omega f)$. This also follows directly from the fact that the operators P and Q transforms spherical monogenics into spherical monogenics of the same type and degree.
Similar to the case of spherical means in $\mathbb{R}^m$ (see [13]) one can show that for θ tending to zero,

$$P_{+,k}f(e^{i\theta},\vec{\nu}) = O(\theta^k) \ , \ P_{-,k}f(e^{i\theta},\vec{\nu}) = O(\theta^{k+1}) \ .$$

Hence it is natural to extend $P_{+,k}f$ and $P_{-,k}f$ for all values of θ by means of the relations

$$P_{+,k}f(e^{-i\theta},\vec{\nu}) = (-1)^k\ P_{+,k}f(e^{i\theta},\vec{\nu}) \ ,$$

$$P_{-,k}f(e^{-i\theta},\vec{\nu}) = (-1)^{k+1}\ P_{-,k}f(e^{i\theta},\vec{\nu}) \ .$$

On the other hand, one still has the relations

$$P_{+,k}f(e^{i(\pi-\theta)},\vec{\nu}) = P_{+,k}f(e^{i\theta},\vec{\nu}) \ , \ P_{-,k}f(e^{i(\pi-\theta)},\vec{\nu}) = -P_{-,k}f(e^{i\theta},\vec{\nu}) \ .$$

Hence $P_{\pm,k}f$ satisfy the relation

$$f(-e^{i\theta},\vec{v}) = (-1)^k f(e^{i\theta},-\vec{v}) ,$$

which means that for k even $P_{\pm,k}f$ are defined on the Lie sphere while for k odd, $P_{\pm,k}f$ are defined on the "anti-Lie sphere". To obtain quantities which are always defined over the Lie sphere, it is natural to introduce the modified spherical means

$$\tilde{P}_{+,k}f(e^{i\theta}\vec{v}) = \frac{1}{\sin^k\theta} P_k(f(\cos\theta\vec{v} + \sin\theta\vec{\mu})) ,$$

$$\tilde{P}_{-,k}f(e^{i\theta}\vec{v}) = \frac{1}{\sin^k\theta} \vec{v}P_k(\vec{\mu}f(\cos\theta\vec{v}+\sin\theta\vec{\mu})) ,$$

which are global sections of the bundle $\mathcal{M}_{+,k}(\vec{v})$ over the Lie sphere. There are no problems about the values for $\theta = 0$ in view of the homogenity of $P_{+,k}f$ and $P_{-,k}f$ for θ tending to zero.
As to the explicit solution of the Darboux equations, we have the following

<u>Theorem 3.</u> Let $\Gamma_\omega f = \lambda f$, $\lambda = -\ell$ or $\lambda = \ell+m-2$, $\ell \in \mathbb{N}$. Then the modified spherical means of f are of the form

$$\tilde{P}_{+,k}f(e^{i\theta}\vec{v}) = C_{\ell-k}^{k+\frac{m}{2}-1}(\cos\theta)\alpha(\vec{v}) , \quad k \leq \ell,$$

$$\tilde{P}_{-,k}f(e^{i\theta}\vec{v}) = \frac{2k+m-2}{k+m-2-\lambda} \sin\theta\, C_{\ell-k-1}^{k+\frac{m}{2}}(\cos\theta)\alpha(\vec{v}) , \quad k < \ell ,$$

where $\alpha(\vec{v})$ is some section of the bundle $\mathcal{M}_{+,k}(\vec{v})$ ouer S^{m-1} .

<u>Proof.</u> When $\Gamma_\omega f = \lambda f$, $\lambda = -\ell$ or $\lambda = \ell+m-2$, then certainly f is spherical harmonic of degree ℓ on S^{m-1} . So, f is polynomial of degree ℓ with respect to $\vec{\mu} \in \bar{S}(\vec{v})$ and therefore $\tilde{P}_{+,k}f = 0$ for $k > \ell$ and $\tilde{P}_{-,k}f = 0$ for $k \geq \ell$.
To find the solution of the Darboux system, we put $\tilde{P}_{+,k}f = A(\cos\theta)$, $\tilde{P}_{-,k}f = \sin\theta\, B(\cos\theta)$, $t = \cos\theta$. Then A and B satisfy the system

$$(1-t^2)B' - (2k+m-1)tB - (k+\lambda)A = 0 , \quad A' = (k+m-2-\lambda)B ,$$

so that

$$(1-t^2)A'' - (2k+m-1)tA' + (\lambda+k)(\lambda-(k+m-2))A = 0 ,$$

which is a Gegenbauer equation (see e. g. [5]).
As for both values of λ .

$$(\lambda+k)(\lambda-(k+m-2)) = (\ell-k)(\ell+k+m-2) ,$$

the solution follows from the theory of Gegenbauer polynomials. □

We still have to give an interpretation for the section $\alpha(\vec{v})$. To that end we introduce for $(\vec{v},\vec{\eta}) \in S^{m-1} \times \bar{S}(\vec{v})$, the following operators

$$D_{+,k}(\vec{v},\vec{\eta})f(\vec{v}) = \lim_{\theta\to 0} \tilde{P}_{+,k}f(e^{i\theta}\vec{v})(\vec{\eta}) ,$$

$$D_{-,k}(\vec{v},\vec{\eta})f(\vec{v}) = \lim_{\theta\to 0} \frac{1}{\sin\theta} \tilde{P}_{-,k}f(e^{i\theta}\vec{v})(\vec{\eta}) .$$

Then these operators are homogeneous differential operators of degrees k and k+1 (see also [13]) for each local section $\vec{\eta}(\vec{v})$ of $\bar{S}(\vec{v})$. Furthermore we have that

$$D_{+,k}(\vec{v},\vec{\eta})f(\vec{v}) = C_{\ell - k}^{k + \frac{m}{2} - 1}(1)\alpha(\vec{v}) ,$$

$$D_{-,k}(\vec{v},\vec{\eta})f(\vec{v}) = \frac{2k+m-2}{k+m-2-\lambda} C_{\ell - k - 1}^{k + \frac{m}{2}}(1)\alpha(\vec{v}) .$$

3. Spherical means of higher codimension

Let $\{\vec{v}_1,\ldots,\vec{v}_p\}$ be an orthonormal p-frame in $\mathbb{R}^m$. Then the p-vector $\nu = \vec{v}_1\ldots\vec{v}_p$ represents an oriented p-dimensional subspace $V(\nu)$ of $\mathbb{R}^m$. Let $\bar{V}(\nu)$ be the orthogonal complement of $V(\nu)$, then by $S(\nu)$ and $\bar{S}(\nu)$ we denote the unit spheres in $V(\nu)$ and $\bar{V}(\nu)$ respectively. A point $\vec{\omega} \in S^{m-1}$ may in a unique way be written as

$$\vec{\omega} = \cos\theta\vec{v} + \sin\theta\vec{\mu} , \ \vec{v} \in S(\nu) , \ \vec{\mu} \in S(\nu) , \ \theta \in [0, \tfrac{\pi}{2}] .$$

Of course one must be careful with the values $\theta = 0$ and $\theta = \frac{\pi}{2}$, which correspond to the spheres $S(\nu)$ and $\bar{S}(\nu)$. A general sphere of codimension p inside S^{m-1} may be given by

$$S(\nu,\vec{v},e^{i\theta}) = \{\vec{\omega} = \cos\theta\vec{v} + \sin\theta\vec{\mu} : \vec{\mu} \in \bar{S}(\nu)\} ,$$

and carries the orientation induced by $\bar{V}(\nu)$. So the totality of oriented spheres of codimension p is parametrized by $(\nu,\vec{v},e^{i\theta}) \in \tilde{G}_{m,p}(\mathbb{R})\times S(\nu)\times S^1$.

By $\mathcal{M}_{+,k}(\nu)$, $\mathcal{M}_{-,k}(\nu)$, $\mathcal{M}_k(\nu)$ and $\mathcal{H}_k(\nu)$ we denote the vector bundles over $\tilde{G}_{m,p}(\mathbb{R})$ of spaces of inner and outer spherical monogenics, spherical monogenics and spherical harmonics of degree k on the sphere $\bar{S}(\nu)$. This leads to the following generalization of the spherical mean transform.

Definition 3. Let $f(\vec{\omega})$ be a function on S^{m-1} . Then the inner and outer spherical means of codimension p are given by

$$P_{+,k}f(e^{i\theta},\nu,\vec{\nu}) = \frac{1}{\omega_{m-p}}\int_{\bar{S}(\nu)} C^{+}_{m-p,k}(\vec{\eta},\vec{\mu})f(\cos\theta\vec{\nu} + \sin\theta\vec{\mu})\,d\vec{\mu} ,$$

$$P_{-,k}f(e^{i\theta},\nu,\vec{\nu}) = \vec{\nu}P_{+,k}(\vec{\mu}f(\cos\theta\vec{\nu} + \sin\theta\vec{\mu})) .$$

Notice that $P_{\pm,k}f(e^{i\theta},\nu,\vec{\nu})(\vec{\eta})$ are inner spherical monogenic of degree k with respect to $\vec{\eta}\in\bar{S}(\nu)$. So $P_{\pm,k}f(e^{i\theta},\nu,\vec{\nu})$ are to be interpreted as sections of the bundle $\mathcal{M}_{+,k}(\nu)$.

Furthermore the projections $\Pi_k f$ of $f(\cos\theta\vec{\nu} + \sin\theta\vec{\mu})$ on the spaces $\mathcal{M}_k(\nu)$ are given by

$$\Pi_k f(e^{i\theta},\nu,\vec{\nu})(\vec{\eta}) = (P_{+,k} + \vec{\eta}\vec{\nu}\,P_{-,k})\; f(e^{i\theta},\nu,\vec{\nu})(\vec{\eta}) .$$

For $\theta = \frac{\pi}{2}$, $P_{+,k}f$ only depends on ν while depends still trivially on $\vec{\nu}$.
For $k = 0$ we use the notations Pf and Qf as before, i. e.

$$Pf(e^{i\theta},\nu,\vec{\nu}) = \frac{1}{\omega_{m-p}}\int_{\bar{S}(\nu)} f(\cos\theta\vec{\nu} + \sin\theta\vec{\mu})\,d\vec{\mu} ,$$

$$Qf(e^{i\theta},\nu,\vec{\nu}) = \frac{\vec{\nu}}{\omega_{m-p}}\int_{\bar{S}(\nu)} \vec{\mu}f(\cos\theta\vec{\nu} + \sin\theta\vec{\mu})\,d\vec{\mu} .$$

As to the Darboux equations we now have

Theorem 4. The spherical means $P_{+,k}f$ and $P_{-,k}f$ satisfy the Darboux system

$$P_{+,k}(\Gamma_\omega f) = (\Gamma_\nu - k)P_{+,k}f - \left(\frac{\partial}{\partial\theta} + (k+m-p-1)\cot\theta + \tan\theta(\Gamma_\nu - p-1)\right)P_{-,k}f ,$$

$$P_{-,k}(\Gamma_\omega f) = (k+m-2-\Gamma_\nu)P_{-,k}f + \left(\frac{\partial}{\partial\theta} - k\cot\theta - \tan\theta\Gamma_\nu\right)P_{+,k}f .$$

Proof. We start from the identity

$$\Gamma_\omega = -\,\vec{\nu}\vec{\mu}\,\frac{\partial}{\partial\theta} + (1+\tan\theta\vec{\nu}\vec{\mu})\Gamma_\nu + (1-\cot\theta\vec{\nu}\vec{\mu})\Gamma_\mu ,$$

where $\vec{\omega} = \cos\theta\vec{\nu} + \sin\theta\vec{\mu}$, $\vec{\nu} \in S(\nu)$, $\vec{\mu} \in \bar{S}(\nu)$ and Γ_ν and Γ_μ are the spherical Dirac operators on $S(\nu)$ and $\bar{S}(\nu)$ respectively.
Similar to the proof of Theorem 2 we again have that

$$P_{+,k}(\Gamma_\mu f) = -kP_{+,k}f \ , \ P_{-,k}(\Gamma_\mu f) = (k+m-p-1)\, P_{-,k}f \ ,$$

while is easy to see that

$$P_{+,k}(\Gamma_\nu f) = \Gamma_\nu\, P_{+,k}f \ , \ P_{-,k}(\Gamma_\nu f) = (p-1-\Gamma_\nu)\, P_{-,k}f \ .$$

Hence it is again sufficient to insert the expression for Γ_ω in the left hand side of the Darboux equations. □

It is rather hard to solve these equations directly, even when f is eigenfunction of Γ_ω with eigenvalue λ . In that case we obtain the system of equations

$$(\Gamma_\nu - k - \lambda)A = \left(\frac{\partial}{\partial\theta} + (k+m-p-1)\cot\theta + \tan\theta\,(\Gamma_\nu - p - 1)\right) B$$

$$(\Gamma_\nu - (k+m-2) + \lambda)B = \left(\frac{\partial}{\partial\theta} - k\cot\theta - \tan\theta\,\Gamma_\nu\right) A \ ,$$

$A = P_{+,k}f$, $B = P_{-,k}f$, in which the operator Γ_ν still occurs. To solve these equations, we need a refinement of the concept of spherical means, which will be the concept of bispherical means.
To that end, we'll introduce some new bundles over the Grassmannian $\tilde{G}_{m,p}(\mathbb{R})$.
Let $k, \ell \in \mathbb{N}$ and let $(\vec{\varkappa},\vec{\eta}) \in S(\nu)\times\bar{S}(\nu)$. Then the spherical Dirac operators $\Gamma_\varkappa$ and Γ_η on $S(\nu)$ and $\bar{S}(\nu)$ may be naturally extended to $\mathbb{R}^m$ by the formulae

$$\Gamma_\varkappa = -\vec{x}\wedge D_x \ , \quad \Gamma_\eta = -\vec{y}\wedge D_y \ , \text{ at } \vec{u} \in \vec{x} + \vec{y} \in \mathbb{R}^m \ ,$$

where $\vec{x} \in V(\nu)$, $\vec{y} \in \bar{V}(\nu)$ and where D_x and D_y are the Dirac operator paralled to $V(\nu)$ and $\bar{V}(\nu)$. Hence it is clear that $\Gamma_\varkappa$ and Γ_η commute, so that we can look for simultaneous eigenfunctions, which are called bispherical monogenics. The eigenvalues can be pairs of the form $(-k,-\ell)$, $(k+p-1,-\ell)$, $(-k,\ell+m-p-1)$ and $(k+p-1,\ell+m-p-1)$ and the corresponding eigenspaces are denoted by $\mathcal{M}^{0,0}_{k,\ell}(\nu)$, $\mathcal{M}^{1,0}_{k,\ell}(\nu)$, $\mathcal{M}^{0,1}_{k,\ell}(\nu)$ and $\mathcal{M}^{1,1}_{k,\ell}(\nu)$ respectively. The total space of bispherical monogenics of "degree" (k,ℓ) is given by

$$\mathcal{M}_{k,\ell}(\nu) = \mathcal{M}^{0,0}_{k,\ell}(\nu) + \mathcal{M}^{1,0}_{k,\ell}(\nu) + \mathcal{M}^{0,1}_{k,\ell}(\nu) + \mathcal{M}^{1,1}_{k,\ell}(\nu) \ ,$$

and every element of this space may be written as

$$A(\vec{x},\vec{\eta}) + \vec{x}\, B(\vec{x},\vec{\eta}) + \vec{\eta}\, C(\vec{x},\vec{\eta}) + \vec{\eta}\,\vec{x}\, D(\vec{x},\vec{\eta}) ,$$

where A, B, C and D belong to $\mathcal{M}^{0,0}_{k,\ell}(\nu)$.
This leads to the following

Definition 4. The bispherical mean of degree (k,ℓ) of a function $f(\vec{\omega})$, $\vec{\omega} = \cos\theta\vec{v} + \sin\theta\vec{\mu} \in S^{m-1}$ is given by

$$\Pi_{k,\ell} f(e^{i\theta},\nu)(\vec{x},\vec{\eta}) = (\Pi^{0,0}_{k,\ell} - \vec{x}\,\Pi^{1,0}_{k,\ell} - \vec{\eta}\,\Pi^{0,1}_{k,\ell} + \vec{\eta}\vec{x}\,\Pi^{1,1}_{k,\ell}) f(e^{i\theta},\nu)(\vec{x},\vec{\eta}) ,$$

where $\theta \in [0, \frac{\pi}{2}]$, $(\vec{x},\vec{\eta}) \in S(\nu)\times \bar{S}(\nu)$ and where

$$\Pi^{0,0}_{k,\ell} f(e^{i\theta},\nu)(\vec{x},\vec{\eta}) = \frac{1}{\omega_p\omega_{m-p}} \int_{S(\nu)\times\bar{S}(\nu)} C^+_{p,k}(\vec{x},\vec{v})\, C^+_{m-p,\ell}(\vec{\eta},\vec{\mu}) f(\vec{\omega})\, d\vec{\mu}\, d\vec{v} ,$$

$$\Pi^{1,0}_{k,\ell} f = \Pi^{0,0}_{k,\ell}(\vec{v}f) , \quad \Pi^{0,1}_{k,\ell} f = \Pi^{0,0}_{k,\ell}(\vec{\mu}f) \quad \text{and} \quad \Pi^{1,1}_{k,\ell} f = \Pi^{0,0}_{k,\ell}(\vec{v}\vec{\mu} f) .$$

Notice that the bispherical means of f are again interpreted as sections of the bundle $\mathcal{M}_{k,\ell}(\nu)$ over $\tilde{G}_{m,p}(\mathbb{R})$.
For $k = \ell = 0$ and $\theta = \frac{\pi}{2}$, we have that $\Pi^{1,0}_{0,0} f$ and $\Pi^{1,1}_{0,0} f$ vanish while

$$\Pi^{0,0}_{0,0} f(i\nu) = \bar{B}_+ f(\nu) = \frac{1}{\omega_{m-p}} \int_{\bar{S}(\nu)} f(\vec{\mu})\, d\vec{\mu} ,$$

$$\Pi^{0,1}_{0,0} f(i\nu) = \bar{B}_- f(\nu) = \frac{1}{\omega_{m-p}} \int_{\bar{S}(\nu)} \vec{\mu} f(\vec{\mu})\, d\vec{\mu}$$

are Radon type transforms of f over all geodesic of codimension p inside S^{m-1}.
Similarly, for $\theta = 0$ we have that $\Pi^{0,1}_{0,0} f$ and $\Pi^{1,1}_{0,0} f$ vanish, while

$$\Pi^{0,0}_{0,0} f(\nu) = B_+ f(\nu) = \frac{1}{\omega_p} \int_{S(\nu)} f(\vec{v})d\vec{v} , \quad \Pi^{1,0}_{0,0} f(\nu) = B_- f(\nu) = \frac{1}{\omega_p} \int_{S(\nu)} \vec{v} f(\vec{v})d\vec{v}$$

are Radon type transforms of f over geodesic spheres of dimension $p-1$ inside S^{m-1}. Hence the bispherical means link together the Radon transforms of dimension $p-1$ with those of codimension p, which we call "dual Radon transform".

As to the Darboux equations we now have

Theorem 5. The bispherical means satisfy the Darboux equations

$$\Pi^{0,0}_{k,\ell}(\Gamma_\omega f) = -(k+\ell)\,\Pi^{0,0}_{k,\ell}f - \left(\frac{\partial}{\partial\theta} + \cot\theta(\ell+m-p-1) - \tan\theta(k+p-1)\right)\Pi^{1,1}_{k,\ell}f \ ,$$

$$\Pi^{1,1}_{k,\ell}(\Gamma_\omega f) = (k+\ell+m-2)\,\Pi^{1,1}_{k,\ell}f + \left(\frac{\partial}{\partial\theta} - \ell\cot\theta + k\tan\theta\right)\Pi^{0,0}_{k,\ell}f \ ,$$

$$\Pi^{0,1}_{k,\ell}(\Gamma_\omega f) = (\ell-k+m-p-1)\,\Pi^{0,1}_{k,\ell}f - \left(\frac{\partial}{\partial\theta} - (k+p-1)\tan\theta - \ell\cot\theta\right)\Pi^{1,0}_{k,\ell}f \ ,$$

$$\Pi^{1,0}_{k,\ell}(\Gamma_\omega f) = (k-\ell+p-1)\,\Pi^{1,0}_{k,\ell}f + \left(\frac{\partial}{\partial\theta} + k\tan\theta + (\ell+m-p-1)\cot\theta\right)\Pi^{0,1}_{k,\ell}f \ .$$

The proof is again bases on the decomposition of Γ_ω corresponding to $\vec{\omega} = \cos\theta\vec{v} + \sin\theta\vec{\mu}$ and the fact that $f(\vec{\omega}) = \sum_{k,\ell=0}^{\infty} \Pi_{k,\ell} f(\vec{\omega})$. It is hence left as exercise to the reader.

We'll now solve the Darboux equations explicitly in the case where $\Gamma_\omega f = \lambda f$, $\lambda = -s$ or $\lambda = s+m-2$, $s \in \mathbb{N}$.

Putting $A = \Pi^{0,0}_{k,\ell}f$, $B = \Pi^{1,0}_{k,\ell}f$, $C = \Pi^{0,1}_{k,\ell}f$ and $D = \Pi^{1,1}_{k,\ell}f$, the Darboux equations lead to the following two separate systems:

$$\text{I.}\ \begin{cases} \left(\frac{\partial}{\partial\theta} - (k+p-1)\tan\theta + (\ell+m-p-1)\cot\theta\right)D = -(k+\ell+\lambda)A \\[2ex] \left(\frac{\partial}{\partial\theta} + k\tan\theta - \ell\cot\theta\right)A = (\lambda-(k+\ell+m-2))D \end{cases}$$

and

$$\text{II.}\ \begin{cases} \left(\frac{\partial}{\partial\theta} + k\tan\theta + (\ell+m-p-1)\cot\theta\right)C = (\lambda+\ell-k-p+1)B \\[2ex] \left(\frac{\partial}{\partial\theta} - \ell\cot\theta - (k+p-1)\tan\theta\right)B = (-\lambda+\ell-k+m-p-1)C \ . \end{cases}$$

Next, we'll put $t = \cos 2\theta$ and

$$A(\theta) = \cos^k\theta\sin^k\theta\, a(\cos 2\theta) \ , \quad D(\theta) = \cos^{k+1}\theta\sin^{k+1}\theta\, d(\cos 2\theta) \ ,$$

$$B(\theta) = \cos^{k+1}\theta\sin^k\theta\, b(\cos 2\theta) \ , \quad C(\theta) = \cos^k\theta\sin^{k+1}\theta\, c(\cos 2\theta) \ ,$$

then systems I and II transform into the systems

$$\text{I}'.\ \begin{cases} (1-t^2)d' - \left[\frac{2k+p}{2}(t-1) + \frac{2\ell+m-p}{2}(t+1)\right]d = (k+\ell+\lambda)\, a, \\[2ex] 4a' = (k+\ell-\lambda+m-2)\, d \ , \end{cases}$$

$$\mathrm{II}'.\begin{cases}2(t-1)c' + (2\ell+m-p)\ c = (\lambda+\ell-k-p+1)\ b\ , \\ 2(t+1)b' + (2k+p)\ b = (\lambda+k-\ell-m+p+1)\ c\ .\end{cases}$$

We first solve I' explicitly. Eliminating "a" from system I' leads to the equation

$$(1-t^2)d'' - [\frac{2k+p+2}{2}(t-1) + \frac{2\ell+m-p+2}{2}(t+1)]\ d'$$

$$- [\ (k+\ell+\frac{m}{2}) + \frac{1}{4}(k+\ell-s)(k+\ell+s+m-2)]\ d = 0\ ,$$

where $\lambda = -s$ or $\lambda = s+m-2$.

Now we put

$$u = \frac{2\ell+m-p+2}{4}\ ,\ v = \frac{2k+p+2}{4}\ ,\ n = \frac{s-k-\ell-2}{2}\ ,$$

then the previous equation becomes the standard differential equation for the Jacobi polynomials (see e. g. [20])

$$(1-t^2)d'' + (2(v-u) - 2(v+u)t)d' + n(n+2(v+u)-1)d = 0\ ,$$

the solution of which is given by

$$P_n^{2u-1,2v-1}(t) = 2^{-n}\sum_{j=0}^{n}\binom{n+2u-1}{j}\binom{n+2u-1}{n-j}(t-1)^{n-j}(t+1)^j\ .$$

Hence we can say that d is of the form

$$d = K(\nu)\ P_{\frac{s-k-\ell-2}{2}}^{\ell+\frac{m-p}{2},\ k+\frac{p}{2}}(t)\ ,$$

where for fixed ν , $K(\nu)$ belongs to $\mathcal{M}_{k,\ell}^{0,0}(\nu)$.

As to "a", we make use of the derivation rule for Jacobi polynomials:

$$\frac{d}{dt}\ P_n^{\alpha,\beta}(t) = \frac{n+\alpha+\beta+1}{2}\ P_{n-1}^{\alpha+1,\beta+1}(t)\ .$$

Putting $\alpha = \ell + \frac{m-p}{2} - 1$, $\beta = k + \frac{p}{2} - 1$, $n = \frac{s-k-\ell}{2}$,

we have that $2(n+\alpha+\beta+1) = s+k+\ell+m-2$

so that

$$a = K(\nu)\ \frac{k+\ell-\lambda+m-2}{k+\ell+s+m-2}\ P_{\frac{s-k-\ell}{2}}^{\ell+\frac{m-p}{2}-1,\ k+\frac{p}{2}-1}(t)\ .$$

It is clear from the solutions "a" and "d", that A and D can only be nonzero in the case where $s-k-\ell$ is even. This can also be shown directly by applying the map $\vec{\omega} \to -\vec{\omega}$ or $\vec{\mu} \to -\vec{\mu}$ and $\vec{\nu} \to -\vec{\nu}$ in Definition 4.

Next, let us attack system II'. Deriving the first equation and eliminating "b" yields

$$(1-t^2)c'' - [\frac{2k+p}{2}(t-1) + \frac{2\ell+m-p+2}{2}(t+1)]c'$$

$$+ [\frac{\ell-k-s-p+1}{2}\,\frac{k-\ell-s-m+p+1}{2} - \frac{2\ell+m-p}{2}\,\frac{2k+p}{2}]c = 0 ,$$

where $\lambda = -s$ or $\lambda = s+m-2$.

Putting

$$u = \frac{2\ell+m-p+2}{4} , \quad v = \frac{2k+p}{4} , \quad n = \frac{s-k-\ell-1}{2} ,$$

this equation again leads us to the Jacobi differential equation, so that we get solutions of the form

$$c = \gamma K(\nu)\, P^{\ell + \frac{m-p}{2},\, k + \frac{p}{2} - 1}_{\frac{s-k-\ell-1}{2}}(t) ,$$

for some section $K(\nu)$ of $\mathcal{H}^{0,0}_{k,\ell}(\nu)$ and $\gamma \in \mathbb{R}_+$.

Comparing this solution with the ones obtained for "a" and "d", it does not take a big guess to see that "b" should be of the form

$$b = \delta K(\nu)\, P^{\ell + \frac{m-p}{2} - 1,\, k + \frac{p}{2}}_{\frac{s-k-\ell-1}{2}}(t) , \quad \delta \in \mathbb{R}_+$$

and we only have to determine the proporionality constants γ and δ .

Putting $\alpha = \ell + \frac{m-p}{2}$, $\beta = k + \frac{p}{2}$, $n = \frac{s-k-\ell-1}{2}$,

we have that

$$c = \gamma\, K\, P^{\alpha,\beta-1}_{n} , \quad b = \delta\, K\, P^{\alpha-1,\beta}_{n} .$$

$$c' = \gamma\, K\, \frac{n+\alpha+\beta}{2} P^{\alpha+1,\beta}_{n-1} , \quad b' = \delta\, K\, \frac{n+\alpha+\beta}{2}\, P^{\alpha,\beta+1}_{n-1}$$

Plugging these expression into the system II' leads to the equation

$$(n+\alpha+\beta)(1-t)P_{n-1}^{\alpha+1,\beta}-(2\ell+m-p)P_n^{\alpha,\beta-1}+\frac{\delta}{\gamma}(\lambda+\ell-k-p+1)P_n^{\alpha-1,\beta}=0\ ,$$

$$(n+\alpha+\beta)(1+t)P_{n-1}^{\alpha,\beta+1}+(2k+p)P_n^{\alpha-1,\beta}-\frac{\gamma}{\delta}(\lambda+k-\ell-m+p+1)P_n^{\alpha,\beta-1}=0\ .$$

Now we have the identity

$$(1-t)P_{n-1}^{\alpha+1,\beta}+(1+t)P_{n-1}^{\alpha,\beta+1}=2\,P_{n-1}^{\alpha,\beta}\ ,$$

so that by adding the previous two equations, we obtain that

$$2(n+\alpha+\beta)P_{n-1}^{\alpha,\beta}=[2\ell+m-p-\frac{\gamma}{\delta}(\lambda+k-\ell-m+p+1)]P_n^{\alpha,\beta-1}$$

$$-[2k+p+\frac{\delta}{\lambda}(\lambda+\ell-k-p+1)]P_n^{\alpha-1,\beta}\ ,$$

which is to coincide with the classical identity

$$P_{n-1}^{\alpha,\beta}=P_n^{\alpha,\beta-1}-P_n^{\alpha-1,\beta}.$$

Now, $2(n+\alpha+\beta)=k+\ell+s+m-1$ and so we must have

$$\frac{\gamma}{\delta}=-\frac{\ell-k-p+1-s}{\lambda+k-\ell-m+p+1}\ ,$$

to make the first term in the right hand side correct. As to the second term, notice that in the expression

$$(\lambda+k-\ell-m+p+1)(\lambda+\ell-k-p+1)\ ,$$

we may replace λ by $-s$ in both cases $\lambda=-s$ or $\lambda=s+m-2$. Hence the complete solution is obtained for

$$\gamma=s+k-\ell+p-1\ ,\quad \delta=\lambda+k-\ell-m+p+1\ .$$

We summarize all these results in the following

Theorem 6. Let $f(\vec{\omega})$ be an eigenfunction of Γ_ω with eigenvalue λ, $\lambda=-s$ or $\lambda=s+m-2$, $s\in\mathbb{N}$.

(i) If $s=k+\ell+2n$ for some $n\in\mathbb{N}$; then $\Pi_{k,\ell}f$ is given by

$$[(k+\ell-\lambda+m-2)P_n^{\alpha-1,\beta-1}(t)+(k+\ell+s+m-2)\vec{y}\,\vec{x}\,P_{n-1}^{\alpha,\beta}(t)]\ K(\nu)(\vec{x},\vec{y})$$

(ii) If $s = k+\ell+2m+1$, for some $n \in \mathbb{N}$; then $\Pi_{k,\ell} f$ is given by

$$[(\lambda+k-\ell-m+p+1)\ \vec{x}\ P_n^{\alpha-1,\beta}(t) + (s+k-\ell+p-1)\ \vec{y}\ P_n^{\alpha,\beta-1}(t)]\ K(\nu)(\vec{x},\vec{y}) \ .$$

Hereby $\alpha = \ell + \frac{m-p}{2}$, $\beta = k + \frac{p}{2}$, $\vec{x} = \cos\theta\vec{v}$, $\vec{y} = \sin\theta\vec{\mu}$, $\vec{v} \in S(\nu)$, $\vec{\mu} \in \bar{S}(\nu)$, $t = \cos 2\theta = \vec{y}^2 - \vec{x}^2$ and $K(\nu)(\vec{x},\vec{y}) = \cos^k\theta \sin^\ell\theta K(\nu)(\vec{v},\vec{\mu}) \in \mathcal{M}_{k,\ell}^{0,0}(\nu)$.

Remarks. (1) Notice that if $\Gamma_\omega f = \lambda f$ with $\lambda = -s$ or $\lambda = s+m-2$, $s \in \mathbb{N}$; then $\Pi_{k,\ell} f = 0$ for all values of $k+\ell$ for which $k+\ell > s$.

(2) $K(\nu)(\vec{v},\vec{\mu})$ can be regarded as "canonical sections" of the vector bundle $\mathcal{M}_{k,\ell}^{0,0}(\nu)$. In view of the Darboux equations, $K(\nu)$ can be determined by evaluating $\Pi_{k,\ell} f$ for any fixed value of $\theta \in]0, \frac{\pi}{2}[$. Another way to see this is by noticing that the curves $(P_n^{\alpha-1,\beta-1}(t)\ ,\ P_n^{\alpha,\beta}(t))$ and $(P_n^{\alpha-1,\beta}(t)\ ,\ P_n^{\alpha,\beta-1}(t))$ are not going through the origin when t varies in $[-1,1]$. To determine $K(\nu)$ explicitly we can introduce boundary value operators, similar to $D_{\pm,k}$ as follows

$$D_{k,\ell}^{\sigma_1,\sigma_2}(\nu) f = \lim_{\theta\to 0} \frac{1}{\sin^{\ell+\sigma_2}\theta}\ \Pi_{k,\ell}^{\sigma_1,\sigma_2} f(e^{i\theta},\nu) \ ,$$

$$E_{k,\ell}^{\sigma_1,\sigma_2}(\nu) f = \lim_{\theta\to\frac{\pi}{2}} \frac{1}{\cos^{k+\sigma_1}\theta}\ \Pi_{k,\ell}^{\sigma_1,\sigma_2} f(e^{i\theta},\nu) \ ,$$

where $(\sigma_1,\sigma_2) \in \{0,1\}^2$. In contrast with the codimension one case, $D_{k,\ell}^{\sigma_1,\sigma_2}$ and $E_{k,\ell}^{\sigma_1,\sigma_2}$ are no longer differential operators. The functions $K(\nu)$ occur in the formulae

$$D_{k,\ell}^{0,0}(\nu) f = (k+\ell-\lambda+m-2) P_n^{\alpha-1,\beta-1}(1) K(\nu) \ , \ s = k+\ell+2n \ ,$$

$$D_{k,\ell}^{1,0}(\nu) f = (\lambda+k-\ell-m+p+1) P_n^{\alpha-1,\beta}(1) \vec{v} K(\nu) \ , \ s = k+\ell+2n+1 \ .$$

(3) Notice that $K(\nu)(\vec{x},\vec{y})$ is homogeneous of degree k in $\vec{x}$ and of degree ℓ in $\vec{y}$. Furthermore, using the identity $\vec{x}^2+\vec{y}^2 = -1$, the factors occuring in the decomposition of $\Pi_{k,\ell} f$ in Theorem 6 are homogeneous in $\vec{x}+\vec{y}$ of degrees $2n$ and $2n+1$ respectively, so that $\Pi_{k,\ell} f$ is homogeneous in $\vec{\omega} = \vec{x}+\vec{y}$ of degree s .

4. Applications to the Radon transform

Theorem 3 and 6 may be applied to the Radon transform in the cases $k=\ell=0$. We'll treat both theorems separately. Furthermore we'll only consider even functions on S^{m-1} in detail since the case of odd functions is essentially similar. This means that we are studying Radon and X-ray transforms on real projective space.

Let $f(\vec{\omega}) = f(-\vec{\omega})$. Then f admits the expansion in spherical monogenics

$$f(\vec{\omega}) = \sum_{s=0}^{\infty} (P_{2s}f + Q_{2s+1}f)(\vec{\omega}) = \sum_{s=0}^{\infty} S_{2s}(f)(\vec{\omega})$$

Hence, in view of Theorem 3 (or Theorem 1) we have that

$$Pf(i\vec{v}) = B_+f(\vec{v}) = \sum_{s=0}^{\infty} [P_{m,2s}(0)P_{2s}f + P_{m,2s+2}(0)Q_{2s+1}f]$$

Now (see[5])

$$P_{m,2s}(0) = \frac{(-1)^s}{\sqrt{\pi}} \frac{\Gamma(\frac{m-1}{2})\Gamma(s+\frac{1}{2})}{\Gamma(s+\frac{m-1}{2})}$$

so that the Radon transform is given by

$$B_+f(\vec{v}) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\sqrt{\pi}} \frac{\Gamma(\frac{m-1}{2})\Gamma(s+\frac{1}{2})}{\Gamma(s+\frac{m-1}{2})} S_{2s}f(\vec{v}) \ .$$

In the even dimensional case $m = 2p$ we put

$$f(\vec{\omega}) = H_+f(\vec{\omega}) + H_-f(\vec{\omega}) \ , \text{ where } H_+f(\vec{\omega}) = \sum_{s=0}^{\infty} Q_{2s+1}f(\vec{\omega}),\ H_-f(\vec{\omega}) = \sum_{s=0}^{\infty} P_{2s}f(\vec{\omega}) \ .$$

Furthermore, notice that the extensions of H_+f and H_-f as nullsolutions of $\mathbf{D}$ to the Lie sphere lead to the formulae

$$H_+f(i\vec{\omega}) = \sum_{s=0}^{\infty} (-1)^{s+p}Q_{2s+1}f(\vec{\omega}) \ , \ H_-f(i\vec{\omega}) = \sum_{s=0}^{\infty} (-1)^s P_{2s}f(\vec{\omega}) \ .$$

Hence we have that

$$B_+(H_-f) = \frac{\Gamma(p-\frac{1}{2})}{\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{(-1)^s P_{2s}f}{(s+p-\frac{3}{2})\dots(s+\frac{1}{2})}$$

so that, in view of $\Gamma_\omega P_{2s} = -2sP_{2s}f$,

$$B_+^{-1}(H_-f) = c_p(\Gamma_\omega-2p+3)\ldots(\Gamma_\omega-1)H_-f(i\vec{\omega}) ,$$

where $c_p = \dfrac{\sqrt{\pi}(-1)^{p-1}}{2^{p-1}\,\Gamma(p-\frac{1}{2})}$.

Similarly one has that

$$B_+^{-1}(H_+f) = c_p(\Gamma_\omega-1)\ldots(\Gamma_\omega-2p+3)H_+f(i\vec{\omega}) .$$

This leads to the following

Theorem 7. In the even dimensional case $m = 2p$, the inverse Radon transform for even functions on the unit sphere is given by

$$B_+^{-1}f(\vec{\omega}) = c_p(\Gamma_\omega-1)\ldots(\Gamma_\omega-2p+3)f(i\vec{\omega}) .$$

Notice that, in view of the identity $\Delta_s = \Gamma_\omega(m-2-\Gamma_\omega)$, this theorem refines the inversion formula by S. Helgason in [3], which says that

$$B_+^{-1}f(\vec{\omega}) = c_p^2(1(2p-3)-\Delta_s)(3(2p-5)-\Delta_s)\ldots((2p-3)1-\Delta_s)B_+f(\vec{\omega}) .$$

Next, let us consider Theorem 6. For $k=\ell=0$ and $f(\vec{\omega}) = f(-\vec{\omega})$ it says that $\Pi_{0,0}(P_{2s}f)$ is of the form

$$[P_s^{\frac{m-p}{2}-1,\frac{p}{2}-1}(\cos 2\theta) + \cos\theta\sin\theta\ \vec{\mu}\vec{\nu}P_{s-1}^{\frac{m-p}{2},\frac{p}{2}}(\cos 2\theta)]K_s(\nu)$$

while $\Pi_{0,0}(Q_{2s-1}f)$ is of the form

$$[P_s^{\frac{m-p}{2}-1,\frac{p}{2}-1}(\cos 2\theta) + \cos\theta\sin\theta c_\lambda\vec{\mu}\vec{\nu}P_{s-1}^{\frac{m-p}{2},\frac{p}{2}}(\cos 2\theta)\ K'_s(\nu) ,$$

where $K_s(\nu)$ and $K'_s(\nu)$ are multivector functions and

$$c_\lambda = \frac{2s+m-2}{-\lambda+m-2} , \quad \lambda = 2s+m-2 .$$

Hence, for $\theta = 0$ we obtain that, putting $L_s(\nu) = K_s(\nu) + K'_s(\nu)$,

$$B_+ f(\nu) = \frac{1}{\omega_p} \int_{S(\nu)} f(\vec{v})d\vec{v} = \sum_{s=0}^{\infty} P_s^{\frac{m-p}{2}-1,\frac{p}{2}-1}(1) L_s(\nu)$$

while for $\theta = \frac{\pi}{2}$ we obtain that

$$\bar{B}_+ f(\nu) = \frac{1}{\omega_{m-p}} \int_{S(\nu)} f(\vec{\mu})d\vec{\mu} = \sum_{s=0}^{\infty} P_s^{\frac{m-p}{2}-1,\frac{p}{2}-1}(-1) L_s(\nu) .$$

Hence we do not obtain from this an inversion formula for the "X-ray" transforms $B_+ f(\nu)$ and $\bar{B}_+ f(\nu)$ but we do obtain a connection between these transforms, which are dual to one another.
Now we have that (see [20])

$$P_s^{\frac{m-p}{2}-1,\frac{p}{2}-1}(1) = \begin{bmatrix} s+\frac{m-p}{2}-1 \\ s \end{bmatrix}, \; P_s^{\frac{m-p}{2}-1,\frac{p}{2}-1}(-1) = (-1)^2 \begin{bmatrix} s+\frac{p}{2}-1 \\ s \end{bmatrix} .$$

This leads to the following theorem which describes the relation between the Radon transform of codimension p and of dimension $p-1$.

Theorem 8. Let f be an even function on S^{m-1} and let $f = \sum_{s=0}^{\infty} S_{2s} f$ be the expansion of f in spherical harmonics. Then the Radon transforms of dimension $p-1$ and of codimension p are linked by the formula

$$\begin{bmatrix} s+\frac{p}{2}-1 \\ s \end{bmatrix} B_+(S_{2s}f) = (-1)^s \begin{bmatrix} s+\frac{m-p}{2}-1 \\ s \end{bmatrix} \bar{B}_+(S_{2s}f) .$$

Notice that in the case where $m = 2q$ is even and $p < q$, we have that

$$B_+(S_{2s}f) = (-1)^s \frac{\Gamma(\frac{p}{2})}{\Gamma(\frac{m-p}{2})} (s+\tfrac{p}{2}+q-p-1)\ldots(s+\tfrac{p}{2}) \, \bar{B}_+(S_{2s}f)$$

and so

$$B_+(P_{2s}f) = c_p \bar{B}_+((\Gamma_\omega-m+p+2)\ldots(\Gamma_\omega-p) \; P_{2s}f(i\vec{\omega})) ,$$

where $c_p = (-1)^{q-p}\Gamma(\frac{p}{2})/\Gamma(\frac{m-p}{2})2^{q-p}$. In a similar way are shows that

$$B_+(Q_{2s-1}f) = (-1)^{p-1} c_p \bar{B}_+((\Gamma_\omega-p)\ldots(\Gamma_\omega-m+p+2)Q_{2s-1}f(i\omega))$$

Now, let us recall that the Hilbert-Riesz transform on the unit sphere is given by

$$H_S f(\vec{\omega}) = (H_+ - H_-) f(\vec{\omega}) = \sum_{s=0}^{\infty} (Q_{2s-1} - P_{2s}) f(\vec{\omega}) \ .$$

Then it is clear that $H_S^2 = 1$. Hence Theorem 8 leads to the following refinement of Theorem 7.

Theorem 9. Let $m = 2q$ be even and $p < q$ be odd. Then the Radon transform of dimension $p-1$ of an even function f may be expressed in terms of the Radon transform of codimension p of f by the formula

$$B_+ f(\nu) = c_p \overline{B}_+ ((\Gamma_\omega - m+p+2) \ldots (\Gamma_\omega - p) f(i\vec{\omega}))(\nu) \ .$$

When p is even and $p < q$ we have the expression

$$B_+ f(\nu) = -c_p \overline{B}_+ ((\Gamma_\omega - m+p+2) \ldots (\Gamma_\omega - p) H_S f(i\vec{\omega}))(\nu) \ .$$

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