Thomas Friedrich

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1. Introduction.

We consider a Riemannian spin manifold (M^n,g) of dimension $n \geq 3$ and denote by S the spinor bundle. The kernel of the Clifford multiplication $T \otimes S \longrightarrow S$ is a subbundle of $T \otimes S$ and there exists a projection of $T \otimes S$ onto this bundle given by the formula

$$p(X \otimes \psi) = X \otimes \psi + \frac{1}{n} \sum_{\alpha=1}^{n} e_\alpha \otimes e_\alpha \cdot X \cdot \psi \quad ,$$

where $X \cdot \psi$ denotes the Clifford multiplication of the vector X by ψ . The twistor operator $\mathcal{D}$ is defined as the composition of the covariant derivative ∇ and the projection p

$$\mathcal{D} = p \circ \nabla : \Gamma(S) \xrightarrow{\nabla} \Gamma(T \otimes S) \xrightarrow{p} \Gamma(T \otimes S)$$

(see [1]). Let D by the Dirac operator acting on sections of the bundle S. Then we have the following formula for the operator $\mathcal{D}$

$$\mathcal{D}\psi = \sum_{\alpha=1}^{n} e_\alpha \otimes (\nabla_{e_\alpha}\psi + \frac{1}{n} e_\alpha \cdot D\psi).$$

The kernel of the twistor operator is given by the equation

$$\nabla_X \psi + \frac{1}{n} X \cdot D\psi = 0 \qquad (1.1.)$$

for any vector $X \in T$. A more symmetric and equivalent form of this equation is

$$X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi = \frac{2}{n} g(X,Y) D\psi .$$

$\mathcal{D}$ is a conformally invariant operator. In particular, if $\bar{g} = \lambda g$ is a conformal change of the metric and $\bar{\ }: S \rightarrow \bar{S}$

denotes the natural isomorphism of the spin bundles, then ψ belongs to the kernel of $\mathcal{D}$ if $\lambda^{\frac{1}{4}}\bar{\psi}$ belongs to the Kernel of $\bar{\mathcal{D}}$ (see [2], [15]). On the other hand, the equation for Killing spinors is given by

$$\nabla_X\psi + \frac{a}{n}X\cdot\psi = 0 \qquad (1.2.)$$

where $a \neq 0$ is a complex number. It is well known (see [7]) that if a Riemannian manifold has a non-trivial Killing spinor, then it must be an Einstein space with scalar curvature $R = \frac{4(n-1)}{n}a^2$. If a is a real (imaginary) number, we call ψ a real (imaginary) Killing spinor. Any Killing spinor is a twistor spinor, i.e. it belongs to the kernel of the twistor operator. In small dimensions we know many spaces with real Killing spinors (see [6],[7],[8],[9],[10],[11],[12],[13]), and there is a classification of complete Riemannian manifolds with imaginary Killing spinors (see [3],[4],[5]).

On the space $\mathrm{Ker}(\mathcal{D})$ of all twistor spinors we have an invariant of order two, namely

$$C_\psi := \mathrm{Re}\langle D\psi,\psi\rangle$$

(see [14]). In this paper we observe that

$$Q_\psi := |\psi|^2\,|D\psi|^2 - C_\psi^2 - \sum_{\alpha=1}^{n}(\mathrm{Re}\langle D\psi,\, e_\alpha\cdot\psi\rangle)^2 \geq 0$$

is an invariant of order four on $\mathrm{Ker}(\mathcal{D})$, too. Using the first integral on $\mathrm{Ker}(\mathcal{D})$ we show in particular that a Riemannian manifold (M^n,g) with a nowhere vanishing twistor spinor ψ is conformally equivalent to a space $(M^n,\bar{g})$ with non-negative scalar curvature

$$\bar{R} = \frac{4(n-1)}{n}(C_\psi^2 + Q_\psi).$$

Moreover, we study the set $N_\psi = \{m \in M^n : \psi(m) = 0\}$ of all zeros of a twistor. It turns out that N_ψ is a discrete subset of M^n. Finally we investigate the question under which conditions a twistor spinor can be conformally deformed into a Killing spinor. For example, $\psi \in \mathrm{Ker}(\mathcal{D})$ can be conformally deformed into a real Killing spinor if and only if $Q_\psi = 0$ and $C_\psi \neq 0$. Similar characterizations we obtain in the imaginary case, too.

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2. The first integral Q_ψ on $\mathrm{Ker}(\mathcal{D})$.

First we collect some formulas that are valid for any twistor spinor $\psi \in \mathrm{Ker}(\mathcal{D})$. A general reference is the paper [14]. Any twistor spinor satisfies

$$D^2\psi = \frac{nR}{4(n-1)}\psi \qquad (2.1.)$$

and

$$\nabla_X(D\psi) = \frac{n}{2(n-2)}\left(\frac{R}{2(n-1)}X - \mathrm{Ric}(X)\right)\cdot\psi \quad , \qquad (2.2.)$$

where $\mathrm{Ric}: T \longrightarrow T$ is the Ricci tensor of the space. Moreover, if $u = |\psi|^2$ denotes the square of the length of ψ, we have

$$\frac{n}{2}\Delta u = \frac{nR}{4(n-1)}u - \langle D\psi, D\psi\rangle. \qquad (2.3.)$$

We denote by S the (1,1)-tensor

$$S(X) := \frac{1}{n-2}\left(\frac{R}{2(n-1)}X - \mathrm{Ric}(X)\right)$$

and we consider the vector bundle $E = S \oplus S$ as well as the connection ∇^E in E defined by the formula

$$\nabla^E_X := \nabla_X + \begin{pmatrix} 0 & , & \frac{1}{n}X \\ -\frac{n}{2}S(X) & , & 0 \end{pmatrix}$$

The twistor equation (1.1.) and formula (2.2.) show that

$$\nabla^E_X \begin{pmatrix} \psi \\ D\psi \end{pmatrix} = 0$$

holds for any solution of the twistor equation.
Conversely, if

$$\nabla^E_X \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = 0,$$

then $\varphi = D\psi$ and ψ is a twistor spinor.
Consequently, the twistor spinors $\psi \in \mathrm{Ker}(\mathcal{D})$ correspond to the

∇^E-parallel sections of the bundle E and we obtain in particular

Proposition 1: Let (M^n,g) be a connected Riemannian manifold of dimension $n \geq 3$. The kernel of the twistor operator $\mathcal{D}$ is a finite-dimensional space,

$$\dim_{\mathbb{C}} \mathrm{Ker}(\mathcal{D}) \leq 2^{\left[\frac{n}{2}\right]+1} .$$

In particular, a twistor spinor ψ is defined by its values $\psi(m_o)$, $D\psi(m_o)$ at some point.

Remark: We understand the Weyl-tensor W of the Riemannian manifold in the usual way as a 2-form with values in the bundle End(S):

$$W(X,Y)\cdot\psi = \sum_{i,j} g(W(X,Y)e_i,e_j)e_i\cdot e_j\cdot\psi .$$

An easy computation yields now the following formula for the curvature tensor R^E of the connection ∇^E

$$R^E(X,Y)\binom{\psi}{\varphi} = \begin{pmatrix} \frac{1}{4}\, W(X,Y)\cdot\psi \\ \frac{1}{4}\, W(X,Y)\cdot\varphi + \frac{n}{2}((\nabla_Y S)(X)-(\nabla_X S)(Y))\cdot\psi \end{pmatrix} .$$

Example 1: We consider a 3-dimensional Riemannian manifold (M^3,g). The Weyl tensor vanishes, $W \equiv 0$, and consequently we obtain the integrability condition

$$(\nabla_X S)(Y)-(\nabla_Y S)(X) = 0,$$

i.e. the space is locally conformally flat (see [16]). Therefore, if a 3-dimensional Riemannian manifold (M^3,g) admits a non-trivial solution of the twistor equation, then (M^3,g) is locally conformally flat.

Example 2: Denote by Δ_n the usual Spin(n)-module and consider a twistor spinor $\psi: R^n \longrightarrow \Delta_n$ on the flat Euclidean space. According to equation (2.2.) we have $\nabla(D\psi) = 0$, i.e. $D\psi = \psi_1$ is constant. Now we integrate the twistor equation

$$0 = \nabla_T\psi + \frac{1}{n}\, T\cdot D\psi = \nabla_T\psi + \frac{1}{n}\, T\cdot\psi_1$$

along the line $\{sx: 0 \leq s \leq 1\}$ and obtain

$$\psi(x) - \psi(0) = -\frac{1}{n}\, x \cdot \psi_1 .$$

Consequently, the kernel of the twistor equation is given by the spinors $\psi : R^n \longrightarrow \Delta_n$

$$\psi(x) = \psi_0 - \frac{1}{n}\, x \cdot \psi_1, \qquad x \in R^n,$$

with $\psi_0, \psi_1 \in \Delta_n$. In particular we have

$$\dim_{\mathbb{C}} \operatorname{Ker}(\mathcal{D}) = 2^{\left[\frac{n}{2}\right]+1} .$$

Proposition 2: Let (M^n, g) be a connected Riemannian manifold and $\psi \not\equiv 0$ a twistor spinor. Then $N_\psi = \{ m \in M^n : \psi(m) = 0 \}$ is a discrete subset of M^n.

Proof: Suppose $\psi(m) = 0$. Using formula (2.2.) we have

$$\nabla(D\psi)(m) = 0 .$$

With respect to

$$(YXu)(m) = 2(Y(\nabla_X \psi, \psi))(m) = -\frac{2}{n}(Y(X \cdot D\psi, \psi))(m) =$$
$$= \frac{2}{n^2}(X \cdot D\psi, Y \cdot D\psi)(m) = \frac{2}{n^2} g(X,Y) |D\psi(m)|^2$$

we see that the Hessian of the function $u = |\psi|^2$ at the point $m \in M^n$ is given by

$$\mathrm{Hess}_m\, u(X,Y) = \frac{2}{n^2} g(X,Y) |D\psi(m)|^2 .$$

In case $D\psi(m) \neq 0$, m is a non-degenerate critical point of u and consequently an isolated zero point of ψ . In case $D\psi(m) = 0$, we obtain $\psi \equiv 0$ by proposition 1.

We consider now a geodesic $\gamma(t)$ in M^n and a twistor spinor ψ . Denote by $u(t)$, $v(t)$ the functions $u(\gamma(t))$, $|D\psi|^2(\gamma(t))$. Moreover, we introduce the functions

$$f_1(t) = g(S(\dot\gamma(t)), \dot\gamma(t))$$
$$f_2(t) = \frac{n^2}{2} |S(\dot\gamma(t))|^2 .$$

Using the twistor equation as well as formula (2.2.) we obtain

$$\left.\begin{aligned} \frac{d^2u(t)}{dt^2} &= f_1(t)u(t) + \frac{2}{n^2}\, v(t) \\ \frac{d^2v(t)}{dt^2} &= f_2(t)u(t) + \frac{n^2}{2}\, \frac{df_1(t)}{dt}\, \frac{du(t)}{dt} + f_1(t)v(t) \end{aligned}\right\} \qquad (2.4.)$$

Proposition 3: Let $\psi \not\equiv 0$ be a twistor spinor and denote by $\gamma : [0,T] \to M^n$ a geodesic joining of two zero points of ψ. Then

a.) $Ric(\dot\gamma)$ is parallel to $\dot\gamma$.

b.) grad u is parallel to $\dot\gamma$.

c.) $\frac{dv}{dt} = \frac{n^2}{2}\, g(S(\dot\gamma),\dot\gamma)\, \frac{du}{dt}$.

d.) $u \cdot v = \frac{n^2}{4} \left(\frac{du}{dt} \right)^2$.

Proof: Using the notation introduced before we have

$$u(0) = \frac{du}{dt}(0) = \frac{dv}{dt}(0) = 0, \qquad v(0) > 0$$

$$u(T) = \frac{du}{dt}(T) = \frac{dv}{dt}(T) = 0, \qquad v(T) > 0.$$

Since $u(t)$ and $v(t)$ satisfy the equations (2.4.), we obtain

$$\frac{d}{dt}\left(\frac{dv}{dt} - \frac{n^2}{2} f_1 \frac{du}{dt}\right) = \left(f_2 - \frac{n^2}{2} f_1^2\right) u.$$

If $f_2 - \frac{n^2}{2} f_1^2 \not\equiv 0$ on the interval $[0,T]$, we have

$$0 = \frac{dv}{dt}(T) - \frac{n^2}{2} f_1(T) \frac{du}{dt}(T) = \int_0^T \left(f_2 - \frac{n^2}{2} f_1^2\right) > 0$$

because $f_2 - \frac{n^2}{2} f_1^2 = \frac{n^2}{2}\left(|S(\dot\gamma)|^2 - g(S(\dot\gamma),\dot\gamma)^2\right) \geq 0$, a contradiction. In case $f_2 - \frac{n^2}{2} f_1^2 \equiv 0$, $Ric(\dot\gamma)$ is parallel to $\dot\gamma$ and $\frac{dv}{dt} = \frac{n^2}{2} g(S(\dot\gamma),\dot\gamma) \frac{du}{dt}$.

Moreover, we calculate

$$\frac{d}{dt}\left(u \cdot v - \frac{n^2}{4}\left(\frac{du}{dt}\right)^2\right) = \frac{du}{dt} v + u \frac{dv}{dt} - \frac{n^2}{2} \frac{du}{dt} \frac{d^2u}{dt^2} =$$

$$= \frac{du}{dt} v + \frac{n^2}{2} f_1\, u \frac{du}{dt} - \frac{n^2}{2} \frac{du}{dt}\left(f_1 u + \frac{2}{n^2} v\right) = 0,$$

i.e. $uv = \frac{n^2}{4}\left(\frac{du}{dt}\right)^2$. Since ψ is a twistor spinor vanishing at some point, we have

$$uD\psi = \frac{n}{2}\,\mathrm{grad}\,u \cdot \psi \quad .$$

This implies $u \cdot v = \frac{n^2}{4}|\mathrm{grad}\,u|^2$ and consequently

$$|\mathrm{grad}\,u|^2 = \left(\frac{du}{dt}\right)^2,$$

i.e. the gradient of u is parallel to $\dot{\gamma}$.

Proposition 4: Let (M^n,g) be a complete connected Riemannian manifold and suppose that the (1,1)-tensor $S := \frac{1}{n-2}\left(\frac{R}{2(n-1)} - \mathrm{Ric}\right)$ is non-negative. Then any twistor spinor $\psi \not\equiv 0$ vanishes at most at one point.

Proof: Suppose $u(p_1) = 0 = u(p_2)$, $p_1 \neq p_2$ and consider a geodesic $\gamma : [0,T] \to M^n$ from p_1 to p_2. Then

$$\frac{d^2}{dt^2}u(t) = f_1(t)u(t) + \frac{2}{n^2}v(t) \geq 0$$

since S is non-negative. With respect to $u(0) = u(T) = 0$ and $\frac{du}{dt}(0) = \frac{d}{dt}u(T) = 0$ we conclude $u(T) \equiv 0$ on $[0,T]$, i.e. ψ vanishes on the curve $\gamma(t)$, a contradiction to proposition 2.

Example 2: The condition $S \geq 0$ is satisfied in particular if (M^n,g) is an Einstein space with scalar curvature $R \leq 0$. On the Euclidean space R^n and on the hyperbolic space H^n there exist twistor spinors vanishing at some point (see example 2).

We denote by (ψ,φ) the real part $\mathrm{Re}\langle\psi,\varphi\rangle$ of the Hermitian product of two spinors. Given an arbitrary spinor $\psi \in \Gamma(S)$ we define the function Q_ψ by the formula

$$Q_\psi = |\psi|^2|D\psi|^2 - (D\psi,\psi)^2 - \sum_{\alpha=1}^{n}(D\psi, e_\alpha\cdot\psi)^2.$$

Denote by V_ψ the real subspace of S given by

$$V_\psi = \{X\cdot\psi \,:\, X \in T\}.$$

Then we have

$$Q_\psi = u\cdot \mathrm{dist}^2(D\psi, \mathrm{Lin}_R(\psi, V_\psi)).$$

Proposition 5: If $\psi \in \mathrm{Ker}(\mathcal{D})$ is a twistor spinor, then Q_ψ is constant.

Proof: Since $(D\psi, \psi)$ is constant for $\psi \in \mathrm{Ker}(\mathcal{D})$ (see [14]), we have

$$\nabla_X(Q_\psi) = 2(\nabla_X\psi, \psi)|D\psi|^2 + 2u\cdot(\nabla_X(D\psi), D\psi)$$
$$- 2\sum_{\alpha=1}^{n}(D\psi, e_\alpha\cdot\psi)(\nabla_X(D\psi), e_\alpha\cdot\psi)$$
$$- 2\sum_{\alpha=1}^{n}(D\psi, e_\alpha\cdot\psi)(D\psi, e_\alpha\cdot \nabla_X\psi).$$

Using the twistor equation (1.1.) and formula (2.2.) we obtain, with respect to

$$\sum_{\alpha=1}^{n}(e_\alpha\cdot\psi, D\psi)(e_\alpha\cdot X\cdot D\psi, D\psi) = -(X\cdot\psi, D\psi)|D\psi|^2$$
$$\sum_{\alpha=1}^{n}(e_\alpha\cdot\psi, D\psi)(e_\alpha\cdot\psi, Z\cdot\psi) = (Z\cdot\psi, D\psi)\,|\psi|^2,$$

that $\nabla_X(Q_\psi) = 0$ immediately.

Remark: For any twistor spinor ψ let us introduce the vector field

$$T^\psi = 2\sum_{\alpha=1}^{n}(\psi, e_\alpha\cdot D\psi)e_\alpha .$$

Then we have

$$T^\psi = -n \operatorname{grad} u$$

(see [14]) and an elementary calculation provides the formula

$$|C_\psi\cdot\psi - uD\psi - \tfrac{1}{2}T^\psi\cdot\psi|^2 = u\,Q_\psi . \tag{2.5.}$$

In particular, if ψ is a twistor spinor such that $C_\psi = 0 = Q_\psi$, then

$$uD\psi = \frac{n}{2}\operatorname{grad}(u)\cdot\psi \tag{2.6.}$$

holds.

Proposition 6: Let (M^n, g) be a Riemannian manifold with a twistor spinor ψ such that $C_\psi = 0 = Q_\psi$ and suppose that ψ does not vanish at any point. Then (M^n, g) is conformally equivalent to a Ricci-flat space $(M^n, \bar{g})$ with parallel spinor.

Proof: Consider the metric $\bar{g} = \frac{1}{u^2} g$, $u = |\psi|^2$. Using the identification $\bar{\ }: S \rightarrow \bar{S}$ of the spin bundles we have (see [2])

$$\bar{\nabla}_t\left(\frac{1}{\sqrt{u}}\bar{\psi}\right) = u\,\overline{\nabla_t\left(\frac{1}{\sqrt{u}}\psi\right)} + \frac{1}{2}\,\overline{t\cdot \mathrm{grad}(u)\cdot \frac{1}{\sqrt{u}}\psi} +$$

$$+ \frac{1}{2}\,du(t)\frac{1}{\sqrt{u}}\bar{\psi} = \frac{1}{\sqrt{u}}\left\{u\,\overline{\nabla_t\psi + \frac{1}{2}\,t\cdot \mathrm{grad}(u)\cdot\psi}\right\} .$$

According to $C_\psi = 0 = Q_\psi$ we can apply equation (2.6.) and then, from the twistor equation

$$0 = \nabla_t\psi + \frac{1}{n}\,t\cdot D\psi = \nabla_t\psi + \frac{1}{2u}\,t\cdot \mathrm{grad}(u)\cdot\psi ,$$

it results that $\bar{\nabla}_t\left(\frac{1}{\sqrt{u}}\bar{\psi}\right) = 0$, i.e. $\frac{1}{\sqrt{u}}\bar{\psi}$ is a parallel spinor with respect to the metric $\bar{g}$.

Corollary 1: Let (M^n,g) be a Riemannian manifold that is not conformally equivalent to a space $(M^n,\bar{g})$ with parallel spinor. Then a twistor spinor $\psi \in \mathrm{Ker}(\mathcal{D})$ vanishes at some point if and only if $C_\psi = 0 = Q_\psi$.

For any twistor spinor ψ we introduce the function

$$H_\psi = \mathrm{dist}^2(i\psi, V_\psi)$$

defined on the set $\{m \in M^n : \psi(m) \neq 0\}$.

Proposition 7: Let ψ be a twistor spinor satisfying $C_\psi = 0 = Q_\psi$. Then

$$\frac{H_\psi}{u}$$

is constant.

Proof: The derivative of the function $f = \sum_{\alpha=1}^{n}(i\psi, e_\alpha\cdot\psi)^2$ is given by

$$df(X) = -\frac{4}{n}\sum_{\alpha=1}^{n}(i\psi, e_\alpha\cdot\psi)(i\psi, e_\alpha\cdot X\cdot D\psi).$$

Since $C_\psi = 0 = Q_\psi$, we have $uD\psi = \frac{n}{2}\,\mathrm{grad}(u)\cdot\psi$ and consequently

$$df = \frac{2}{u}\,f\,du.$$

Finally we obtain

$$d\left(\frac{H_\psi}{u}\right) = d\left(1-\frac{f}{u^2}\right) = 0.$$

Let $f: M^n \longrightarrow \mathbb{C}$ be a complex valued function on M^n and consider the equation

$$\nabla_X \psi + \frac{f}{n} X \cdot \psi = 0.$$

A. Lichnerowicz (see [14]) proved that if $\psi \not\equiv 0$ and $\mathrm{Re}(f) \not\equiv 0$, then $\mathrm{Re}(f)$ is constant and ψ is a real Killing spinor. We consider now the case $f = ib$.

Proposition 8: If $\nabla_X\psi + \frac{ib}{n} X \cdot \psi = 0$ with a real function $b: M^n \longrightarrow R^1$, then

a.) $u \cdot H_\psi$ is constant

b.) $Q_\psi = b^2 u H_\psi$.

Proof: Suppose $\nabla_X \psi + \frac{ib}{u} X \cdot \psi = 0$. Then $D\psi = ib\psi$ and we obtain $Q_\psi = b^2 u H_\psi$ by definition of Q_ψ . Since $u H_\psi = u^2 - \sum_{\alpha=1}^{n} (i\psi, e_\alpha \cdot \psi)^2$ we calculate

$$\nabla_X(u H_\psi) = 4u(\nabla_X\psi, \psi) - 2\sum_{\alpha=1}^{n}(i\psi, e_\alpha \psi)(i\nabla_X\psi, e_\alpha \cdot \psi)$$

$$-2\sum_{\alpha=1}^{n}(i\psi, e_\alpha \cdot \psi)(i\psi, e_\alpha \cdot \nabla_X\psi) =$$

$$= -\frac{4b}{n} u \cdot (iX \cdot \psi, \psi) - \frac{2b}{n}\sum_{\alpha=1}^{n}(i\psi, e_\alpha \cdot \psi)(X \cdot \psi, e_\alpha \cdot \psi)$$

$$+\frac{2b}{n}\sum_{\alpha=1}^{n}(i\psi, e_\alpha \psi)(i\psi, e_\alpha \cdot X \cdot i\psi) =$$

$$= \frac{4b}{n} u(i\psi, X \cdot \psi) - \frac{2b}{n}(i\psi, X \cdot \psi) - \frac{2b}{n}(i\psi, X \cdot \psi)$$

$$= 0,$$

i.e. $|\psi|^2 H_\psi$ is constant.

Corollary 2: If ψ is a solution of the equation $\nabla_X\psi + \frac{ib}{n} X \cdot \psi = 0$ and $Q_\psi \neq 0$, then b is constant and ψ is an imaginary Killing spinor.

Corollary 3: If ψ is a non-trivial solution of the equation $\nabla_X\psi + \frac{ib}{n} X \cdot \psi = 0$ and $Q_\psi = 0$, then $\frac{1}{\sqrt{u}}\psi$ is a parallel spinor with respect to the metric $\overline{g} = \frac{1}{u^2} g$.

Proposition 9: Let $\psi \in \mathrm{Ker}(\mathcal{D})$ be a twistor spinor and denote by u the square of the length of ψ , $u = |\psi|^2$. Then u is a solution of the following equation

$$\frac{nR}{4(n-1)} u^2 = C_\psi^2 + Q_\psi + \frac{n}{2} u^2 \Delta(\ln u) + \frac{n(n-2)}{4} u^2 |\mathrm{grad}(\ln u)|^2.$$

Proof: We consider the vector field

$$T^\psi = 2 \sum_{\alpha=1}^{n} (\psi, e_\alpha \cdot D\psi) e_\alpha .$$

Then we have

$$Q_\psi = u|D\psi|^2 - C_\psi^2 - \frac{1}{4}|T^\psi|^2$$

and $\quad \mathrm{grad}\, u = -\frac{1}{n} T^\psi$

(see [14]). Consequently we obtain by equation (2.3.)

$$\Delta(\ln u) = \frac{1}{u^2}|\mathrm{grad}\, u|^2 + \frac{1}{u}\Delta(u) =$$

$$= \frac{1}{u^2 n^2}|T^\psi|^2 + \frac{R}{2(n-1)} - \frac{2}{n \cdot u}|D\psi|^2$$

$$|\mathrm{grad}(\ln u)|^2 = \frac{1}{u^2 n^2}|T^\psi|^2.$$

Finally we have

$$\frac{n}{2} u \Delta(\ln u) + \frac{n(n-2)}{4} u |\mathrm{grad}(\ln u)|^2 =$$

$$= \frac{1}{4u}|T^\psi|^2 + \frac{nR}{4(n-1)} u - |D\psi|^2 =$$

$$= \frac{1}{4u}|T^\psi|^2 + \frac{nR}{4(n-1)} u - \frac{C_\psi^2 + Q_\psi}{u} - \frac{1}{4u}|T^\psi|^2 =$$

$$= \frac{nR}{4(n-1)} u - \frac{C_\psi^2 + Q_\psi}{u}$$

and this is the equation we claimed.

Theorem 1: Let (M^n, g) be a Riemannian spin manifold of dimension $n \geq 3$ with a nowhere vanishing twistor spinor ψ . The Riemannian metric

$$\overline{g} = \frac{1}{|\psi|^4} g$$

has constant and non-negative scalar curvature

$$\bar{R} = \frac{4(n-1)}{n}\,(C_{\psi}^{2}+Q_{\psi}).$$

Proof: Denote by $h := u^{-n/2+1}$. Then $h^{\frac{4}{n-2}} = \frac{1}{u^2}$ and the metrics $\bar{g}$ and g are related by

$$\bar{g} = h^{\frac{4}{n-2}}\,g.$$

Then the scalar curvatures are related by the formula

$$\bar{R}\,h^{\frac{4}{n-2}} = \frac{4(n-1)}{n-2}\,\frac{\Delta h}{h} + R.$$

The result follows now by a direct calculation using the formula of proposition 9.

3. The conformal deformation of twistor spinors into Killing spinors

Consider a Riemannian spin manifold (M^n,g) and a twistor spinor $\psi \in \mathrm{Ker}(\mathcal{D})$. We say that ψ is conformally equivalent to a Killing spinor if there exists a conformal change of the metric $\bar{g} = \lambda g$ such that $\lambda^{\frac{1}{4}}\bar{\psi}$ is a Killing spinor with respect to the metric $\bar{g}$. We introduce the function $f = \frac{1}{2}\lambda^{-\frac{1}{2}}$. Then the equation

$$\bar{\nabla}_X(\lambda^{\frac{1}{4}}\bar{\psi}) + \frac{a}{n}\,X\cdot(\lambda^{\frac{1}{4}}\bar{\psi}) = 0$$

becomes equivalent to

$$a\psi - 2fD\psi + n\,\mathrm{grad}(f)\cdot\psi = 0. \qquad (3.1.)$$

Indeed, with respect to $\nabla_X\psi + \frac{1}{n}X\cdot D\psi = 0$ we use only the well-known formulas describing the change of the covariant derivative (see [2]) in order to derive (3.1.). Consequently, ψ is conformally equivalent to a Killing spinor iff (3.1.) has a positive solution f for some constant $0 \neq a \in \mathbb{C}$.

<u>Theorem 2:</u> Let (M^n,g) be a Riemannian spin manifold and $0 \neq \psi \in \mathrm{Ker}(\mathcal{D})$ a twistor spinor. Then ψ is conformally

equivalent to a real Killing spinor if and only if $C_\psi \neq 0$ and $Q_\psi = 0$. In this situation there exists - up to a constant - precisely one metric

$$\bar{g} = \frac{1}{|\psi|^4}\, g$$

with respect to which ψ becomes a Killing spinor.

Proof: Let $0 \neq a$ be a real number and suppose f is a solution of (3.1.). Then

$$a\,|\psi|^2 - 2f\, C_\psi = 0$$

and, consequently, $C_\psi \neq 0$. Moreover, in this case we have $\mathrm{dist}^2(D\psi, \mathrm{Lin}_R(\psi, V_\psi)) = 0$, i.e. $Q_\psi = 0$. Conversely, suppose $C_\psi \neq 0$ and $Q_\psi = 0$. Again we consider the vector field T^ψ defined by

$$T^\psi = 2 \sum_{\alpha=1}^{n} (\psi, e_\alpha \cdot D\psi) e_\alpha .$$

With respect to (2.5.) we have

$$C_\psi \psi - u\, D\psi - \frac{1}{2} T^\psi \cdot \psi = 0,$$

and since ψ is a twistor spinor, we know

$$T^\psi = - n\, \mathrm{grad}(u).$$

Consequently

$$C_\psi \psi - u\, D\psi + \frac{n}{2} \mathrm{grad}(u) \cdot \psi = 0,$$

i.e. $f = \frac{u}{2}$ is a solution of (3.1.). Finally we remark that any solution f^* of (3.1.) is proportional to u in case $a \in R^1$ since f^* must satisfy the relation $a \cdot u - 2f^* C_\psi = 0$.

For an arbitrary spinor field ψ we introduce the 1-form

$$\eta_\psi(X) = \frac{1}{i} \langle X \cdot \psi, \psi \rangle = \mathrm{Im} \langle X \cdot \psi, \psi \rangle .$$

Theorem 3: Let (M^n, g) be a Riemannian spin manifold and $0 \neq \psi$ a twistor spinor with $Q_\psi = 0$. Then ψ is conformally equivalent to an imaginary Killing spinor if and only if

a.) $C_\psi = 0$, $H_\psi \equiv 0$

b.) $\pm \frac{\eta_\psi}{|\psi|^4}$ is the differential of a positive function.

In this situation, for any function $k > 0$ with $\pm \frac{\eta_\psi}{|\psi|^4} = dk$, the twistor spinor becomes a Killing spinor in the metric

$$\bar{g} = \frac{1}{|\psi|^4 k^2}\, g.$$

Proof: By equation (2.5.) $Q_\psi = 0$ implies

$$u\, D\psi = C_\psi \psi + \frac{n}{2}\, \mathrm{grad}(u)\cdot\psi \quad .$$

Suppose now that

$$\pm i\psi - 2f\, D\psi + n\, \mathrm{grad}(f)\cdot\psi = 0$$

has a solution $f > 0$. Then $-2f(D\psi, \psi) = 0$, i.e. $C_\psi = 0$ and $u\, D\psi = \frac{n}{2}\, \mathrm{grad}(u)\cdot\psi$. This implies $D\psi \in V_\psi$ and, finally, $i\psi \in V_\psi$. Thus we have the necessary condition $H_\psi \equiv 0$. Furthermore, we calculate η_ψ and obtain

$$\eta_\psi(X) = -\langle i\, X\cdot\psi, \psi\rangle = \pm\langle 2fX\cdot D\psi - nX\cdot\mathrm{grad}(f)\cdot\psi, \psi\rangle$$

$$= \pm\left\{-n2f(\nabla_X \psi, \psi) + n(\mathrm{grad}(f)\cdot\psi, X\cdot\psi)\right\} =$$

$$= \pm\, n\left\{-f\, du(X) + df(X)\, u\right\}$$

$$\frac{\eta_\psi}{u^2} = \pm\, n\, d(f\cdot\tfrac{1}{u}),$$

i.e. $\pm \frac{\eta_\psi}{u^2}$ is the differential of a positive function.

Conversely, suppose $Q_\psi = 0$, $C_\psi = 0$, $H_\psi \equiv 0$ and $\pm \frac{\eta_\psi}{u^2} = dk$. Then $f = uk$ is a solution of equation (3.1.). Indeed, we have $u\, D\psi = \frac{n}{2}\, \mathrm{grad}(u)\cdot\psi$ (since $C_\psi = 0 = Q_\psi$) and, consequently, we obtain with $f = uk$

$$-2f\, D\psi + n\, \mathrm{grad}(f)\cdot\psi = n\cdot u\, \mathrm{grad}(k)\cdot\psi =$$

$$= n\cdot u\cdot \sum_{\alpha=1}^{n} dk(e_\alpha) e_\alpha\cdot\psi =$$

$$= \pm\, n\cdot\frac{1}{u} \sum_{\alpha=1}^{n} (ie_\alpha\psi, \psi) e_\alpha\psi =$$

$$= \pm\, ni\psi .$$

The latter equation follows from $H_\psi \equiv 0$, i.e. $i\psi \in V_\psi$.

Theorem 4: Let (M^n,g) be a Riemannian spin manifold and $0 \neq \psi$ a twistor spinor with $Q_\psi \neq 0$. Then ψ is conformally equivalent to an imaginary Killing spinor if and only if

$$C_\psi = 0 \text{ and } \mathrm{dist}^2(D\psi, \mathrm{Lin}_R(i\psi, V_\psi)) \equiv 0.$$

In this case there exists exactly one positive function k with $\pm \frac{\eta_\psi}{u^2} = dk$ such that ψ becomes a Killing spinor with respect to the metric

$$\bar{g} = \frac{1}{|\psi|^4 k^2}\, g.$$

Proof: Suppose $Q_\psi \neq 0$ and that equation (3.1.) has a positive solution f for some imaginary number a. Then

$$\mathrm{dist}^2(D\psi, \mathrm{Lin}_R(i\psi, V_\psi)) \equiv 0$$

and $C_\psi = 0$, and we obtain the necessary conditions mentioned above. On the other hand, if $Q_\psi \neq 0$, $C_\psi = 0$ and $\mathrm{dist}(D\psi, \mathrm{Lin}_R(i\psi, V_\psi)) \equiv 0$, then there exist a function A and a vector field ξ such that $D\psi = Ai\psi + \xi\psi$. Using the twistor equation $\nabla_X\psi = -\frac{1}{n} X\cdot D\psi$ we obtain

$$\nabla_X(D\psi) = \left\{dA(X) + \frac{2}{n} A \langle \xi, X\rangle\right\} i\psi +$$
$$+\left\{\frac{A^2}{n} X + \nabla_X\xi + \frac{2}{n}\langle \xi, X\rangle \xi - \frac{1}{n} |\xi|^2 X\right\}\psi .$$

With respect to $Q_\psi \neq 0$ we know that $i\psi$ is linearly independent (over R^1) of V_ψ. The latter formula as well as formula (2.2.) yield now

$$dA(X) = -\frac{2}{n} A \langle \xi, X\rangle .$$

Consider the function $f := \frac{1}{2}\frac{1}{|A|}$ (since $Q_\psi \neq 0$, A cannot vanish). Then we have

$$\mathrm{grad}(f) = \frac{1}{n} \frac{1}{|A|}\xi = \frac{1}{n} 2f\xi$$

and, consequently,

$$D\psi = Ai\psi + \xi\cdot\psi = \mathrm{sgn}(A) \frac{1}{2f} i\psi + \frac{n}{2f} \mathrm{grad}(f)\cdot\psi ,$$

i.e. f is a solution of equation (3.1.) and the corresponding conformally equivalent metric is given by

$$\bar{g} = A^2 g.$$

Furthermore, $D\psi = Ai\psi + \xi\cdot\psi$ implies

$$\frac{n}{2}\, du = A\eta_\psi + u\,\xi =$$
$$= A\eta_\psi - \frac{n}{2}\, u\, \frac{dA}{A}$$

and, finally,

$$-\frac{n}{2}\, d\left(\frac{1}{Au}\right) = \frac{\eta_\psi}{u^2}\,.$$

This means that A^2 is given by $A^2 = \frac{1}{u^2 k^2}$ for a unique function $k > 0$ satisfying $dk = \pm \frac{\eta_\psi}{u^2}$.

On a manifold of dimension $n = 3,5$ we have $H_\psi \equiv 0$ for an arbitrary spinor field. Therefore Theorem 3 and Theorem 4 provide the following

Corollary 4: Let (M^n,g) be a Riemannian spin manifold of dimension $n=3,5$ and let $\psi\in \mathrm{Ker}(\mathcal{D})$ be a twistor spinor. Then ψ is conformally equivalent to an imaginary Killing spinor if and only if

a.) $Q_\psi = 0$, $C_\psi = 0$

b.) $\pm \frac{\eta_\psi}{u^2}$ is the differential of a positive function.

REFERENCES

[1] Atiyah M.F., Hitchin N.J., Singer I.M. "Self-duality in four-dimensional Riemannian Geometry", Proc. R. Soc. London A 362 (1978), 425-461.

[2] Baum H. "Spin-Strukturen und Dirac-Operatoren über pseudo-riemannschen Mannigfaltigkeiten", Teubner-Verlag Leipzig 1981.

[3] Baum H. "Variétés riemanniennes admettant des spineurs de Killing imaginaires", C.R. Acad. Sci. Paris Serie I (1989).

[4] Baum H. "Odd-dimensional Riemannian manifolds with imaginary Killing spinors", Ann. Global Anal. Geom. 7 (1989).

[5] Baum H. "Complete Riemannian manifolds with imaginary Killing spinors", Ann. Global Anal. Geom. 7 (1989),

[6] Duff M.J., Nilsson B., Pope C.N., "Kaluza-Klein supergravity", Phys. Rep. 130 (1986), 1-142.

[7] Friedrich, Th. "Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung", Math. Nachr. 97(1980), 117-146.

[8] Friedrich Th., Grunewald R. "On the first eigenvalue of the Dirac operator on 6-dimensional manifolds", Ann. Global Anal. Geom. 3 (1985), 265-273.

[9] Friedrich Th., Kath I. "Einstein manifolds of dimension five with small eigenvalue of the Dirac operator" Journ. Diff. Geom. 29 (1989)

[10] Friedrich Th., Kath I. "Variétés riemanniennes compactes de dimension 7 admettant des spineurs de Killing", C.R. Acad. Sci. Paris Serie I . 307 (1988), 967-969.

[11] Friedrich Th., Kath I. "7-dimensional compact Riemannian manifolds with Killing spinors", to appear.

[12] Grunewald R. "Untersuchungen über Einstein Metriken auf 6-dimensionalen Mannigfaltigkeiten", Dissertation (A) Humboldt-Universität zu Berlin 1986.

[13] Grunewald R. "Six-dimensional compact Riemannian manifolds with Killing spinors" , to appear.

[14] Lichnerowicz A. "spin manifolds, Killing spinors and universality of the Hijazi-inequality", Lett. Math. Phys. 13 (1987), 331-344.

[15] Lichnerowicz A. "Les spineurs-twisteurs sur une variété spinorielle compacte", C.R. Acad. Sci. Paris Serie I 306(1988), 381-385.

[16] Raschewski P.K. "Riemannsche Geometrie und Tensoranalysis", Berlin 1959.

Author's address:

THOMAS FRIEDRICH
SEKTION MATHEMATIK
HUMBOLDT-UNIVERSITÄT
1086 BERLIN, PSF 1297
G.D.R.