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GRADED RIEMANN SURFACES AND KRICHEVER-NOVIKOV ALGEBRAS.

PETER BRYANT

Abstract. Following the work of Krichever & Novikov, Bonora, Martellini, Rinaldi & Russo defined a superalgebra associated to each compact Riemann surface with spin structure. Noting that this data determines a graded Riemann surface, we find a natural interpretation of the BMRR-algebra in terms of the geometry of graded Riemann surfaces. We also discuss the central extensions of these algebras (correcting the form of the central extension given by Bonora et al.). It is hoped that this work will the first step towards defining Krichever-Novikov algebras for (the more general) super Riemann surfaces ; in particular we emphasise the importance of *graded conformal* vectorfields.

§1 INTRODUCTION.

BRS quantisation in the context of string theory [9] is usually concerned with infinite dimensional Lie algebras such as the Virasoro algebra. Krichever & Novikov [10] have noted that it is more natural to consider algebras adapted to the nontrivial worldsheet geometry which we expect for interacting strings. BRS quantisation, using the Krichever-Novikov algebra associated with a Riemann surface in genus g, was successfully carried out in [5] (in the critical dimension).

The success of this approach leads one to believe that it is worthwhile to extend these considerations to the fermionic string. Unfortunately there are some technical obstacles to overcome in defining the "superversions" of Krichever-Novikov algebras because of our limited understanding of the geometry of super Riemann surfaces. Instead I shall consider Krichever-Novikov algebras associated with a graded Riemann surface of genus g.

Graded Riemann surfaces (first defined in [3]) should be regarded as simple examples of super Riemann surfaces [8]. They are useful because they provide a solid mathematical framework for the first approach to superconformal geometry that simply uses λ-differentials where λ can be integral *or* half-integral. More precisely both these approaches use the same initial data (a Riemann surface Σ together with a spin structure on Σ) whilst graded Riemann surfaces are related in a simple way with superconformal manifolds ; there is a natural correspondence between graded Riemann surfaces and split super Riemann surfaces given by z-extension [4].

We reinterpret the work of Bonora, Martellini, Rinaldi & Russo [6] on graded Krichever-Novikov algebras in terms of the geometry of graded Riemann surfaces, providing justification for their confidence that the rather ad hoc algebra they write down is a natural generalisation of Krichever & Novikov's. Our results demonstrate that the category of graded Riemann surfaces is an adequate category for the purposes of describing the geometry of fermionic superstring theory.

We correct the impression given in the preliminary announcement [6] that the odd part of the cocycle which gives the central extension of the graded Krichever-Novikov algebra is the same in all genus. The correct version of this cocycle must include a Schwarzian connection contribution (because one of the cocycle identities tells us that the odd and even cocycles may not be chosen independently). That this choice is actually correct depends upon the surprising fact that the inhomogeneous transformation laws for both odd and even cocycles contain a Schwarzian derivative term ; they can be corrected *simultaneously* using a Schwarzian connection.

Finally we should point out that this gives the first step towards defining a super Krichever-Novikov algebra on a general superconformal manifold. The most important point here is that we should use meromorphic *superconformal* – as opposed to superanalytic – vectorfields. Just as in §4, superconformal vectorfields can be constructed from superanalytic sections of $\mathcal{D}^2$ [11,12]. The use of a super Riemann-Roch theorem (see e.g. [11]) should enable us to confirm the existence of superconformal vectorfields with appropriate orders at two chosen points. This will be the subject of a separate paper.

The organisers of the Winter school are warmly thanked for their conscientious hospitality during my visit to Czechoslovakia. I would like to thank the Science & Engineering Research Council of Great Britain for financial support and Paolo Teofillatto for first suggesting the possibility that the BMRR-algebra might have a geometrical interpretation using graded Riemann surfaces.

§2 GRADED RIEMANN SURFACES .

Riemann surfaces in the category of (holomorphic) graded manifolds were first formulated by Marjorie Batchelor and myself [3]. We later discovered that essentially the same notion was given in the appendix of a paper by Baranov,Manin,Frolov & Schwarz [1] a year earlier.

Definition 2.1. *A Graded Riemann surface is a (1,1)-dimensional holomorphic graded manifold $(X,\mathcal{A})$ equipped with an isomorphism*

$$k : \mathcal{A}_1 \otimes_{\mathcal{O}_X} \mathcal{A}_1 \longrightarrow \Omega^1_X$$

of sheaves of $\mathcal{O}_X$-Modules.

X is a one-dimensional complex manifold (= Riemann surface) with structure sheaf $\mathcal{O}_X$ (holomorphic functions) and Ω^1_X is the sheaf of holomorphic 1-forms on X. $\mathcal{A}$ is a sheaf of graded-commutative algebras ("generalised or graded functions") which satisfies the following properties. There is an epimorphism of sheaves $\mathcal{E} : \mathcal{A} \to \mathcal{O}_X$ (augmentation) and an open cover $\{U_i\}$ of X together with isomorphisms $\mathcal{A}_{|U_i} \to \mathcal{O}_X \otimes \Lambda \mathbf{C}_{|U_i}$ (local triviality).

In dimension (1,1), graded manifolds are very simple to describe. The even part of the augmentation identifies $\mathcal{A}_0$ with $\mathcal{O}_X$. Then $\mathcal{A}_1$ becomes a (locally free) $\mathcal{O}_X$-Module ; we can construct a holomorphic line bundle $E \longrightarrow X$ with $\mathcal{A}_1 \simeq \mathcal{O}_X(-, E)$. These facts together imply that there is a canonical isomorphism (of sheaves of graded algebras)

$$\mathcal{A} \simeq \mathcal{A}_E \equiv \mathcal{O}_X(-, \Lambda E)$$

The GRS condition amounts then to a bundle isomorphism $E \bigotimes E \simeq \kappa_X$ where κ_X denotes the canonical (= holomorphic cotangent) bundle of X ; E is a spin reduction of the cotangent bundle , $E \equiv \kappa_X^{1/2}$. From the well-known fact that every (compact or noncompact) Riemann surface admits a spin structure, we deduce the existence of a graded Riemann surface $(X, \mathcal{A})$ for every Riemann surface X. More precisely there is a unique GRS structure if X is noncompact and 2^{2g} GRS structures if X is compact and of genus g.

Example 2.2 (Graded Riemann sphere). *Suppose we take $X = \mathbf{C}P^1$, E_1 =tautological bundle. Then $C(\kappa_X) = -2, C(E_1) = -1$ implies the existence of an isomorphism $E_1 \bigotimes E_1 \simeq \kappa_X$. $(\mathbf{C}P^1, \mathcal{A}_{E_1})$ is the unique graded Riemann surface over the Riemann sphere (= projective line = $\mathbf{C} \cup \{\infty\}$).*

Suppose that θ denotes a zerofree local section of E over U open in X. Then, for suitable local complex coordinate z on X, the map k is generated by the assignment $\theta \otimes \theta \mapsto dz$. These θ (taken over some open cover of X) will be the odd coordinates for $(X, \mathcal{A}_E)$ and we treat z and θ on equal footing. To illustrate what is meant by this, we consider the section "$\frac{\partial}{\partial \theta}$" of E^* dual to θ (over $U \subset X$). This operates on $\mathcal{A}_E(U)$ according to the rule $p_0 + p_1\theta \mapsto p_1$. We claim that it makes sense to describe $\frac{\partial}{\partial \theta}$ as an "odd" vectorfield on the graded Riemann surface. What we mean is that $\frac{\partial}{\partial \theta}$ is an odd derivation of the graded algebra $\mathcal{A}_E(U)$:

$$\frac{\partial}{\partial \theta}((p_0 + p_1\theta).(q_0 + q_1\theta)) = p_0q_1 + p_1q_0 = (\frac{\partial}{\partial \theta}(p_0 + p_1\theta).(q_0 + q_1\theta) + (p_0 - p_1\theta).(\frac{\partial}{\partial \theta}(q_0 + q_1\theta))$$

Thus the coordinate vectorfield $\frac{\partial}{\partial \theta}$ can be placed on the same footing as $\frac{\partial}{\partial z}$.

At this point it is relevent to mention that if we make the definition $\mathbf{D} \equiv \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$, $\mathbf{D}$ satisfies $(\mathbf{D})^2 = \frac{\partial}{\partial z}$ and has the transformation rule

$$\mathbf{D} = (\frac{\partial \tilde{z}}{\partial z})^{1/2}.\tilde{\mathbf{D}}$$

as we pass from one coordinate patch to another. Whenever $(X, \mathcal{A})$ is a (1,1)-dimensional holomorphic graded manifold, the existence of a graded $\mathcal{A}$-Module $\mathcal{D}$ locally generated by $\mathbf{D}$ is equivalent to the GRS condition. We call $\mathcal{D}$ a *graded conformal structure* on $(X, \mathcal{A})$. This viewpoint emphasises the close connections between graded and super Riemann surfaces.

§3 THE SUPERVIRASORO ALGEBRA .

The superVirasoro algebra $\mathcal{SV}$ is the graded Lie algebra with even generators $\{L_m; m \in \mathbf{Z}\}$ and odd generators $\{G_r; r \in \mathbf{Z} + 1/2\}$ (Neveu-Schwarz sector). The defining brackets are as follows [9] :

$$\begin{aligned} [L_m, L_n] &= (m-n)\,L_{m+n} \\ [L_m, G_r] &= (\frac{m}{2} - r)\,G_{m+r} \\ \{G_r, G_s\} &= 2\,L_{r+s} \end{aligned}$$

Remark : Central extensions of $\mathcal{SV}$ and other algebras will be considered in §5.

Theorem 3.1 . *There is a representation of* $\mathcal{SV}$ *by graded meromorphic vectorfields of* $(\mathbf{C}P^1, \mathcal{A}_{E_1})$ *given by*

$$\begin{aligned} L_m &= -z^{m+1}\frac{\partial}{\partial z} - \frac{(m+1)}{2}z^m\theta\frac{\partial}{\partial\theta} \\ G_r &= z^{r+1/2}(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z}). \end{aligned}$$

Proof : Direct calculation of the brackets for L_m , G_r as derivations of $R \otimes \Lambda\theta$ where $R = \{rational functions\} = \mathcal{M}(\mathbf{C}P^1)$. We emphasise that our formulae do give *globally defined* graded vectorfields. $\triangle$

Remark : These formulae have appeared in the physics literature (p. 253 of [7]). They were obtained (in the present setting) by Marjorie Batchelor and myself (cf. last section of [3]).

§4 BMMR-algebra .

Let X be a Riemann surface of genus $g \geq 1$ with fixed spin structure. Pick points P_+, P_- (in general position) on X. These points are supposed to represent the points of compactification of the string worldsheet so there is no a priori reason why we should consider fields that are holomorphic over all of X including P_+P_- ; indeed X has no nonconstant global holomorphic functions. From now on *meromorphic functions or fields on* X *are assumed to be holomorphic and zerofree away from the distinguished points* P_+, P_-.

We claim that there exist meromorphic vectorfields e_n and spinorfields ($= -\frac{1}{2}$-differentials) f_r on X that are determined up to normalisation (in a coordinate system near P_+ say) by their orders at P_+, P_- which are specified as follows :

$$\begin{aligned} Ord_{P_\pm}(e_n) &= \mp n + (1 - \frac{3g}{2}) \qquad [n \in \mathbf{Z} + \frac{1}{2}(g\, mod\, 2)] \\ Ord_{P_\pm}(f_r) &= \mp r + (\frac{1}{2} - g) \qquad [r \in \frac{1}{2}\mathbf{Z}] \end{aligned}$$

This is most easily checked by a simple application of the Riemann-Roch theorem (which we give in some generality).

Proposition 4.1 . *Fix $\lambda \in \frac{1}{2}\mathbf{Z}$. For suitable i [running over either $\mathbf{Z}$ or $\frac{1}{2}\mathbf{Z}$ depending on the parity of $k(\lambda)$], there is a one-dimensional space of meromorphic λ-differentials ω on X with $Ord_{P_\pm}(\omega) = \mp i + k(\lambda)$ provided that*

$$k(\lambda) = (\lambda - \frac{1}{2})g - \lambda.$$

Proof: Let S_i^λ denote the line bundle on X associated with the divisor $(-i + k(\lambda))P_+ + (+i + k(\lambda))P_-$. Riemann-Roch $\Longrightarrow$

$$\begin{aligned} h^0(\kappa^\lambda \otimes S_i^\lambda) - h^0(\kappa^{1-\lambda} \otimes (S_i^\lambda)^{-1}) &= C(\kappa^\lambda \otimes S_i^\lambda) + (1 - g) \\ &= 2\lambda(g - 1) - 2k(\lambda) + (1 - g). \\ &= 1 \end{aligned}$$

But $C(\kappa^{1-\lambda} \otimes (S_i^\lambda)^{-1}) = g - 2 < g \implies (genericity)\ h^0(\kappa^{1-\lambda} \otimes (S_i^\lambda)^{-1}) = 0$.
Hence $h^0(\kappa^\lambda \otimes S_i^\lambda) = 1$. Δ

Bonora, Martellini, Rinaldi & Russo [6] gave the graded vectorspace, whose even part is spanned by the e_n and whose odd part is spanned by the f_r, the following graded Lie bracket structure turning it into a graded Lie algebra [which we will refer to as the "BMRR-algebra"].

$$\begin{aligned} [e_m, e_n] &= \mathcal{L}_{e_m}(e_n) \\ [e_m, f_r] &= \mathcal{L}_{e_m}(f_r) \\ \{f_r, f_s\} &= -(f_r \otimes f_s + f_s \otimes f_r) \end{aligned}$$

where $\mathcal{L}$ is the Lie derivative. In particular we recall that if we have some local coordinate z and $\underline{e} = g(z)\frac{\partial}{\partial z}, \underline{f} = \rho(z)(dz)^{-\frac{1}{2}}$, we have $\mathcal{L}_{\underline{e}}(\underline{f}) = (g(z)\rho'(z) - \frac{1}{2}g'(z)\rho(z))(dz)^{-\frac{1}{2}}$.

The main theorem of this lecture is the following interpretation of the BMRR-algebra.

Theorem 4.2 . *The BMRR-algebra associated with a spin structure on X is isomorphic with the graded Lie algebra of (meromorphic)* **graded conformal** *vectorfields on $(X, \mathcal{A}_{\kappa_X^{1/2}})$.*

Let us first explain the terminology used here. Suppose that ξ is a graded vectorfield on $(X, \mathcal{A})$, $\xi \in Der\mathcal{A}(X)$. When U is an open subset of X on which $\kappa_X^{1/2}$ is trivial, we may take the *graded* bracket of $\xi_{|U}$ with the odd vectorfield $\mathsf{D} = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$ (determined by graded coordinates (z, θ) over U). ξ is **graded conformal** for $(X, \mathcal{A})$ if each point has some coordinate neighbourhood U for which $[\xi\,,\,\mathsf{D}]$ *is a multiple of* D. It is clear that the definition depends only upon the graded conformal structure $\mathcal{D}$ of $(X, \mathcal{A})$ rather than the particular choices of graded coordinates used to write down the above condition [12].

Write $\xi = a(z, \theta)\frac{\partial}{\partial z} + b(z, \theta)\mathsf{D}$ where $a(z, \theta) = a_0(z) + \theta a_1(z)$ and $b(z, \theta) = b_0(z) + \theta b_1(z)$ are graded meromorphic functions on U [i.e. a_0, a_1, b_0, b_1 are ordinary meromorphic functions on U]. Then

$$[\xi, \mathsf{D}] = (-1)^{|b|}\mathsf{D}b.\mathsf{D} + (2b - (-1)^{|a|}\mathsf{D}a)\frac{\partial}{\partial z}$$

implies that ξ *is graded conformal precisely when* $b = (-1)^{|a|}\frac{1}{2}Da$. Because this is a crucial point let us spell out what this means for graded conformal vectorfields ξ that are homogeneous :

$$\begin{aligned}\xi_0 &= p_0(z)\frac{\partial}{\partial z} + \frac{1}{2}p_0'(z)\theta\frac{\partial}{\partial\theta}\\ \xi_1 &= p_1(z)(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})\end{aligned}$$

Proof of 4.2.

Given an ordinary vectorfield $\underline{e}$ on X, we construct an even graded conformal vectorfield $\chi(\underline{e})$ on $(X,\mathcal{A})$ as follows. Given a coordinate neighbourhood U and graded coordinates (z,θ) write $\underline{e}_{|U} = a_0(z)\frac{\partial}{\partial z}$ and take

$$\chi(\underline{e})_{|U} = a_0\frac{\partial}{\partial z} + \frac{1}{2}a_0'\theta\frac{\partial}{\partial\theta}.$$

It is guaranteed that the $\chi(\underline{e})_{|U}$ patch together to give a graded vectorfield $\chi(\underline{e})$. For $a_0(z)$ becomes $\hat{a_0}(w) = (w')a_0(z)$ in a new patch whilst $Da_0(z)$ becomes $D\hat{a}_0(w) = (w')^{1/2}Da_0(z)$. Hence the combination given transforms correctly as we pass from one patch to the next. The resulting graded vectorfield is clearly graded conformal. Similarly from each $-\frac{1}{2}$-differential (spinorfield) $\underline{f}$ on X we can form an odd graded conformal vectorfield by the local assignment

$$\chi(\underline{f})_{|U} = b_1(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})$$

whenever we can write $\underline{f}_{|U} = b_1(z)(dz)^{-\frac{1}{2}}$.

It follows that we can find even graded conformal vectorfields $\chi(e_m)$, and odd graded conformal vectorfields $\chi(f_r)$ from the basis elements of the BMRR-algebra. Since χ clearly gives an isomorphism between meromorphic -1-differentials (respectively $-\frac{1}{2}$-differentials) on X and even (resp. odd) meromorphic graded conformal vectorfields on $(X,\mathcal{A})$, it suffices to check that it sends the BMRR-brackets to the standard brackets given by the graded Lie brackets of graded derivations. More specifically we claim that

$$\begin{aligned}[\chi(a_0\frac{\partial}{\partial z}),\chi(b_0\frac{\partial}{\partial z})] &= \chi([a_0\frac{\partial}{\partial z},b_0\frac{\partial}{\partial z}])\\ [\chi(a_0\frac{\partial}{\partial z}),\chi(b_1\frac{\partial}{\partial\theta})] &= \chi(\mathcal{L}_{a_0\frac{\partial}{\partial z}}(b_1\frac{\partial}{\partial\theta}))\\ \{\chi(a_1\frac{\partial}{\partial\theta}),\chi(b_1\frac{\partial}{\partial\theta})\} &= -\chi(2a_1b_1\frac{\partial}{\partial z})\end{aligned}$$

These checks are straightforward. We will compute the second of these relations to illustrate what the checks involve.

$$\begin{aligned}[\,a_0\frac{\partial}{\partial z} + \frac{1}{2}a_0'\theta\frac{\partial}{\partial\theta}, b_1(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})\,]\,(p_0 + p_1\theta) &= (a_0b_1' - \frac{1}{2}a_0'b_1)\,(p_1 - p_0'\theta)\\ &= (a_0b_1' - \frac{1}{2}a_0'b_1)(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z})\,(p_0 + \theta p_1)\end{aligned}$$

This completes the proof of 4.2. Δ

Example 4.3.

In genus 0 we can make an assertion which completely analoguous to 4.2. Indeed the formulae that appear in theorem 3.1 describe graded conformal vectorfields on the graded Riemann sphere (see the formulae for homogeneous graded conformal vectorfields immediately prior to the previous proof). These correspond (under χ^{-1}) with the geometric objects that in a coordinate system near 0 (i.e. the south pole), look like $e_m = -z^{m+1}\frac{\partial}{\partial z}$ (vectorfield) and $f_r = z^{r+\frac{1}{2}}\frac{\partial}{\partial\theta}$ (spinorfield). These are globally defined geometric objects on the Riemann sphere (cf. the statement of Proposition 4.1), meromorphic in the sense that they are holomorphic away from the north and south poles (our special choices for P_+, P_-) with zeroes and poles of the appropriate orders at these two points. In summary the results of §4 are consistent with those of §3, the super Virasoro algebra being the graded Krichever-Novikov algebra in genus 0.

For general genus $g \geq 2$, theorem 4.2 justifies the interpretation of the BMRR-algebra of X (plus a spin structure) as the *graded Krichever-Novikov algebra* associated with the graded Riemann surface $(X, \mathcal{A}_{\kappa_X^{1/2}})$.

§5 CENTRAL EXTENSIONS .

It is well-known that $\mathcal{SV}$, the super Virasoro algebra, has a central extension $\widetilde{\mathcal{SV}}$ of the following form [9].

$$
\begin{aligned}
[\,L_m, L_n\,] &= (m-n)\,L_{m+n} + \frac{1}{12}(m^3-m)\delta_{m,-n}.t \\
[\,L_m, G_r\,] &= (\frac{1}{2}m - r)G_{m+r} \\
\{\,G_r, G_s\,\} &= 2.L_{r+s} + \frac{1}{3}(r^2-\frac{1}{4})\delta_{r,-s}.t
\end{aligned}
$$

In an irreducible representation of $\mathcal{SV}$ the central term, written "t" above, just becomes a constant (usually written "c" and called the central charge).

The 'anomalous' terms are normally given by residues :

$$
\begin{aligned}
\tilde{\chi}(L_m, L_n) &= \frac{1}{24\pi i}\oint \chi(L_m, L_n) \\
\tilde{\varphi}(G_r, G_s) &= \frac{1}{6\pi i}\oint \varphi(G_r, G_s)
\end{aligned}
$$

where we take $\tilde{\chi}(f,g) = \frac{1}{2}(f'''(z)g(z) - f(z)g'''(z))dz$ and $\tilde{\varphi}(\rho,\sigma) = -\rho'(z)\sigma'(z)dz$. These formulae define *coordinate-independent* 1-forms on the Riemann sphere [here and in future, we restrict to a local coordinate system, f is short for $f(z)\frac{\partial}{\partial z}$, ρ is short for $\rho(z)(dz)^{-1/2}$ and a prime denotes differentiation w.r.t. z].

The cocycle relations (which follow from the Jacobi identity) are

$$
\begin{aligned}
\tilde{\chi}(f,[g,h]) + \tilde{\chi}(g,[h,f]) + \tilde{\chi}(h,[f,g]) &= 0 \\
\tilde{\varphi}(\rho,[\sigma,f]) + \tilde{\varphi}(\sigma,[f,\rho]) - \tilde{\chi}(f,\{\rho,\sigma\}) &= 0
\end{aligned}
$$

One checks that this holds by manipulating χ and φ remembering that the contour integral of an exact 1-form vanishes. We are thus able to talk of χ and φ as 'cocycles' (i.e. cocycles modulo irrelevent exact terms). It is in this sense that the ('integration by parts') identities hold :

$$\chi(f,g) = f'''g = -f''g' = \frac{1}{2}(f'g'' - f''g').$$

These give alternative forms for the 'cocycle' χ (of course they differ only by 'coboundary' terms).

Naively one might expect that these formulae give cocycles defining a central extension of the graded Krichever-Novikov algebras. It turns out however that the 1-forms defined are not independent of the choice of local coordinate z on a general Riemann surface X. Suppose we consider a coordinate overlap on X with associated transition function $w = w(z)$. Since $\frac{\partial}{\partial w} = (w')^{-1}\frac{\partial}{\partial z}$, a vectorfield ξ which appears in the first coordinate system as $f(z)\frac{\partial}{\partial z}$ will become $\hat{f}(w)\frac{\partial}{\partial w}$ in the second where $\hat{f}(w) = f(z).w'$. We can then calculate

$$\frac{1}{2}\left(\frac{\partial^2}{\partial w^2}\hat{f}(w).\frac{\partial}{\partial w}\hat{g}(w) - \frac{\partial}{\partial w}\hat{f}(w).\frac{\partial^2}{\partial w^2}\hat{g}(w)\right) =$$

$$= \left(\frac{1}{2}(f''(z)g'(z) - f'(z)g''(z)) - (f'(z)g(z) - g'(z)f(z))\right.$$

$S(w,z))(w')^{-1}$

(up to exact terms) where

$$S(w,z) = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$$

is the *Schwarzian derivative* of w with respect to z.

Similarly $\hat{\rho}(w) = \rho(z)(w')^{1/2}$ implies,

$$\frac{\partial}{\partial w}\hat{\rho}(w).\frac{\partial}{\partial w}\hat{\sigma}(w) = \left(\rho'(z)\sigma'(z) - \frac{1}{2}\rho(z)\sigma(z)S(w,z)\right)(w')^{-1}.$$

Observe that, in the case of the Riemann sphere, the single transition function is $w = w(z) = 1/z$ which is a Mobius transformation (i.e. $S(w,z) = 0$). This demonstrates the validity of the previous statement concerning the coordinate independence of $\tilde{\chi}, \tilde{\varphi}$ on the Riemann sphere.

The appearance of the Schwarzian derivative $S(w,z)$ in *both* formulae is crucial. The second cocycle identity tells us that it is not legitimate to amend independently $\tilde{\chi}$ and $\tilde{\varphi}$ (to obtain coordinate independent cocycles). It is well-known that one should amend the 'cocycle' χ by adding a *Schwarzian connection* term.

Definition 5.1 . *A Schwarzian connection on X is a collection $\mathcal{R} = \{R_U(z) : U \in \mathcal{U}\}$ of locally defined functions satisfying*

(i) $\mathcal{U}$ is an open cover of X by coordinate charts

(ii) $R_U(z)$ is a meromorphic function on the chart U [if U does not contain P_+ or P_- then this means that $R_U(z)$ is holomorphic]

(iii) on chart overlaps we have the following inhomogeneous transformation rule:

$$\hat{R}(w).(w')^2 = R(z) + S(w,z).$$

We are now in a position to find a central extension of the graded Krichever-Novikov algebra associated to $\{(X, \kappa_X^{1/2}), P_+, P_-\}$.

Theorem 5.2 . *Suppose that $\mathcal{R}$ is a Schwarzian connection on X. Then there is a central extension of the graded Krichever-Novikov algebra corresponding to the following cocycles :*

$$\begin{aligned}\chi(f,g) &= \big(\tfrac{1}{2}(f'''(z)g(z) - f(z)g'''(z)) - R(z).(f'(z)g(z) - f(z)g'(z))\big)dz \\ \varphi(\rho,\sigma) &= -\big(\rho'(z).\sigma'(z) + \tfrac{1}{2}R(z).\rho(z)\sigma(z)\big)dz\end{aligned}$$

Proof : The cocycles given by taking the residues of χ and φ are coordinate independent as we can easily show by combining the transformation laws given above [note : a residue is found by contour integration about any closed contour on X that separates P_+ from P_-]. It is instructive to check the second 'cocycle' identity :

$$\begin{aligned}&-\big(\varphi(\rho,[f,\sigma]) + \varphi(\sigma,[f,\rho]) - \chi(f,\{\rho,\sigma\})\big) \\ &= 4\big(\rho'(f\sigma'-\tfrac{1}{2}f'\sigma)' + \sigma'(f\rho'-\tfrac{1}{2}f'\rho)' + \tfrac{1}{2}R(z)\rho(f\sigma'-\tfrac{1}{2}f'\sigma) + \tfrac{1}{2}R(z)\sigma(f\rho'-\tfrac{1}{2}f'\rho)\big) \\ &\qquad -\tfrac{1}{2}(2f'(\rho\sigma)'' - 2f''(\rho\sigma)') + 2R(z)(f'(\rho\sigma) - f(\rho\sigma)') \\ &= 4\big((f(\rho'\sigma')'-\tfrac{1}{2}f''(\rho\sigma)') + 2R(z)(f(\rho\sigma)'-f'(\rho\sigma)\big) - \big((f'(\rho\sigma)')' - 2f''(\rho\sigma)'\big) + 2R(z)(f'(\rho\sigma)-f(\rho\sigma)') \\ &= 0\end{aligned}$$

(up to exact terms). Δ

When X is the Riemann sphere, we may take $\mathcal{R} = 0$ and it is strightforward to confirm that we get the standard anomalous terms when we plug in basis elements (as in 3.1) : for example

$$\tilde{\varphi}(G_r,G_s) = -\frac{1}{6\pi i}\oint(r+\frac{1}{2})(s+\frac{1}{2})z^{r+s-1}dz = \frac{1}{3}(r^2-\frac{1}{4})\delta_{r,-s}$$

because the residue picks out the coefficient of z^{-1} and $(r+\frac{1}{2})(-r+\frac{1}{2}) = -(r^2-\frac{1}{4})$.

Remarks : (i) the two calculations given actually fix the relative coefficients of the amended cocycle terms (with no arbitrariness if we insist on obtaining the standard extension of $\mathcal{SV}$ given at the begining of this section). (ii) In [6], the calculation of the central extension of (what we call) the BMRR-algebra fails to take into account the fact that we need to amend the naive cocycle φ. In fact, as we have pointed out, it is not consistent to amend χ without also amending φ. The situation is saved due to the remarkable fact that both inhomogeneous transformation rules for the naive 'cocycles' can be rendered homogeneous by adding a multiple of the Schwarzian connection term. (iii) Of course, uniformisation allows to choose a coordinate system in which all transition functions are Mobius. With respect to such a coordinate system, the 'cocycles' are coordinate independent (just as for the Riemann sphere). However, in genus $g \geq 2$, there is no *natural* uniformisation. In fact the space of Schwarzian connections on X is an affine space modelled on the vectorspace of quadratic differentials on X : these parametrise the space of uniformisations. See the appendix of [13] for a similar argument.

1 Baranov A., Manin Y., Frolov I. & Schwarz A., ≪A superanalog of the Selberg trace formula and multiloop contributions for fermionic strings≫, Comm.Math.Phys., **111** (1988), 373.

2 Batchelor M. ≪Graded manifolds and supermanifolds≫, in *Mathematical aspects of Superspace* (eds. H-J.Seifert, C.Clarke & A.Rosenblum), Reidel Publ., 1985.

3 Batchelor M. & Bryant P. ≪Graded Riemann Surfaces≫, Comm.Math.Phys., **114** (1988), 243.

4 Bryant P. ≪Deformations of super Riemann surfaces≫ , Cambridge University preprint, August 1988.

5 Bonora L., Bresola M., Cotta-Ramusino P. & Martellini M. , ≪Virasoro type algebras and BRST operators on Riemann surfaces≫, Phys.Lett., **205B** (1988), 53.

6 Bonora L., Martellini M., Rinaldi M. & Russo J., ≪Neveu-Schwarz and Ramond-type superalgebras≫, Trieste preprint, April 1988.

7 Cohn J., ≪Modular geometry of superconformal field theory≫, Nucl. Phys., **B306** (1988), 239.

8 Crane L. & Rabin J., ≪Super Riemann Surfaces : Uniformisation and Teichmuller theory≫, Comm.Math.Phys., **113** (1988), 601.

9 Green M., Schwarz J. & Witten E. ≪Superstring Theory I≫, Cambridge University Press, 1987.

10 Krichever I. & Novikov S., Funk.Anal. i Pril., **21(2)** (1987), 46 ; **21(4)** (1987), 47.

11 Rosly A., Schwarz A. & Voronov A., ≪Geometry of superconformal manifolds≫, Comm.Math.Phys., **119** (1988), 129.

12 Rothstein M. & LeBrun C., ≪Moduli of super Riemann surfaces≫, Comm.Math.Phys., **117** (1988), 159.

13 Witten E. ≪Quantum field theory, Grassmannians and algebraic curves≫, Comm.Math.Phys., **113** (1988), 529.

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