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REPRESENTATIONS, DUALS AND QUANTUM DOUBLES OF MONOIDAL CATEGORIES

Shahn Majid[1]

ABSTRACT Let H be a finite-dimensional Hopf algebra. It has double quantum group $D(H)\cong H{}_\beta\bowtie_\alpha H^{*\mathrm{op}}$, a double cross product of H and H^* by mutual coadjoint actions α,β. Motivated by the representation theory of $D(H)$ we proceed to generalize the construction to double cross products of monoidal categories. Let $F:\mathcal{C}\to\mathcal{V}$ be a monoidal functor between monoidal categories. We define a dual functored monoidal category $F^\circ:\mathcal{C}^\circ\to\mathcal{V}$. If $\mathcal{V}$ is a quasitensor (i.e. braided monoidal) category, we define a coadjoint action of $\mathcal{C}$ on $\mathcal{C}^\circ$.

1 INTRODUCTION AND PRELIMINARIES Double cross products $H_1{}_\beta\bowtie_\alpha H_2$ of Hopf algebras H_1,H_2 were introduced in [14] as a generalization of double cross products $G_1{}_\beta\bowtie_\alpha G_2$ of groups. In this paper we generalize further to double cross products of monoidal categories. We recall the group case first and explain some of the motivation from physics. The Hopf algebra case and the example $D(H)$ are reviewed in Section 2. The monoidal category constructions follow in Sections 3,4.

Double cross products (or "bicrossproducts") of groups G_1,G_2 by actions α,β are a generalization of group cross (i.e. semidirect) products in which both groups act on the spaces of each other. They have been introduced independently by several authors, among the recent ones [20][14][6]. See [15] for full references. Suitable data are α a left action of G_1 on G_2 and β a right action of G_2 on G_1, such that $\alpha_u(st)=\alpha_u(s)\alpha_{\beta_s(u)}(t)$, $\alpha_u(e)=e$, $\beta_s(uv)=\beta_{\alpha_v(s)}(u)\beta_s(v)$, $\beta_s(e)=e$ for all $u,v\in G_1$, $s,t\in G_2$. Here e denotes the relevant identity element. In this situation the associated double cross product $G_1{}_\beta\bowtie_\alpha G_2$ is defined on $G_1\times G_2$ by $(u,s)(v,t)=(\beta_t(u)v,s\alpha_u(t))$. $G_1\bowtie G_2$ contains both G_1 and G_2 and indeed any situation in which a group factorizes into subgroups $G=G_2G_1$ is such a double cross product. The ordinary cross product case occurs precisely when one of the factor groups is normal, which is clearly a very special case. Also, in [13] it was shown how to explicitly construct examples by exponentiation of Lie algebra double cross products by integrating a certain matched pair of connections. Such Lie algebra data arise for example from solutions of the classical Yang-Baxter equations. Many solutions of these are known in the context of classical inverse scattering.

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The factorisation data of a double cross product has many interesting applications. For example in the locally compact case let $\mathcal{M}(G_1)$ be the group Kac algebra generated by the left regular representation on $L^2(G_1)$. Let $L^\infty(G_2)$ be the Kac algebra of functions acting by multiplication on $L^2(G_2)$. For regular α we have a weakly continuous action of G_1 and $\mathcal{M}(G_1)$ on $L^\infty(G_2)$ and can form the usual von Neumann algebra cross product $\mathcal{M}(G_1)\ltimes_\alpha L^\infty(G_2)$. Likewise we have dually a coaction of $L^\infty(G_2)$ on $\mathcal{M}(G_1)$ induced by β. In virtue of the above matching equations, the corresponding cross coproduct fits together with the cross product algebra to form a *bicrossproduct* Hopf-von Neumann algebra $M = \mathcal{M}(G_1)^\beta\bowtie_\alpha L^\infty(G_2)$. See [9]. A purely algebraic version of this but for general Hopf algebras $H_1{}^\beta\bowtie_\alpha H_2$ appeared in [14], and a version for algebraic groups independently in [20] based on work of [18]. These bicrossproducts are related to the double cross products $H_1{}_\beta\bowtie_\alpha H_2$ studied below. Under favourable circumstances M is a Kac algebra and

$$(\mathcal{M}(G_1)^\beta\bowtie_\alpha L^\infty(G_2))\hat{} = \mathcal{M}(G_2)^\alpha\bowtie_\beta L^\infty(G_1).$$

Physically, M can be interpreted as the quantum algebra of observables of a particle moving on a homogeneous spacetime: if G_i are Lie groups, G_1 semisimple and compact, α effective and transitive with reductive isotropy group then α corresponds to a metric on G_2 such that the geodesics are the one-parameter flows. The cross product algebra M is then the natural quantum algebra of observables[8], cf. Mackey's quantization using systems of imprimitivity. The Kac algebra property means that there is a symmetry between observables (in M) and states (roughly speaking, in $\hat{M}$). This is part of the author's algebraic approach [14][9][13][8] to quantum mechanics combined with gravity. The above equations for α (with β as an auxiliary variable) are like "Einstein's equation" in that α plays the role of the spacetime metric on G_2. What is interesting is that these equations are forced by the observable-state symmetry.

Clearly, for physically more realistic models than those above, we are going to need a much more general framework[10]. This is one motivation for the category-theoretic work of the present paper. A second motivation comes from rational conformal field theory[11]. Here the "chiral algebra" plays the role of the maximal "group of symmetries" of the theory. It contains the Virasoro algebra (corresponding to symmetry under conformal transformations). However, in general it is not a group or even a quantum group, but something more general. This greater generality is provided by monoidal categories. In the case when they are equipped with monoidal functors to $\mathcal{V} = Vec$ these correspond to quantum groups: Our results are formulated in such a way that a functor is not necessarily required.

This is the final form of a preprint of similar title. The case of more general $\mathcal{V}$ is now stated more explicitly. I have added the observation that the example $(\mathcal{C}, \mathrm{id})^\circ$ below corresponds to a center construction defined by V.G. Drinfeld in another context: I thank him for advising me of this.

PRELIMINARIES k denotes an arbitrary ground field. We adopt the usual notations for Hopf algebras (H, Δ, ϵ, S)[19]. Here H is a unital algebra and $\Delta : H \to H\otimes H$ the coproduct and $\epsilon : H \to k$ the counit. These define a bialgebra: by Hopf algebra we mean a bialgebra equipped also with an antipode $S : H \to H$. We often write $\Delta h \equiv \sum h_{(1)} \otimes h_{(2)}$ for all $h \in H$. 1 denotes the unit element. H^* denotes the dual and H^{*op} denotes H^* with the opposite algebra. $<\ ,\ >$ denotes evaluation. Algebra maps are supposed unital. A coalgebra map f is a linear map between coalgebras for which $(f \otimes f) \circ \Delta = \Delta \circ f$, $\epsilon \circ f = \epsilon$. A Hopf algebra H is *quasitriangular* if it possesses an invertible element $\mathcal{R} \in H \otimes H$ such that (i) $(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$, $(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$, (ii) $\sum h_{(2)} \otimes h_{(1)} = \mathcal{R}(\Delta h)\mathcal{R}^{-1}$ for all $h \in H$. This definition is due to [3]. An introduction is [15]. There is an analogous definition of H dual quasitriangular with $\mathcal{R} \in (H \otimes H)^*$.

${}_H\mathcal{M}$ denotes the category of finite-dimensional left H-modules. A coalgebra C is a left H-module *coalgebra* if the action $H \otimes C \to C$ is a coalgebra map. ${}^H\mathcal{M}$ denotes the dual notion of the category of finite-dimensional left H-comodules.

A category $\mathcal{C}$ is *monoidal* if there is a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and functorial isomorphisms $\Phi : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ for all objects X, Y, Z, and a unit object $\underline{1}$ with functorial isomorphisms $l : X \to \underline{1} \otimes X, r : X \to X \otimes \underline{1}$ for all objects X. The Φ should obey a pentagon coherence identity while the l and r obey triangle identities of compatibility with Φ.

A monoidal category $\mathcal{C}$ is *rigid* if for each object X there is an object X^* and functorial morphisms $ev : X^* \otimes X \to \underline{1}$, $\pi : \underline{1} \to X \otimes X^*$ such that [1]

$$X \cong \underline{1} \otimes X \overset{\pi \otimes \mathrm{id}}{\to} (X \otimes X^*) \otimes X \cong X \otimes (X^* \otimes X) \overset{\mathrm{id} \otimes ev}{\to} X \otimes \underline{1} \cong X$$

$$X^* \cong X^* \otimes \underline{1} \overset{\mathrm{id} \otimes \pi}{\to} X^* \otimes (X \otimes X^*) \cong (X^* \otimes X) \otimes X^* \overset{ev \otimes \mathrm{id}}{\to} \underline{1} \otimes X^* \cong X^*$$

compose to id_X and id_{X^*} respectively. We shall also sometimes have recourse to a *quasisymmetry* or "braiding" Ψ. This is a functorial isomorphism $\Psi_{X,Y} : X \otimes Y \to Y \otimes X$ obeying two hexagon coherence identities with Φ and triangle coherence identities with l and r. If $\Psi^2 = \mathrm{id}$ then one of the hexagons is superfluous and we have an ordinary symmetric monoidal category or tensor category as in [1]. Categories with quasisymmetry appeared under the heading "braided monoidal categories" in connection with low dimension topology [5][4], as well as more recently in physics in the context of quantum groups and conformal field theories[15, Sec. 7] where they were called *quasitensor* categories because they generalized the tensor category case. Some relevant early work in this area is [17].

Let $\mathcal{V}$ be a monoidal category. A monoidal category $\mathcal{C}$ is *functored* over $\mathcal{V}$ if there is a functor $F : \mathcal{C} \to \mathcal{V}$ and functorial isomorphisms $c_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ such that

$$\begin{array}{ccc} F(X) \otimes (F(Y) \otimes F(Z)) \overset{\mathrm{id} \otimes c}{\to} F(X) \otimes F(Y \otimes Z) & \overset{c}{\to} & F(X \otimes (Y \otimes Z)) \\ \downarrow \Phi^{\mathcal{V}} & & \downarrow F(\Phi) \\ (F(X) \otimes F(Y)) \otimes F(Z) \overset{c \otimes \mathrm{id}}{\to} F(X \otimes Y) \otimes F(Z) & \overset{c}{\to} & F((X \otimes Y) \otimes Z) \end{array}$$

and $F(X) \overset{F(l_X)}{\to} F(\underline{1} \otimes X) \overset{c^{-1}}{\to} F(\underline{1}) \otimes F(X)$ should compose to $l_{F(X)}$ under $F(\underline{1}) \cong \underline{1}$. Similarly for $F(r_X)$. Such a functor is called monoidal. If $\mathcal{C}$ and $\mathcal{V}$ are rigid then $d_X : F(X)^* \cong F(X^*)$ by functorial isomorphisms with $\pi_{F(X)} = (\mathrm{id} \otimes d_X^{-1}) \circ c^{-1} \circ F(\pi_X)$ and $ev_{F(X)} = F(ev_X) \circ c \circ (d_X \otimes \mathrm{id})$[21]. The isomorphisms Φ will be left implicit for brevity. Vec denotes finite-dimensional vector spaces.

2 QUANTUM DOUBLES Let H be a Hopf algebra over a field k. The finite-dimensional comodules of H, ${}^H\mathcal{M}$, form a rigid monoidal category with tensor product of comodules defined via the algebra structure of H. The duals are provided by the antipode: this coverts the adjoint of a left comodule (which is a right comodule) to a left comodule. In addition, there is a forgetful functor $F : {}^H\mathcal{M} \to Vec$. If H is quasitriangular, then the category is a quasitensor one. The associated quasisymmetry or braiding $\Psi_{V,W} : V\otimes W \to W\otimes V$ is given by $v\otimes w \mapsto \sum \mathcal{R}^{(2)} \cdot w\otimes\mathcal{R}^{(1)} \cdot v$ where the action of $\mathcal{R}$ is obtained by dualizing the comodule structures.

THEOREM 2.1[21]*Suppose $\mathcal{C}$ is a rigid monoidal category equivalent to a small one and functored by F over Vec. Then there is a Hopf algebra H, unique up to isomorphism, such that F factors through the forgetful functor of ${}^H\mathcal{M}$. If $\mathcal{C}$ is k-linear, abelian and F is k-linear, exact, faithful then $\mathcal{C}$ is equivalent to ${}^H\mathcal{M}$.*

Proof The main idea of the proof is to consider the functors ${}_VF : \mathcal{C} \to Vec : X \mapsto V\otimes F(X)$ for V an object in Vec, and show that the functor $\tilde{F} : V \mapsto Mor(F, {}_VF)$ is representable (here Mor denotes natural transformations). I.e. there exists a vector space H such that $Lin(H, V)\cong Mor(F, {}_VF)$. By choosing V suitably we obtain the various linear maps to make H a Hopf algebra[21]. id $\in Lin(H, H)$ corresponds to comodule structures on each $F(X)$. The $\otimes$ structure corresponds to the product of H.

This establishes a well-known Tannaka-Krein type equivalence between Hopf algebras and functored monoidal categories. If $\mathcal{C}$ has a quasisymmetry, it is easy to see that the underlying Hopf algebra is dual quasitriangular. This was explained while reviewing the proof of [21] in [15, Section 7.4-7.5].

This general equivalence provides the heuristic strategy of the present paper. We formulate Hopf algebra constructions in terms of monoidal categories $\mathcal{C}$ and functors $F : \mathcal{C} \to Vec$ and generalize by replacing Vec by another monoidal category $\mathcal{V}$. In particular the functor can be done away with altogether by considering $\mathcal{V} = \mathcal{C}$ and $F = \mathrm{id}$. In this way quantum group constructions now make sense for more general monoidal categories.

We now recall the double cross products of Hopf algebras as introduced in [14, Section 3.2]. Further examples were given in [7]. Let H_1, H_2 be Hopf algebras with H_2 a left H_1-module coalgebra by α and H_1 a right H_2-module coalgebra by β. These should be matched according to

$$\alpha(h\otimes ab) = \sum \alpha(h_{(1)}\otimes a_{(1)})\alpha(\beta(h_{(2)}\otimes a_{(2)})\otimes b), \quad \alpha(h\otimes 1) = \epsilon(h)1$$
$$\beta(hg\otimes a) = \sum \beta(h\otimes\alpha(g_{(1)}\otimes a_{(1)}))\beta(g_{(2)}\otimes a_{(2)}), \quad \beta(1\otimes a) = 1\epsilon(a)$$
$$\sum \alpha(h_{(1)}\otimes a_{(1)})\otimes\beta(h_{(2)}\otimes a_{(2)}) = \sum \alpha(h_{(2)}\otimes a_{(2)})\otimes\beta(h_{(1)}\otimes a_{(1)})$$

for all $h, g \in H_1, a, b \in H_2$. For such data $(H_1, H_2, \alpha, \beta)$ we can define on $H_1\otimes H_2$ a double cross product Hopf algebra $H_1{}_\beta\bowtie_\alpha H_2$

$$(h\otimes a)(g\otimes b) = \sum \beta(h_{(2)}\otimes b_{(2)})g\otimes a\alpha(h_{(1)}\otimes b_{(1)}), \; 1 = 1\otimes 1$$

with the tensor product coalgebra structure and a suitable antipode. Here $H_i \hookrightarrow H_1{}_\beta\bowtie_\alpha H_2$ by the canonical inclusions. There is an analog of the factorization property[14].

EXAMPLE 2.2[14]*Let $H_1 = H$ a finite-dimensional Hopf algebra and $H_2 = H^{*\mathrm{op}}$ its dual with the opposite algebra. We let α and β be the mutual coadjoint actions ad^*. These are explicitly defined by*

$$\alpha(h\otimes\phi) = \sum \phi_{(2)} < (S\phi_{(1)})\phi_{(3)}, h >, \quad \beta(\phi\otimes h) = \sum h_{(2)} < (Sh_{(1)})h_{(3)}, \phi >$$

for all $h \in H$, $\phi \in H^*$. Here β is a left action of H^*, i.e. a right action of $H^{*\mathrm{op}}$. The resulting $H_{ad^*} \bowtie_{ad^*} H^{*\mathrm{op}}$ is isomorphic to Drinfeld's $D(H)$.

Explicitly, $D(H)$ can be built on $H\otimes H^*$ as

$$(h \otimes \phi)(g \otimes \psi) =< Sh_{(1)}, \psi_{(1)} > \sum h_{(2)}g \otimes \psi_{(2)}\phi < h_{(3)}, \psi_{(3)} >$$

for $(h\otimes\phi), (g\otimes\psi) \in H\otimes H^*$. (This is isomorphic to the definition of $D(H)$ in [3] and to that in [14] via $(\mathrm{id}\otimes S)$.) A quasitriangular structure on $D(H)$ is given by $\mathcal{R}_{D(H)} = \sum_a (1\otimes f^a)\otimes(e_a\otimes 1)$ where $\{e_a\}$ is a basis of H and $\{f^a\}$ a dual basis[3]. Therefore the representations of $D(H)$ form a quasitensor category. Applications of this are in [16]. An explicit description of the category and its relation with crossed bimodules is in [12].

3 DUALS OF MONOIDAL CATEGORIES Let $\mathcal{V}$ be a monoidal category. If $(\mathcal{C}_i, F_i)$ are functored monoidal categories over $\mathcal{V}$, we can consider the natural transformations $Mor(\tilde{F}_2, \tilde{F}_1)$ where $\tilde{F}_i : V \mapsto Mor(F_i, {}_VF_i)$ where ${}_VF_i(X) = V\otimes F_i(X)$. From Theorem 2.1 we see that if $\mathcal{C}_i = {}^{H_i}\mathcal{M}$ (and the F_i forgetful to Vec) then the set $Mor(\tilde{F}_2, \tilde{F}_1)$ coincides with $Lin(H_1, H_2)$.

LEMMA 3.1*Let $(\mathcal{C}_i, F_i)$ be functored monoidal categories. Every functor $f : \mathcal{C}_1 \to \mathcal{C}_2$ which is compatible with F_i in the sense $F_2(f(X)) = F_1(X)$ for all objects X in $\mathcal{C}$ induces an element $\tilde{f} \in Mor(\tilde{F}_2, \tilde{F}_1)$.*

Proof Explicitly, $\tilde{f}_V : Mor(F_2, {}_VF_2) \to Mor(F_1, {}_VF_1)$ is defined by $\tilde{f}_V(t)_X = t_{f(X)}$ for $t \in Mor(F_2, {}_VF_2)$.

In the case when $\mathcal{C}_i = {}^{H_i}\mathcal{M}$ and F is forgetful the corresponding linear maps are the coalgebra maps. If the functor is monoidal, the corresponding maps are Hopf algebra maps. This motivates the definitions below.

DEFINITION 3.2*Let $\mathcal{V}$ be a monoidal category. Let $(\mathcal{C}, F)$ be a functored monoidal category over $\mathcal{V}$. A right $(\mathcal{C}, F)$-module is an object V in $\mathcal{V}$ and a natural transformation $\lambda_V \in Mor({}_VF, F_V)$ such that the $\lambda_{V,X} : V\otimes F(X) \to F(X)\otimes V$ obey*

$$\begin{array}{ccccc}
V\otimes F(X)\otimes F(Y) & \stackrel{\lambda_{V,X}\otimes\mathrm{id}}{\longrightarrow} & F(X)\otimes V\otimes F(Y) & \stackrel{\mathrm{id}\otimes\lambda_{V,Y}}{\longrightarrow} & F(X)\otimes F(Y)\otimes V \\
\mathrm{id}\otimes c_{X,Y} \searrow & & & & \swarrow c_{X,Y}\otimes\mathrm{id} \\
 & V\otimes F(X\otimes Y) & \stackrel{\lambda_{V,X\otimes Y}}{\longrightarrow} & F(X\otimes Y)\otimes V. &
\end{array}$$

We also require $\lambda_{V,\underline{1}} = \mathrm{id}$.

If $\mathcal{C} = {}^H\mathcal{M}$ with F the forgetful functor then these are the right H-modules or, if H is finite-dimensional, objects in ${}^{H^*}\mathcal{M}$. This motivates the following,

THEOREM 3.3*Let $\mathcal{V}$ be a monoidal category. Let $(\mathcal{C}, F)$ be a functored monoidal category over $\mathcal{V}$. Let $\mathcal{C}^*, F^*$ be defined as follows. The objects of $\mathcal{C}^*$ are right $(\mathcal{C}, F)$-modules. The morphisms $Mor((V, \lambda_V), (W, \lambda_W))$ are morphisms $\phi : V \to W$ in $\mathcal{V}$ such that*

$$\begin{array}{ccc} V \otimes F(X) & \stackrel{\phi\otimes\mathrm{id}}{\to} & W \otimes F(X) \\ \lambda_{V,X} \downarrow & & \downarrow \lambda_{W,X} \\ F(X) \otimes V & \stackrel{\mathrm{id}\otimes\phi}{\to} & F(X) \otimes W \end{array}$$

for all X in $\mathcal{C}$. $F^* : \mathcal{C}^* \to \mathcal{V}$ is the forgetful functor. Then $(\mathcal{C}^*, F^*)$ is a functored monoidal category. We call it the full right dual category, $(\mathcal{C}, F)^*$. We also define the full subcategory $(\mathcal{C}, F)^\circ \subseteq (\mathcal{C}, F)^*$ as those modules (V, λ_V) for which the $\lambda_{V,X}$ are isomorphisms.

Proof The monoidal structure is given by $(V, \lambda_V) \otimes (W, \lambda_W) = (V \otimes W, \lambda_{V\otimes W})$ where

$$\lambda_{V\otimes W,X} = \lambda_{V,X} \circ \lambda_{W,X}$$

acting on $V \otimes W \otimes F(X)$. The identity object is $(V = \underline{1}, \lambda_{\underline{1},X} = \mathrm{id})$. Tensor product of morphisms is given by tensor product of morphisms in $\mathcal{V}$. That morphisms compose (as morphisms in $\mathcal{V}$) follows from composing the commuting squares in their definition. That the tensor product $\lambda_{V\otimes W,X}$ is functorial in X follows from functoriality of the λ_V, λ_W. That it respects tensor products in $\mathcal{C}$ follows since the λ_V, λ_W do. The associativity morphisms are induced from those of $\mathcal{V}$. That F^* is a monoidal functor is automatic.

EXAMPLE 3.4*Let $\mathcal{C}$ be a monoidal category. Set $\mathcal{V} = \mathcal{C}$ and $F = \mathrm{id}$. Then $(\mathcal{C}, F)^\circ = Z(\mathcal{C})$ along with $F^\circ : Z(\mathcal{C}) \to \mathcal{C}$. Here $Z(\mathcal{C})$ denotes the center of $\mathcal{C}$ as defined in another context by V.G. Drinfeld[2]. This is of interest because it has a quasisymmetry (i.e. braiding) $\Psi_{(X,\lambda_X),(Y,\lambda_Y)} = \lambda_{X,Y} : X \otimes Y \to Y \otimes X$[2]. If $\mathcal{C} = {}_H\mathcal{M}$ with H finite-dimensional then $Z(\mathcal{C}) = {}_{D(H)}\mathcal{M}$[2]. So $({}_H\mathcal{M}, \mathrm{id})^\circ = {}_{D(H)}\mathcal{M}$ along with a monoidal functor $F^\circ : {}_{D(H)}\mathcal{M} \to {}_H\mathcal{M}$.*

PROPOSITION 3.5*Let $\mathcal{V}$ be a rigid quasitensor category. Let $(\mathcal{C}, F)$ be a rigid monoidal category functored over $\mathcal{V}$. Then $(\mathcal{C}, F)^\circ = (\mathcal{C}, F)^*$.*

Proof Let $(V, \lambda_V) \in (\mathcal{C}, F)^*$. Using the rigidity and the quasisymmetry in $\mathcal{V}$ it is possible to construct for each $\lambda_{V,X}$ an adjoint map $(\lambda_{V,X^*})^* : F(X) \otimes V \to V \otimes F(X)$ (adjoint to λ_{V,X^*} viewed as $V \otimes F(X)^* \to F(X)^* \otimes V$). From the definition we have that $c^{-1} \circ \lambda_{V,X\otimes X^*} \circ c = (\lambda_{V,X^*})^* \circ \lambda_{V,X}$. Considering the morphism $\pi_X : \underline{1} \to X \otimes X^*$ and using functoriality we obtain a commuting square between $\lambda_{V,\underline{1}} = \mathrm{id}$ and $c^{-1} \circ \lambda_{V,X\otimes X^*} \circ c$. From this and the rigidity axioms it is not hard to see that $(\lambda_{V,X^*})^*$ is a left inverse for $\lambda_{V,X}$. Similarly on the right.

In the case when $\mathcal{C} = {}^H\mathcal{M}$ and F is the forgetful functor, the elements of $(\mathcal{C}, F)^\circ$ are the convolution-invertible H-modules $(V, \lambda : H \to EndV)$. The category is rigid as H has an antipode S. The last proposition corresponds to the fact that every algebra map λ from a Hopf algebra is convolution-invertible with inverse $\lambda^{-1} = \lambda \circ S$.

We can likewise define left modules and left duals $^*(\mathcal{C},F)$ of $(\mathcal{C},F)$ as (V,λ_V) where $\lambda\in Mor(F_V,{}_VF)$ and $\lambda_{V,X}\circ\lambda_{V,Y}=c^{-1}\circ\lambda_{V,X\otimes Y}\circ c$. Similarly $^\circ(\mathcal{C},F)$. However, it is easy to see that $\iota:{}^\circ(\mathcal{C},F)\to(\mathcal{C},F)^\circ$ sending (V,λ_V) to (V,λ_V^{-1}) is an isomorphism of monoidal categories.

PROPOSITION 3.6 *Let $\mathcal{V}$ be a monoidal category. Let $(\mathcal{C},F)$ be a functored monoidal category over $\mathcal{V}$. Then there is a monoidal functor $\psi:\mathcal{C}\to{}^*(\mathcal{C}^*)$ given by*

$$X\mapsto(F(X),\lambda_{\psi(X)}),\quad \lambda_{\psi(X),(V,\lambda_V)}=\lambda_{V,X}$$

and $\psi(\phi)=F(\phi)$ for $\phi:X\to Y$. Using the functor ι we also have $\mathcal{C}\to\mathcal{C}^{\circ\circ}$.

Proof That $\lambda_{\psi(X),(V,\lambda_V)}$ is functorial in (V,λ_V) follows from the definition of morphisms in $\mathcal{C}^*$. That it respects tensor products in $\mathcal{C}^*$ follows from the $\otimes$ in $\mathcal{C}^*$. That $\psi(\phi)$ is an intertwiner in $^*(\mathcal{C}^*)$ follows from functoriality of $\lambda_{V,X}$ in X. ψ is monoidal with $\psi(X)\otimes\psi(Y)\cong\psi(X\otimes Y)$ induced by $c:F(X)\otimes F(Y)\cong F(X\otimes Y)$. That this c is an intertwiner in $^*(\mathcal{C}^*)$ follows from Definition 3.2. That this makes ψ monoidal reduces to F monoidal.

We say $(\mathcal{C},F)$ is *reflexive* if this functor $\mathcal{C}\to\mathcal{C}^{\circ\circ}$ is an isomorphism (we can also consider here equivalence). For example $\mathcal{C}={}^H\mathcal{M}$ if H is a finite-dimensional Hopf algebra.

We now consider actions of a category on another category. A Hopf algebra H_1 can act on another H_2 in a variety of ways. In view of Lemma 3.1, we consider H_2 a left H_1-module coalgebra, i.e. a left action $\alpha:H_1\otimes H_2\to H_2$ which is also a coalgebra map. This motivates,

DEFINITION 3.7 *A left action of a monoidal category $\mathcal{C}_1$ on a monoidal category $\mathcal{C}_2$ is a functor $\alpha:\mathcal{C}_1\times\mathcal{C}_2\to\mathcal{C}_2$ such that*

$$\alpha(X,\alpha(Y,A))\overset{\Gamma_{X,Y,A}}{\cong}\alpha(X\otimes Y,A)$$

for all objects X,Y in $\mathcal{C}_1$, A in $\mathcal{C}_2$. Here Γ is a natural equivalence of the two functors $\mathcal{C}_1\times\mathcal{C}_1\times\mathcal{C}_2\to\mathcal{C}_2$ and is required to obey an obvious coherence identity analogous to that for the pentagon identity for the Φ.

If H is any finite-dimensional Hopf algebra, H^* becomes a left H-module coalgebra by the left coadjoint action as in Section 2. In view of Lemma 3.1, this has the following analog.

PROPOSITION 3.8 *Let $\mathcal{V}$ be a quasitensor category with quasisymmetry Ψ. If $(\mathcal{C},F)$ is a rigid functored monoidal category over $\mathcal{V}$, then there is a coadjoint action $\alpha:\mathcal{C}\times\mathcal{C}^\circ\to\mathcal{C}^\circ$ defined by*

$$(X,(V,\lambda_V))\mapsto(F(X)\otimes V,\lambda_{\alpha(X,(V,\lambda_V))}),\quad \lambda_{\alpha(X,(V,\lambda_V)),Y}=\lambda_{V,X}\circ\lambda_{V,Y}\circ\Psi_{F(X),F(Y)}\circ\lambda^{-1}_{V,X}.$$

The action on morphisms $\phi\times\psi$ in $\mathcal{C}\times\mathcal{C}^\circ$ is induced by $\alpha(\phi,\psi)=F(\phi)\otimes\psi$.

Proof It is straightforward to verify that $\alpha(X,(V,\lambda_V))$ is indeed an element of $\mathcal{C}^\circ$ as defined in Definition 3.2. This follows after cancelling adjacent λ, λ^{-1}, commuting certain operators to collect the Ψ's and then using one of the two hexagon coherence identities for Ψ. Similarly (after some computation) it follows that $\Gamma_{X,Y,(V,\lambda_V)} = c_{X,Y}\otimes\mathrm{id} : F(X)\otimes F(Y)\otimes V \to F(X\otimes Y)\otimes V$ is an intertwiner, i.e. a morphism $\alpha(X,\alpha(Y,(V,\lambda_V))) \to \alpha(X\otimes Y,(V,\lambda_V))$ as in Theorem 3.3.

Likewise, there is a functor $\widetilde{\beta} : \mathcal{C}\times\mathcal{C}^\circ \to \mathcal{C}^{\circ\circ}$ sending object $(X,(V,\lambda_V))$ to $F(X)\otimes V$ and the natural transformation

$$\lambda_{\widetilde{\beta}(X,(V,\lambda_V)),(W,\lambda_W)} = \lambda_{V,X}\circ\Psi_{V,W}\circ\lambda^{-1}_{W,X}\circ\lambda^{-1}_{V,X}.$$

If $\mathcal{C}$ is reflexive then there is an object $\beta(X,(V,\lambda_V))$ in $\mathcal{C}$ with $F(\beta(X,(V,\lambda_V))) = F(X)\otimes V$ such that the above expression is $\lambda^{-1}_{W,\beta(X,(V,\lambda_V))}$. This defines a right action $\beta : \mathcal{C}\times\mathcal{C}^\circ \to \mathcal{C}$ analogous to the coadjoint right action in Section 2.

4 DOUBLE CROSS PRODUCTS OF MONOIDAL CATEGORIES In this section we introduce the notion of a double cross product of monoidal categories. The case that we formulate first is strictly analogous to that for groups as in Section 1. After stating this, the work of the preceding section then goes towards constructing an example analogous to the representations of the quantum double $D(H) = H \bowtie H^{*\mathrm{op}}$. We will not formulate this in maximal generality. Rather, the construction demonstrates an example of a novel phenomenon: when functored over a category $\mathcal{V}$, our monoidal category $\mathcal{C}$ has a "Hopf algebra-like" structure. $\mathcal{V}$ plays the role of ground field, F of counit, and the monoidal product in $\mathcal{C}^\circ$ (i.e. composition) plays the role of coproduct in $\mathcal{C}$.

PROPOSITION 4.1*Let $\mathcal{C}_1,\mathcal{C}_2$ be monoidal categories acting on each other by functors $\alpha : \mathcal{C}_1\times\mathcal{C}_2 \to \mathcal{C}_2$ and $\beta : \mathcal{C}_1\times\mathcal{C}_2\to\mathcal{C}_1$ such that*

$$\alpha(X,A\otimes B) = \alpha(X,A)\otimes\alpha(\beta(X,A),B), \qquad \alpha(X,\underline{1}) = \underline{1}$$

$$\beta(X\otimes Y,A) = \beta(X,\alpha(Y,A))\otimes\beta(Y,A), \qquad \beta(\underline{1},A) = \underline{1}$$

and likewise for the morphisms. Then there is a double cross product monoidal category $\mathcal{C}_1{}_\beta\bowtie_\alpha \mathcal{C}_2$ built on $\mathcal{C}_1\times\mathcal{C}_2$ with the monoidal structure

$$(X,A)\otimes(Y,B) = (\beta(X,B)\otimes Y, A\otimes\alpha(X,B))$$

$$(\phi_1,\psi_1)\otimes(\phi_2,\psi_2) = (\beta(\phi_1,\psi_2)\otimes\phi_2, \psi_1\otimes\alpha(\phi_1,\psi_2)).$$

This corresponds to the group double cross product of Section 1. The proof is entirely straightforward. We can also allow here for functorial isomorphisms rather than equalities in the matching equations. We now consider an analog of the Hopf algebra double cross product of Section 2. We concentrate on the example analogous to Example 2.2.

THEOREM 4.2 *Let $\mathcal{V}$ be a quasitensor category with quasisymmetry Ψ. Let $(\mathcal{C}, F)$ be a reflexive functored monoidal category over $\mathcal{V}$, $(\mathcal{C}^\circ, F^\circ)$ the dual and α, β the coadjoint actions of $\mathcal{C}$ on $\mathcal{C}^\circ$ and $\mathcal{C}^\circ$ on $\mathcal{C}$ as in Section 3. Then α, β are matched as*

$$\alpha(X, A \otimes B) = \alpha(X_{(1)}, A_{(1)}) \otimes \alpha(\beta(X_{(2)}, A_{(2)}), B)$$

$$\beta(X \otimes Y, A) = \beta(X, \alpha(Y_{(2)}, A_{(2)})) \otimes \beta(Y_{(1)}, A_{(1)})$$

etc. for all X, Y in $\mathcal{C}$ and A, B in $\mathcal{C}^\circ$. We mean by these equations simply that

$$\lambda_{\alpha(X, A\otimes B), Y} = \lambda_{\alpha(X,A),Y} \circ \Psi^{-1}_{F(X)\otimes A, F(Y)} \circ \lambda_{\alpha(\beta(X,A),B),Y}, \qquad \forall Y$$

$$\lambda_{B, \beta(X\otimes Y, A)} = \lambda_{B,\beta(Y,A)} \circ \Psi_{F(Y)\otimes A, B} \circ \lambda_{B, \beta(X, \alpha(Y,A))}, \qquad \forall B.$$

The first equation is defined on $F(X) \otimes A \otimes B \otimes F(Y)$ and the second on $B \otimes F(X) \otimes F(Y) \otimes A$ with c suppressed. There is a monoidal category $D(\mathcal{C}, F)$ defined as $\mathcal{C} \times \mathcal{C}^\circ$ with the product

$$(X, A) \otimes (Y, B) = (\beta(X_{(2)}, B_{(2)}) \otimes Y, A \otimes \alpha(X_{(1)}, B_{(1)}))$$

understood in the sense

$$\lambda_{A\otimes\alpha(X,B),Z} \circ \Psi^{-1}_{A\otimes F(X)\otimes B, F(Z)} \circ \Psi^{-1}_{F(X)\otimes B\otimes F(Y), C} \circ \lambda^{-1}_{C, \beta(X,B)\otimes Y}$$

when viewed in $\mathcal{C}^{\circ\circ} \times \mathcal{C}^\circ$ and evaluated at $(C, Z) \in \mathcal{C}^\circ \times \mathcal{C}$ on $F(Z) \otimes A \otimes F(X) \otimes B \otimes F(Y) \otimes C$.

Proof These are proven using similar techniques as those for the coadjoint actions in Section 3. Extensive use is made of the fact that composites of Ψ, Ψ^{-1} have the same composition if they correspond to the same braid (the coherence theorem for quasitensor categories).

The notation introduced in this proposition corresponds to a kind of "coproduct" $\Delta(X, A) = (X_{(1)}, A_{(1)}) \otimes (X_{(2)}, A_{(2)})$ where actions due to $(X_{(1)}, A_{(1)})$ and $(X_{(2)}, A_{(2)})$ are composed as in Theorem 3.3 but share *the same* objects $F(X)$ and A. Compare with Example 2.2. Also, putting in the definitions of α, β from Section 3, collecting Ψ's and using coherence the monoidal structure $(X, A) \otimes (Y, B)$ viewed in $\mathcal{C}^{\circ\circ} \times \mathcal{C}^\circ$ at $(C, Z) \in \mathcal{C}^\circ \times \mathcal{C}$ is (suppressing c)

$$\lambda_{B,X} \circ \lambda_{A\otimes B, Z} \circ \Psi^{-1}_{A\otimes B, F(Z)} \circ \Psi^{-1}_{F(X\otimes Y), C} \circ \lambda^{-1}_{C, X\otimes Y} \circ \lambda^{-1}_{B,X}.$$

If H is a finite-dimensional Hopf algebra then $D({}^H\mathcal{M}, \text{Forgetful}) = ({}^{D(H)}\mathcal{M}, \text{Forgetful})$. For another example suppose that $\mathcal{C}$ is a quasitensor category and take $\mathcal{C} = \mathcal{V}$ and $F = \text{id}$. If $(\mathcal{C}, \text{id})$ is reflexive we have a double $D(\mathcal{C}, \text{id})$. We remark that even if $(\mathcal{C}, F)$ is not reflexive, the last formula gives a well-defined object of $\mathcal{C}^{\circ\circ} \times \mathcal{C}^\circ$. In analogy with $D(H)$ we can also expect that $D(\mathcal{C}, F)$ is the dual of a quasitensor category.

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