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ON GELFAND-ZETLIN MODULES

Yu.A.Drozd, S.A.Ovsienko, V.M.Futorny

1⁰. GELFAND-ZETLIN SUBALGEBRA

Let $\mathfrak{G} = \mathfrak{GL}(n, \mathbb{C})$, e_{ij} be the matrix units. Consider the standard inclusions $\mathfrak{G}_1 \subset \mathfrak{G}_2 \subset \dots \subset \mathfrak{G}_n = \mathfrak{G}$ where $\mathfrak{G}_k = \langle e_{ij} \mid i, j = 1..k \rangle$. Denote U_k the universal enveloping algebra of \mathfrak{G}_k , Z_k the center of U_k , $U = U_n$ and put $\Gamma = \langle Z_k \mid k = 1..n \rangle$, the Gelfand-Zetlin (GZ-) subalgebra of U . One knows [6] that Z_k is the polynomial ring in k variables c_{kj} ($j = 1..k$) where $c_{kj} = \sum e_{t_1 t_2} e_{t_2 t_3} \dots e_{t_{j-1} t_j}$ for all sequences (t_1, t_2, \dots, t_j) with

$t_s \in \{1..k\}$; moreover, Γ is the polynomial ring in $n(n+1)/2$ variables c_{kj} ($k = 1..n, j = 1..k$).

Proposition 1. Γ is a maximal commutative subalgebra in U .

Proof. Following the known proof of the Harish-Chandra theorem ([2], 2.5.7), one can show that $u \in \Gamma$ if and only if $\rho(u) \in \rho(\Gamma)$ for any finite-dimensional representation ρ of \mathfrak{G} . But the Gelfand-Zetlin formulae [6] imply that $\rho(\Gamma)$ coincides with the set of all diagonal matrices. Hence if u commutes with all $\alpha \in \Gamma$ then $\rho(u) \in \rho(\Gamma)$ which accomplishes the proof.

A \mathfrak{G} -module V will be called a GZ-module provided $V = \bigoplus_{\chi} V(\chi)$ where χ runs through the space Γ^{\wedge} of characters of Γ and $V(\chi) = \{v \in V \mid \forall \alpha \in \Gamma \exists m (\alpha - \chi(\alpha))^m v = 0\}$. Denote \mathfrak{G} the category of all GZ-modules (it contains all finite-dimensional \mathfrak{G} -modules [6]). For $V \in \mathfrak{G}$ put $\text{supp} V = \{\chi \in \Gamma^{\wedge} \mid V(\chi) \neq 0\}$, $V_{\chi} = \{v \in V \mid \forall \alpha \in \Gamma \alpha v = \chi(\alpha)v\}$.

Consider another polynomial ring L in $n(n+1)/2$ variables

l_{kt} ($k=1..n, t=1..k$) and the homomorphism $\iota: \Gamma \rightarrow L$ which maps c_{kj} to $\sum_t l_{kt}^j \prod_{p=t} (1 - (l_{kt} - l_{kp})^{-1})$. The symmetric group S_k acts on L permuting l_{kt} ($t=1..k$). Thus the direct product $S = \prod_k S_k$ acts on L .

Proposition 2. ι is an inclusion and its image coincides with the ring of invariants L^S .

Proof. It is easy to check that $\iota(c_{kj})$ is a symmetric polynomial in l_{kt} ($t=1..k$) of the form $\sum_t l_{kt}^{j+f}$ with $\deg f < j$. As the power sums are algebraically independent and generate L^S , it proves the statement.

From now on we identify Γ with its image in L . It is convenient to choose new generators $\sigma_{kj} = \sigma_j(l_{k1}, \dots, l_{kk})$ where σ_j are the elementary symmetric functions. The inclusion ι induces the surjection $\pi: L^{\wedge} \rightarrow \Gamma^{\wedge}$ which identifies Γ^{\wedge} with $S \setminus L^{\wedge}$. For any $\lambda \in L^{\wedge}$ we shall write $V(\lambda)$ instead of $V(\pi(\lambda))$ and put $\lambda_{kt} = \lambda(l_{kt})$, $\sigma_{kj}(\lambda) = \sigma_j(\lambda_{k1}, \dots, \lambda_{kk}) = \lambda(\sigma_{kj})$.

Denote $X_k^+ = e_{k, k+1}$, $X_k^- = e_{k+1, k}$. Then the set $\{X_k^{\pm} \mid k=1..n-1\}$ generates U . Of course, X_k^{\pm} commutes with elements of Z_t if $t \neq k$. Define the polynomials $f_{kjm}^{\pm}(\sigma_{k1}, \dots, \sigma_{kj})$ by the formula:

$F_{kj}^{\pm}(T) = \prod_t (T - \sigma_{kj}(\lambda \pm \delta_{kt})) = T^k + \sum_m f_{kjm}^{\pm}(\sigma_{k1}(\lambda), \dots, \sigma_{kj}(\lambda)) T^m$
 where δ_{kt} ($t=1..k$) are given by the rule: $\delta_{kt}(l_{qp}) = 1$ if $q=k, p=t$ and 0 otherwise.

Proposition 3. (i) $X_k^{\pm} V(\lambda) \subset \sum_t V(\lambda \pm \delta_{kt})$ for any $\lambda \in L^{\wedge}$.

(ii) $\sigma_{kj} X_k^{\pm} + \sum_m \sigma_{kj}^m X_k^{\pm} f_{kjm}^{\pm}(\sigma_{k1}, \dots, \sigma_{kj}) = 0$.

(iii) If V is a simple GZ-module, $\lambda \in L^{\wedge}$ and $\lambda_{kt} - \lambda_{kp} \neq 0$ for all indices $t \neq p$, then $V(\lambda) = V_{\lambda}$.

Proof. If V is finite-dimensional, (i) is known [6]. Moreover, in this case $V(\lambda) = V_{\lambda}$. Thus $F_{kj}^{\pm}(\sigma_{kj}) X_k^{\pm} v = 0$ for any $v \in V$. This means that the left part of the equality (ii) annihilates V . By the Harish-Chandra theorem it proves (ii). (i) and (iii) are easy consequences of (ii).

Corollary 1. If $V(\chi) \neq 0$ for some $\chi \in \Gamma^{\wedge}$ and the module V is simple, then V is a GZ-module.

Corollary 2. If V is an indecomposable GZ-module and $V(\lambda) \neq 0$ for some $\lambda \in L^{\wedge}$ then $\text{supp } V \subset \pi(\lambda + \Delta)$ where Δ is the subgroup of L^{\wedge} generated by all δ_{kj} ($k=1..n-1, j=1..k$).

Obviously, if P and P' are cosets modulo Λ , then either $\pi(P)=\pi(P')$ or $\pi(P)\cap\pi(P')=\emptyset$. For $D=\pi(P)$ denote \mathfrak{G}_D the complete subcategory of \mathfrak{G} consisting of all modules V with $\text{supp}V \subset D$ and \equiv the equivalence relation on $\Gamma^\wedge: \chi \equiv \chi'$ provided both of them belong to $\pi(P)$ for some coset $P \in L^\wedge/\Lambda$.

Corollary 3. $\mathfrak{G} = \bigsqcup_{D \in \Gamma^\wedge/\equiv} \mathfrak{G}_D$.

For $\lambda \in L^\wedge$ put $\sigma_{k,j}, t_p = \sigma_{k,j}(\lambda + \delta_{k,t} - \delta_{t,p})$ and define the polynomials $g_{k,j,m}(\sigma_{k,1}, \dots, \sigma_{k,j})$ by the formula:

$$g_{k,j}(T) = \prod_{t=p} (T - \sigma_{k,j}, t_p) = T^K + \sum_m g_{k,j,m}(\sigma_{k,1}(\lambda), \dots, \sigma_{k,j}(\lambda)) T^m$$
 where $K = k(k-1)/2$.

Proposition 4. For any $\alpha \in \Gamma$ the element

$$Y_{k,j}(\alpha) = \sigma_{k,j}^{-K} X_k^{-\alpha} X_k^{+\sum_m \sigma_{k,j}^m X_k^{-\alpha} X_k^{+}} g_{k,j,m}(\sigma_{k,1}, \dots, \sigma_{k,j})$$
 belongs to Γ and the same is true if we permute $+$ and $-$.

The proof is quite analogous to that of proposition 3.

2⁰. GELFAND-ZETLIN CATEGORIES

For $\chi \in \Gamma^\wedge$ denote $I_\chi = \text{Ker} \chi$, Γ_χ the I_χ -adique completion of Γ . Consider for any pair $\chi, \psi \in \Gamma^\wedge$ the Γ -bimodule $U(\psi, \chi) = \{u \in U \mid \forall m \exists n I_\psi^n u \in U I_\chi^m\}$. The adique topologies of Γ induce a topology on $U(\psi, \chi)$, so we can form its completion $\mathfrak{S}(\psi, \chi)$. Now we can define a category \mathfrak{S}_D for $D \in \Gamma^\wedge/\equiv$ whose objects are characters $\chi \in D$ and sets of morphisms are $\mathfrak{S}(\psi, \chi)$. Surely, the category \mathfrak{G}_D is equivalent to the category $\mathfrak{S}_D\text{-mod}$ of \mathfrak{S}_D -modules, i.e. continuous linear functors from \mathfrak{S}_D to the category of vector spaces over \mathbb{C} with discrete topologies. The following result is a simple corollary of the abstract nonsense.

Proposition 5. Denote $\mathfrak{S}_\chi = \mathfrak{S}(\chi, \chi)$ and $\mathfrak{F}_\chi = \mathfrak{S}_\chi / \text{rad} \mathfrak{S}_\chi$. Then:

(i) If V is a simple GZ-module with $V(\chi) \neq 0$, then $V(\chi)$ is a simple \mathfrak{F}_χ -module.

(ii) For any simple \mathfrak{F}_χ -module M there exists the unique (up to isomorphism) simple GZ-module V with $V(\chi) \cong M$.

Denote $\nu(D)$ (resp. $\nu(\chi)$) the number of non-isomorphic simple GZ-modules V with $\text{supp}V \subset D$ (resp. $V(\chi) \neq 0$). Define two open dense subsets Ω_1, Ω_2 of Γ^\wedge in the following way:

$$\begin{aligned} \Omega_1 &= \{\chi = \pi(\lambda) \mid \lambda_{k,t} - \lambda_{k,p} \neq 0 \text{ for all } t \neq p \text{ and } k=2..n-1\} \\ \Omega_2 &= \Omega_1 \cap \{\chi = \pi(\lambda) \mid \lambda_{k,t} - \lambda_{k-1,p} \neq 0 \text{ for all } t, p \text{ and } k=2..n\} \end{aligned}$$

Remark that both Ω_1 and Ω_2 are stable under the relation \equiv .

THEOREM 1. (i) If $\chi \in \Omega_1$, then $v(\chi)=1$ and $\dim V(\chi)=1$ for the unique simple GZ-module V with $V(\chi) \neq 0$.

(ii) If $D \in \Omega_2$, then $v(D)=1$.

Proof. Let $\chi = \pi(\lambda)$ and $\chi \in D$. Denote x_{kt}^\pm the morphism from $\mathfrak{S}(\chi, \pi(\lambda + \delta_{kt}))$ generated by X_k^\pm . Of course, if χ runs through D , the images of Γ_χ and all possible x_{kt}^\pm generate \mathfrak{S}_D . If $\chi \in \Omega_1$, proposition 3(iii) implies that the image of Γ_χ in \mathfrak{S}_χ consists only of scalars: the class of $a \in \Gamma$ coincides with that of $\chi(a)$. Let $y_{kt}(a)$ for any $a \in \Gamma$ be the image in \mathfrak{S}_χ of $\Sigma_i x_{kt}^- a x_{kt}^+$. We know then from proposition 4 that $G_{R_j}(\sigma_{R_j}(\lambda)) y_{R_j}(a)$ is also a scalar, namely, the image of $Y_{R_j}(a)$. If $\chi \in \Omega_1$, then $G_{R_j}(\sigma_{R_j}(\lambda)) \neq 0$, hence $y_{R_j}(a)$ is also a scalar. Putting $a=1, \sigma_{R_2}, \dots, \sigma_{R_k}$ we obtain k linear equations for k products $x_{kt}^+ x_{kt}^-$ and one can check that their determinant is not 0 provided $\lambda_{kt} \neq \lambda_{kp}$ for $t \neq p$. Thus these products are scalars too. The same is true for $x_{kt}^- x_{kt}^+$. But it is easy to see that all these products generate \mathfrak{S}_χ (together with Γ_χ). Hence \mathfrak{S}_χ is either 0 or \mathbb{C} and $v(\chi)$ is respectively either 0 or 1. But using GZ-formulae as in [6] one can construct a GZ-module W with $W(\chi) \neq 0$ for all $\chi \in D$ which proves (i). The same formulae show that if $D \in \Omega_2$, then W is simple and hence it is the only simple GZ-module in \mathfrak{S}_D which proves (ii).

Remark. It follows from proposition 3 that if V is a simple GZ-module and $v \in V(\chi)$ then in any case $\sigma_{R_1} v = \chi(\sigma_{R_1}) v$, $\sigma_{n_j} v = \chi(\sigma_{n_j}) v$ and $(\sigma_{R_j} - \chi(\sigma_{R_j}))^k v = 0$ in other cases.

Conjectures. (i) $0 < v(\chi) < \infty$ for any $\chi \in \Gamma^\wedge$ and $v(D) < \infty$ for any $D \in \Gamma^\wedge$.

(ii) $\dim V(\chi) \leq n$ for any simple GZ-module and any $\chi \in \Gamma^\wedge$.

(iii) The image of Γ in $\text{End}_{\mathbb{C}} V(\chi)$ is a maximal commutative subalgebra and coincides with the subalgebra generated by a Jordan cell (where V and χ are as in (ii)).

These conjectures are true if $n \leq 3$. Really, if $n=2$, it follows from [3] that $v(\chi)=1$ and $\dim V(\chi)=1$ for all V and χ ; $v(D)$ can equal 1, 2 or 3; if $v(D)=3$, then one of the simple modules in \mathfrak{S}_D is finite-dimensional and all finite-

dimensional simple \mathfrak{g} -modules are of this type (cf. also [2], 7.0.9).

If $n=3$, the following statements hold (cf. [4]).

THEOREM 2. For any $\chi \in \Gamma^{\wedge}$ and any $D \in \Gamma^{\wedge} / \equiv$

(i) $0 < v(\chi) \leq 2$ and $v(D) < \infty$.

(ii) If V is a simple GZ-module with $V(\chi) \neq 0$, then $\dim V(\chi) \leq 2$ and if $\dim V(\chi) = 2$, then $v(\chi) = 1$.

(iii) If $\dim V(\chi) = 2$, then σ_{22} acts on $V(\chi)$ as a Jordan cell.

Proof. A straightforward calculation using the results of [1] shows that \mathfrak{g}_{χ} equals either \mathfrak{c} or $\mathfrak{c} \oplus \mathfrak{c}$ or $M_2(\mathfrak{c})$. According to proposition 5 it proves (ii). If all objects of the category \mathfrak{G}_D are isomorphic, then it implies also (i). Otherwise, if V is a simple module from \mathfrak{G}_D , then $V(\chi) = 0$ for some $\chi \in D$. But it follows from [5] that there exist only finitely many such modules in \mathfrak{G}_D which accomplishes the proof of (i). The statement (iii) can be easily checked.

REFERENCES

1. BRITTEN D.J., LEMIRE F.W. "Irreducible representations of A_n with a 1-dimensional weight subspace", Trans.Amer.Math.Soc., 273 (1982), 509-540.
2. DIXMIER J. "Algebres enveloppantes", Gauthier-Villard, Paris, 1974.
3. DROZD Yu.A. "On representations of Lie algebra $SL(2)$ ", Visnyk Kiev.Gos.Univ., 25 (1983), 70-77.
4. DROZD Yu.A., OVSIENKO S.A., FUTORNY V.M. "Irreducible weight $SL(3)$ -modules", Funk.Anal.Prilozh., (3) 23 (1989), 57-58.
5. FUTORNY V.M., "One generalization of Verma modules and irreducible representations of Lie algebra $SL(3)$ ", Ukrain. Mat.J., 38 (1986), 492-497.
6. ZHELOBENKO D.P. "Compact Lie groups and their representations", Nauka, Moscow, 1974.

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