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DERIVATIONS ON THE NIJENHUIS-SCHOUTEN BRACKET ALGEBRA

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This is an announcement of results. The proofs will appear elsewhere.

All structures appearing in this paper are of class C^∞. Let M be a connected and paracompact orientable manifold, $\dim M = m$. As usual we denote by TM the tangent bundle of M, and by $\Lambda^i TM$ its i-th exterior power. We set

$$L_i = \Gamma\Lambda^{i+1}TM \quad \text{for } -1 \leq i \leq m-1,$$

where Γ denotes the functor of sections over M. In order to avoid technical complications we set

$$L_i = 0 \quad \text{for } i < -1 \text{ or } i > m-1.$$

Obviously for any $i \in \mathbf{Z}$ L_i is a real vector space. To complete our notation we set

$$L = \sum_{i=-\infty}^{\infty} L_i.$$

If $\alpha \in L_i$ we call α homogenous element and write $|\alpha| = i$. Let us notice that L_{-1} is the vector space of functions on M, and L_0 is the vector space of vector fields on M.

Using a result of Schouten [2], Nijenhuis [1] defined a bilinear mapping

$$[\ ,\ ] \colon L \times L \to L$$

which is now called Nijenhuis-Schouten bracket. This bracket is characterized by the following properties (All elements are homogenous.):

(a) $[L_i, L_j] \subset L_{i+j}$

(b) $[\alpha, \beta] = -(-1)^{|\alpha|\cdot|\beta|}[\beta, \alpha]$

(c) $(-1)^{|\gamma|\cdot|\alpha|}[\alpha,[\beta,\gamma]] + (-1)^{|\alpha|\cdot|\beta|}[\beta,[\gamma,\alpha]] + (-1)^{|\beta|\cdot|\gamma|}[\gamma,[\alpha,\beta]] = 0$

(d) $[\alpha, f] = \iota_{df}\alpha$, where $f \in L_{-1}$ and ι denotes the inner product operator.

(e) $[\alpha, \beta\wedge\gamma] = [\alpha,\beta]\wedge\gamma + (-1)^{|\alpha|\cdot|\beta|}\beta\wedge[\alpha,\gamma]$

The properties (b) and (c) show that L is a graded Lie algebra. Further using (b)-(e) we find easily that for $X \in L_0$, $\alpha \in L$ there is $[X, \alpha] = \mathcal{L}_X \alpha$, where $\mathcal{L}_X$ denotes the Lie derivative. Consequently for X, $Y \in L_0$ $[X, Y]$ is the ordinary Lie bracket.

Let us recall that a derivation of degree $k \in \mathbf{Z}$ on L is a linear mapping $D\colon L \to L$ such that

(1) $DL_i \subset L_{i+k}$

(2) $D[\alpha, \beta] = [D\alpha, \beta] + (-1)^{k.|\alpha|}[\alpha, D\beta]$.

A derivation D is called local if it has the following property: If $\alpha \in L_i$, $U \subset M$ is an open subset and $\alpha|U = 0$, then $D\alpha|U = 0$. We shall denote by Der_k the vector space of all local derivations of degree k on L. The goal of this paper is to describe Der_k for $k \in \mathbf{Z}$.

PROPOSITION 1. *$Der_k = 0$ for $k < -1$.*

For the sake of formulation of the next propositions we shall recall some facts about the forms of higher order. By a k-form on M we shall mean a local skew-symmetric k-linear (over the reals) mapping

$$\omega\colon \underbrace{L_0 \times \ldots \times L_0}_{k\text{-times}} \to L_{-1}.$$

(ω is called local if it has the following property: If $X_1, \ldots, X_k \in L_0$, $U \subset M$ is an open subset, and $X_1|U = 0$, then $\omega(X_1, \ldots, X_k)|U = 0$.) The usual formula

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1})$$

defines the exterior derivative $d\omega$ of ω, which is a $(k+1)$-form (i.e. it is again local). Ordinary k-forms on M we shall call k-forms of order 0.

We shall fix a volume element μ on M (i.e. an everywhere nonzero m-form of order 0). For any $X \in L_0$ there exists a unique function, which we shall denote by $\mathrm{div} X$ such that

$$\mathcal{L}_X \mu = \mathrm{div} X . \mu,$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. The linear mapping $\mathrm{div}\colon X \mapsto \mathrm{div} X$ is a closed 1-form. (We remark that this is not a 1-form of order 0. In fact the order of div is 1.)

Obviously any derivation $D \in \mathrm{Der}_{-1}$ determines a 1-form ω_D on M defined by the formula

$$\omega_D(X) = DX.$$

PROPOSITION 2. *If* $dim M = 1$, *then the mapping* $D \mapsto \omega_D$ *defines an isomorphism between* Der_{-1} *and the vector space of closed 1-forms on* M.

PROPOSITION 3. *If* $dim M > 1$, *then the mapping* $D \mapsto \omega_D$ *defines an isomorphism between* Der_{-1} *and the vector space consisting of all 1-forms*

$$\omega = c.div + \omega'$$

on M, *where* $c \in \mathbf{R}$, *and* ω' *is a closed 1-form of order 0.*

PROPOSITION 4. *Let* $D \in Der_0$. *Then there exist unique* $X_D \in L_0$ *and* $c \in \mathbf{R}$ *such that*

$$D\alpha = \mathcal{L}_{X_D}\alpha + ic\alpha, \quad \alpha \in L_i, \ -1 \leq i \leq m-1,$$

where $\mathcal{L}_{X_D}$ *denotes the Lie derivative with respect to* X_D.

Conversely for any $X \in L_0$ *and* $c \in \mathbf{R}$ *the formula*

$$D\alpha = \mathcal{L}_X\alpha + ic\alpha, \quad \alpha \in L_i, \ -1 \leq i \leq m-1$$

defines a derivation of degree 0 on L.

PROPOSITION 5. *Every derivation* $D \in Der_k$, $k > 0$ *is inner.*

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