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THE WEDGE SUM OF DIFFERENTIAL SPACES

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ABSTRACT. In this paper we study some geometric properties of the wedge sum [10] of differential spaces in the sense of Sikorski [7],[8]. In Section 1 we review some of the standard facts on Sikorski's differential spaces. In Section 2 we describe some basic notions and facts concerning the singularity which is obtained by taking the wedge sum of differential spaces.

1. PRELIMINARIES. Let M be any set and let C be any non-empty set of real functions on M. By τ_C we shall denote the weakest topology on M in which all functions from C are continuous. For any subset $A \subset M$, let C_A be the set of all real functions β on A such that, for any $p \in A$, there exist an open neighbourhood $U \in \tau_C$ of p and a function $\alpha \in C$ such that $\beta | A \cap U = \alpha | A \cap U$. By scC we shall denote the family of all real functions on M of the form $\omega \circ (\alpha_1, \ldots, \alpha_n) \in C$, where $\omega \in \mathcal{E}_n$, $\alpha_1, \ldots, \alpha_n \in C$, $n \in N$, and $\mathcal{E}_n = C^{\infty}(R^n)$.

A set C of real functions on M is called a differential structure on M if $C = C_M = scC$ [8]. The pair (M,C) is said to be a differential space; the family C is then a linear ring[8] and its elements are called smooth functions on M. For a set C_0 of real functions on M, the set $(scC_0)_M$ is the smallest differential structure on M containing C_0. A differential space (M,C) is said to be generated by C_0 if $C = (scC_0)_M$.

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If (M,C) is a differential space and A is a subset of M, then (A,C_A) is also a differential space, which is called the differential subspace of (M,C). By a tangent vector to (M,C) at a point $p\in M$ we shall mean any linear mapping $v\colon C \longrightarrow R$ which satisfies the condition

$$v(\alpha\cdot\beta) = v(\alpha)\cdot\beta(p) + \alpha(p)v(\beta) \quad \text{for } \alpha,\beta\in C.$$

By T_pM we shall denote the linear space of all tangent vectors to (M,C) at $p\in M$, called the tangent space to (M,C) at $p\in M$.

Let (M,C) and (N,D) be differential spaces. A mapping $f\colon M \longrightarrow N$ is said to be smooth if $f^*(\alpha) := \alpha\circ f\in C$ for every $\alpha\in D$. A mapping $f\colon M \longrightarrow N$ is said to be a diffeomorphism of (M,C) onto (N,D) if f is a smooth bijection and f^{-1} is smooth.

If $f\colon M \longrightarrow N$ is smooth and $v\in T_pM$, then the formula

$$(f_{*p}v)(\alpha) = v(\alpha\circ f) \quad \text{for } \alpha\in D,$$

defines a vector $f_{*p}v$ tangent to (N,D) at $f(p)$.

Let $TM := \bigsqcup_{p\in M} T_pM$ be the disjoint union of tangent spaces to (M,C) and let $\pi\colon TM \longrightarrow M$ be the canonical projection. We denote by TC the differential structure on TM generated by the set $\{\alpha\circ\pi : \alpha\in C\} \cup \{d\alpha : \alpha\in C\}$, where $d\alpha\colon TM \longrightarrow R$ is given by

$$(d\alpha)(v) = v(\alpha) \quad \text{for } v\in TM.$$

Let $\mathfrak{X}(M)$ be the C-module of all smooth vector fields tangent to (M,C). Every vector field $X\in\mathfrak{X}(M)$ is a smooth section of $\pi\colon TM \longrightarrow M$ [7],[8].

We shall denote by $\mathcal{L}^k(M)$ the C-module of pointwise smooth k-forms (see[2]). Every element θ of $\mathcal{L}^k(M)$ is a smooth mapping $\theta\colon TM\oplus\ldots\oplus TM \longrightarrow R$ such that the restriction $\theta|T_pM\times\ldots\times T_pM$ is a k-linear form for each $p\in M$.

A sequence $W_1,\ldots,W_n\in\mathfrak{X}(M)$ is said to be a vector basis of the C-module $\mathfrak{X}(M)$ if for every point $p\in M$ the sequence $W_1(p)$, ..., $W_n(p)$ is a basis of T_pM. We say that the differential space (M,C) is of constant differential dimension n if every point $p\in M$ has a neighbourhood $U\in\tau_C$ such that there is a vector basis of $\mathfrak{X}(U)$ composed of n vector fields. A point p of (M,C) is called regular if there exists a neighbourhood $V\in\tau_C$ of p such that the differential subspace (V,C_V) is of constant differential dimension. A point $p\in M$ is called singular if p is not regular.

Now, let ϱ be an equivalence relation on (M,C) [4]. A function $f\in C$ is said to be consistent with ϱ if $x\varrho y$ implies $f(x) = f(y)$ for any $x,y\in M$. We denote by C_ϱ the set of all $f\in C$ consistent with ϱ. One can easily show that C is a differential structure on M. Let M/ϱ denote the set of all equivalence classes of ϱ and let $\pi_\varrho: M \longrightarrow M/\varrho$ be the canonical mapping. We denote by $C/\varrho := (\pi_\varrho^*)^{-1}(C)$ the differential structure on M/ϱ coinduced on M/ϱ by the mapping π_ϱ [11],[4]. It is easy to show that $\pi_\varrho^*|(C/\varrho): C/\varrho \longrightarrow C_\varrho$ is an isomorphism of algebras. A subset $A\subset M$ is called ϱ-saturated if $\pi_\varrho^{-1}(\pi_\varrho(A)) = A$. Let us observe that the mapping $M/\varrho \supset A \xrightarrow{I} \pi_\varrho^{-1}(A)\subset M$ is a bijection between the family of ϱ-saturated sets in M and the family of all subsets of M/ϱ. Let us put $\mathfrak{M}_\varrho := \{U\in\tau_C : U = \pi_\varrho^{-1}(\pi_\varrho(U))\}$. It is easy to see that $\mathfrak{M}_\varrho = I(\tau_C/\varrho)$, where τ_C/ϱ is the quotient topology in the set M/ϱ and $\tau_{C_\varrho} = I(\tau_{C/\varrho})$, where $\tau_{C/\varrho}$ is the weakest topology on M/ϱ such that all functions belonging to C/ϱ are continuous. We have $\tau_C/\varrho = \tau_{C/\varrho}$ if and only if $\mathfrak{M}_\varrho = \tau_{C_\varrho}$. Moreover, $\mathfrak{M}_\varrho = \tau_{C_\varrho}$ iff for any $U\in\mathfrak{M}_\varrho$ and for any $p\in U$ there is a function $\varphi\in C_\varrho$ such that $\varphi(p) = 1$ and $\varphi|M - U = 0$.

2. MAIN RESULTS. Let (M_i,C_i), $i = 1,\dots,k$, be differential spaces and let $p_i\in M_i$, $i = 1,\dots,k$, be arbitrary points. Let $(N,D) = \left(\bigsqcup_{i=1}^{k} M_i, \bigsqcup_{i=1}^{k} C_i\right)$ be the disjoint union [10]. By definition $f\in D$ iff $f|M_i\in C_i$ for $i = 1,\dots,k$. For a family $f_i\in C_i$, $i = 1,\dots,k$, we denote by $f_1\sqcup\dots\sqcup f_k$ the real function on N such that $(f_1\sqcup\dots\sqcup f_k)|M_i = f_i$ for $i = 1,\dots,k$.

Let ϱ be the equivalence relation on (N,D) identifying the points $p_1,\dots,p_k$. We denote by p_* the equivalence class containing the points $p_1,\dots,p_k$. Of course equivalence classes different from p_* are one-element.

The quotient space $(N/\varrho, D/\varrho)$ is called the wedge sum of the differential spaces $(M_1,C_1),\dots,(M_k,C_k)$ and it will be denoted by $(M_1\vee\dots\vee M_k, C_1\vee\dots\vee C_k)$. It can be seen that $D_\varrho = \{f\in D: f|\{p_1,\dots,p_k\} = \text{const}\}$.

LEMMA 1. $\tau_D/\varrho = \tau_{D/\varrho}$.

Proof. Let $U \in \mathfrak{M}_\varrho$. It suffices to show that for any point $p \in U$ there exists a function $\varphi \in D_\varrho$ such that

(1) $\varphi(p) = 1$ and $\varphi(q) = 0$ for $q \notin U$.

Assume that $p \in \{p_1, \ldots, p_k\}$. For any $i \in \{1, \ldots, k\}$, there exists a function $f_i \in C_i$ such that $f_i(p_i) = 1$ and $f_i | M_i - (U \cap M_i) = 0$ (see [8] for instance). It is evident that the function $\varphi = f_1 \sqcup \ldots \sqcup f_k$ is consistent with ϱ and satisfies (1).

Now let $p \notin \{p_1, \ldots, p_k\}$ and let $p \in U \cap M_j$ for some $j \in \{1, \ldots, k\}$. There exists a function $g \in C_j$ such that $g(p) = 1$, $g(p_j) = 0$ and $g | M_j - (U \cap M_j) = 0$. Let $\varphi : N \longrightarrow R$ be given by

(2) $\varphi | M_j = g$ and $\varphi | M_i = 0$ for $i \neq j$, $i \in \{1, \ldots, k\}$. It is clear that $\varphi \in D_\varrho$ and φ satisfies (1). This finishes the proof.

Now for $j \in \{1, \ldots, k\}$ and $f \in C_j$ let $\tilde{f}: N \longrightarrow R$ be the function defined by

$$(3) \qquad \tilde{f}(q) = \begin{cases} f(q) & \text{for } q \in M_j, \\ f(p_j) & \text{for } q \notin M_j. \end{cases}$$

Of course $\tilde{f}$ is consistent with ϱ. Let $\hat{f} \in D/\varrho$ be the function corresponding to $\tilde{f}$ by the isomorphism $\pi_\varrho^* | (D/\varrho): D/\varrho \longrightarrow D_\varrho$. $\hat{f}$ satisfies the condition

(4) $\tilde{f} = \hat{f} \circ \pi_\varrho$.

Now one can prove

PROPOSITION 2. Let (M_i, C_i) be a differential space generated by a set C_i^o, $i = 1, \ldots, k$. Then the wedge sum $(M_1 \vee \ldots \vee M_k, C_1 \vee \ldots \vee C_k)$ is generated by the set $\bigcup_{i=1}^{k} \{\hat{f} : f \in C_i^o\}$.

Proof. Let $f \in D/\varrho$ be an arbitrary function. It suffices to show that f smoothly depends on a finite number of functions from the set $\bigcup_{i=1}^{k} \{\hat{f} : f \in C_i^o\}$, in a neighbourhood of p_*.

For $i \in \{1, \ldots, k\}$ let $U_i \in \tau_{C_i}$ be an open neighbourhood of p_i such that there exist functions $f_1^i, \ldots, f_n^i \in C_i^o$, $\theta^i \in \mathcal{E}_n$ satisfying

$$f \circ \pi_\varrho | U_i = \theta^i \circ (f_1^i, \ldots, f_n^i) | U_i .$$

Clearly the set $U := \pi_\varrho^{-1}\left(\bigcup_{i=1}^{k} U_i\right)$ is an open neighbourhood of p_*.

It is easily seen that

$$f|U = \left(\sum_{i=1}^{k} \theta^{i}\circ(\hat{f}_1^{i},\dots,\hat{f}_n^{i}) - \sum_{i=1}^{k-1} \theta^{i}\left(f_1^{i}(p_i),\dots,f_n^{i}(p_i)\right)\right)\Big|U.$$

From Proposition 2 we deduce

COROLLARY 3. If (M_i, C_i), $i = 1,\dots,k$, are differential spaces locally finitely generated [3], then $(M_1\vee\dots\vee M_k, C_1\vee\dots\vee C_k)$ is locally finitely generated.

PROPOSITION 4. For $i\in\{1,\dots,k\}$ the restriction $\pi_\varrho|M_i$ is a diffeomorphism onto its image and

$$T_{p_*}(N/\varrho) = \bigoplus_{i=1}^{k}(\pi_\varrho|M_i)_{*p_i}T_{p_i}M_i.$$

Proof. It is clear that $\pi_\varrho|M_i$ is bijective for $i\in\{1,\dots,k\}$. Let $\Psi_i:\pi_\varrho(M_i)\longrightarrow M_i$ be the inverse of $\pi_\varrho|M_i$ for $i = 1,\dots,k$. It is easy to see that

$$f\circ\Psi_i = \hat{f}\mid\pi_\varrho(M_i) \quad \text{for any } f\in C_i,\ i = 1,\dots,k.$$

So Ψ_i is smooth for $i = 1,\dots,k$.

Now let $w\in T_{p_*}(N/\varrho)$ be an arbitrary vector. For $i\in\{1,\dots,k\}$ let $v_i: C_i \longrightarrow R$ be the mapping defined by

(5) $\quad v_i(\alpha) := w(\hat{\alpha}) \quad$ for $\alpha\in C_i$.

It is easy to verify that $v_i\in T_{p_i}M_i$ for $i = 1,\dots,k$.

One can check that every function $g\in D/\varrho$ can be represented as a sum

(6) $$g = \sum_{i=1}^{k}\widehat{g\circ(\pi_\varrho|M_i)} - (k-1)g(p_*),$$

where $\widehat{g\circ(\pi_\varrho|M_i)}$ is the function defined by (4).

From (5) and (6) it follows that

$$w(g) = \sum_{i=1}^{k}v_i\left(g\circ(\pi_\varrho|M_i)\right) = \sum_{i=1}^{k}\left[(\pi_\varrho|M_i)_{*p_i}v_i\right](g)$$

for any $g\in D/\varrho$. Hence

(7) $$w = \sum_{i=1}^{k}(\pi_\varrho|M_i)_{*p_i}v_i.$$

It remains to show the uniqueness of the decomposition (7).

Note that for any $v\in T_{p_i}M_i$ and $\beta\in C_j$, $i,j\in\{1,\dots,k\}$, if $i\neq j$, then

(8) $\left[(\pi_\varrho|M_i)_{*p_i}v\right](\hat{\beta}) = 0.$

Let $(u_1,\dots,u_k)\in T_{p_1}M_1\times\dots\times T_{p_k}M_k$ be a sequence of vectors such that

(9) $$w = \sum_{i=1}^{k}(\pi_\varrho|M_i)_{*p_i}u_i.$$

Now from (7)-(9) it follows that

$$w(\hat{\beta}) = u_j(\beta) = v_j(\beta) \quad \text{for any } \beta\in C_j,\ j = 1,\dots,k.$$

Hence $u_j = v_j$ for $j = 1,\dots,k$.

In the sequel we denote by $\varrho_i\colon T_{p_*}(N/\varrho) \longrightarrow T_{p_i}M_i$, $i=1,\dots,k$, the projection defined by

(10) $\varrho_i(w) = v_i$ for $w\in T_{p_*}(N/\varrho)$,

where $v_i\in T_{p_i}M_i$ is defined by (5).

<u>LEMMA 5</u>. For any $X\in\mathfrak{X}(N/\varrho)$ there exists a unique sequence $(X_1,\dots,X_k)\in\mathfrak{X}(M_1)\times\dots\times\mathfrak{X}(M_k)$ such that

(11) $X(q) = (\pi_\varrho|M_i)_{*\psi_i(q)}X_i(\psi_i(q))$ for $q\in\pi_\varrho(M_i) - p_*, i=1,\dots,k,$

(12) $$X(p_*) = \sum_{i=1}^{k}(\pi_\varrho|M_i)_{*p_i}X_i(p_i).$$

<u>Proof</u>. For $i\in\{1,\dots,k\}$ let $X_i\in\mathfrak{X}(M_i)$ be the vector field defined by

(13) $X_i(\alpha) = X(\hat{\alpha})\circ(\pi_\varrho|M_i)$ for $\alpha\in C_i$,

where $\hat{\alpha}$ is the function defined by (4).

It can be seen that $X_1,\dots,X_k$ satisfy (11) and (12). The uniqueness of the sequence $X_1,\dots,X_k$ is a consequence of the uniqueness of the decomposition (7).

<u>COROLLARY 6</u>. If p_i is not an isolated point in (M_i, τ_{C_i}) for $i = 1,\dots,k$, then $X(p_*) = 0$ for every $X\in\mathfrak{X}(N/\varrho)$.

<u>Proof</u>. Let $(X_1,\dots,X_k)\in\mathfrak{X}(M_1)\times\dots\times\mathfrak{X}(M_k)$ be the unique sequence satisfying (11) and (12). We will show that $X_i(p_i) = 0$ for $i = 1,\dots,k$.

Fix $i\in\{1,\dots,k\}$. From (11) it follows that

$$X(\hat{\alpha})\circ\pi_\varrho|M_j - \{p_j\} = 0 \quad \text{for } \alpha\in C_i,\ j\neq i,\ j\in\{1,\dots,k\}.$$

Since p_j is not isolated in (M_i, τ_{C_i}), $X(\hat{\alpha})\circ\pi_\varrho|M_j = 0$ for $j\in\{1,\dots,k\}$, $j\neq i$. Of course $X(\hat{\alpha})\circ\pi_\varrho\in D_\varrho$. Thus $X(\hat{\alpha})\circ\pi_\varrho(p_i) = 0$

and, by (8), $X_i(p_i)(\alpha) = 0$. We have thus proved that $X_i(p_i) = 0$ for $i = 1,\dots,k$. Hence (12) gives $X(p_*) = 0$.

REMARK 7. From Lemma 5 and Corollary 6 it follows that if p_i is not isolated in $(M_i, \mathcal{T}_{C_i})$ for $i = 1,\dots,k$, then the D/ϱ -module $\mathfrak{X}(N/\varrho)$ is isomorphic to the D/ϱ - module $\mathfrak{X}_0(M_1,\dots,M_k) := \{(X_1,\dots,X_k)\in\mathfrak{X}(M_1)\times\dots\times\mathfrak{X}(M_k): X_i(p_i) = 0 \text{ for } i = 1,\dots,k\}$.

In the sequel the vector field $X\in\mathfrak{X}(N/\varrho)$ corresponding to a sequence $(X_1,\dots,X_k)\in\mathfrak{X}_0(M_1,\dots,M_k)$ will be denoted by $X_1*\dots*X_k$. Clearly, for any sequence $(f_1,\dots,f_k)\in C_1\times\dots\times C_k$ such that $f_1(p_1) =\dots= f_k(p_k)$ there exists a unique function $f_1*\dots*f_n\in D/\varrho$ satisfying the condition

(14) $\qquad (f_1*\dots*f_n)\circ(\pi_\varrho | M_i) = f_i \quad$ for $i = 1,\dots,k$.

It is easy to verify that the mapping $\Psi:\{(f_1,\dots,f_k)\in C_1\times\dots\times C_k: f_1(p_1) = \dots = f_k(p_k)\} \longrightarrow D/\varrho$, $\Psi(f_1,\dots,f_k) = f_1*\dots*f_k$, is an isomorphism of linear rings over R.

The following equalities hold:

(15) $\quad f_1*\dots*f_k\; X_1*\dots*X_k = f_1X_1*\dots*f_kX_k$,

(16) $\quad (X_1*\dots*X_k)(f_1*\dots*f_k) = X_1f_1*\dots*X_kf_k$,

(17) $\quad X_1*\dots*X_k + Y_1*\dots*Y_k = (X_1 + Y_1)*\dots*(X_k + Y_k)$,

(18) $\quad [X_1*\dots*X_k, Y_1*\dots*Y_k] = [X_1,Y_1]*\dots*[X_k,Y_k]$,

for any $(X_1,\dots,X_k),(Y_1,\dots,Y_k)\in\mathfrak{X}_0(M_1,\dots,M_k)$ and $(f_1,\dots,f_k)\in C_1\times\dots\times C_k$ such that $f_1(p_1) =\dots= f_k(p_k)$.

Now we can prove

PROPOSITION 8. Let p_i be a regular and non-isolated point in (M_i,C_i) for $i = 1,\dots,k$. If $\overset{i}{\nabla}$ is a covariant derivative [6] in the C_i-module $\mathfrak{X}(M_i)$, $i = 1,\dots,k$, then the mapping $\nabla:\mathfrak{X}(N/\varrho)\times\mathfrak{X}(N/\varrho)\longrightarrow\mathfrak{X}(N/\varrho)$ defined by

(19) $$\nabla_{X_1*\dots*X_k} Y_1*\dots*Y_k = \overset{1}{\nabla}_{X_1}Y_1*\dots*\overset{k}{\nabla}_{X_k}Y_k$$

for any $(X_1,\dots,X_k),(Y_1,\dots,Y_k)\in\mathfrak{X}_0(M_1,\dots,M_k)$, is a covariant derivative in the D/ϱ -module $\mathfrak{X}(N/\varrho)$. Moreover, if $R_1,\dots,R_k$ is the curvature tensor of $\overset{1}{\nabla},\dots,\overset{k}{\nabla}$ respectively and $T_1,\dots,T_k$ are the respective torsion tensors, then the curvature tensor R and the torsion tensor T of ∇ satisfy:

(20) $\quad R(X_1*\dots*X_k, Y_1*\dots*Y_k)Z_1*\dots*Z_k = R_1(X_1,Y_1)Z_1*\dots*R_k(X_k,Y_k)Z_k$

(21) $\quad T(X_1*\dots*X_k, Y_1*\dots*Y_k) = T_1(X_1,Y_1)*\dots*T_k(X_k,Y_k)$,

for any $(X_1,\dots,X_k),(Y_1,\dots,Y_k),(Z_1,\dots,Z_k)\in\mathfrak{X}_0(M_1,\dots,M_k)$.

Proof. Since p_i is a regular point in (M_i,C_i) and $X_i(p_i)=0$ for $i=1,\dots,k$, $(\overset{i}{\nabla}_{X_i}Y_i)(p_i)=\overset{i}{\nabla}_{X_i(p_i)}Y_i=0$ for $i=1,\dots,k$. Thus $(\overset{1}{\nabla}_{X_1}Y_1,\dots,\overset{k}{\nabla}_{X_k}Y_k)\in\mathfrak{X}_0(M_1,\dots,M_k)$ and ∇ is well defined. Using the formulas (15)-(17) it is easy to verify that ∇ is a covariant derivative in the D/ϱ -module $\mathfrak{X}(N/\varrho)$. The proof of (20) and (21) is straightforward.

COROLLARY 9. If (M_i,C_i) for $i\in\{1,\dots,k\}$, is a C^∞- manifold, then on the wedge sum $(M_1\vee\dots\vee M_k,C_1\vee\dots\vee C_k)$ there exists a covariant derivative.

For any sequence $(\omega_1,\dots,\omega_k)\in\mathcal{Y}^r(M_1)\times\dots\times\mathcal{Y}^r(M_k)$ of smooth pointwise r-forms let $\omega: T(N/\varrho)\oplus\dots\oplus T(N/\varrho)\longrightarrow R$ be the r-form defined by

$$(22)\quad \omega(w_1,\dots,w_r):=$$
$$=\begin{cases}\sum_{i=1}^{k}\omega_i(\varrho_i(w_i)) & \text{if } \pi^r(w_1,\dots,w_r)=p_*,\\ \omega_i((\Psi_i)_*w_1,\dots,(\Psi_i)_*w_r) & \text{if } \pi^r(w_1,\dots,w_r)\in\pi_\varrho(M_i)-\{p_*\},\\ & i=1,\dots,k,\end{cases}$$

where $\pi^r: T(N/\varrho)\oplus\dots\oplus T(N/\varrho)\longrightarrow N/\varrho$ is the canonical projection, ϱ_i is defined by (10) and Ψ_i is the inverse of $\pi_\varrho|M_i$ for $i=1,\dots,k$.

One can verify that ω is a smooth r-form on $(N/\varrho,D/\varrho)$. It is enough to prove the smoothness of ω in a neighbourhood of the point p_*. For $i\in\{1,\dots,k\}$ let U_i be a neighbourhood of p_i such that there exist smooth functions $f_1^i,\dots,f_n^i\in C_i$, $\theta_i\in\mathcal{E}_{2n}$ satisfying

$$(23)\quad \omega_i|\pi_i^{-1}(U_i)=\theta_i\circ(df_1^i,\dots,df_n^i,f_1^i\circ\pi_i,\dots,f_n^i\circ\pi_i)|\pi_i^{-1}(U_i),$$

for $i=1,\dots,k$.

From (22) and (23) it follows that

$$(24)\quad \omega|U=\sum_{i=1}^{k}\theta_i\circ(d\hat f_1^i,\dots,d\hat f_n^i,\hat f_1^i\circ\pi,\dots,\hat f_n^i\circ\pi)|U,$$

where $U := \pi_\varrho^{-1}\left(\bigcup_{i=1}^{k} U_i\right)$, $\pi : T(N/\varrho) \longrightarrow N/\varrho$ is the canonical projection.

In the sequel the r-form corresponding to $(\omega_1,\dots,\omega_k)$ by means of (22) will be denoted by $\omega_1 * \dots * \omega_k$.

Now one can prove

PROPOSITION 10. If g_i is a riemannian metric on (M_i, C_i) for $i = 1,\dots,k$, then $g_1 * \dots * g_k$ is a riemannian metric on the wedge sum $(M_1 \vee \dots \vee M_k, C_1 \vee \dots \vee C_k)$. Moreover, if (M_i, C_i) is of constant differential dimension for $i = 1,\dots,k$, $\overset{i}{\nabla}$ is the Levi-Civita connection corresponding to g_i [8], then the torsion tensor T of the connection ∇ corresponding to $\overset{1}{\nabla},\dots,\overset{k}{\nabla}$ by (19) is equal to 0.

Proof is straightforward.

EXAMPLE. Let $M_i = \{(t,i) : t \in R\} \subset R^2$, $i = 1,2$, be equipped with the standard differential structures C_1 and C_2 generated by $\{\tau_1\}$ and $\{\tau_2\}$ respectively, where $\tau_i : M_i \longrightarrow R$ is defined by

$$\tau_i(t,i) = t \quad \text{for } t \in R,\ i = 1,2.$$

Let us take the point $p_1 = (0,1)$ and $p_2 = (0,2)$. It can be proved that the wedge sum $(M_1 \vee M_2, C_1 \vee C_2)$ is diffeomorphic to the differential subspace $(M, \mathcal{E}_{2M})$ of $(R^2, \mathcal{E}_2)$, where $M := \{(x,y) \in R^2 : xy = 0\}$. One can verify that the mapping $\gamma : M_1 \vee M_2 \longrightarrow M$ given by

$$\gamma([t,1]) = (t,0) \quad \text{for } t \in R, \tag{25}$$
$$\gamma([t,2]) = (0,t) \quad \text{for } t \in R,$$

is a diffeomorphism. One can see that the $C_1 \vee C_2$-module $\mathfrak{X}(M_1 \vee M_2)$ is free with the basis $\{V_1, V_2\}$, where $V_1 = \tau_1 * 0 \cdot \frac{d}{d\tau_1} * 0$, $V_2 = 0 * \tau_2 \cdot 0 * \frac{d}{d\tau_2}$.

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