Yan S. Soibelman

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QUANTUM WEYL GROUP AND SOME ITS APPLICATIONS

Ya.S.Soibelman

1. This paper contains some results from author's talk at the school "Geometry and Physics" (January, 1990).

2. Let $\mathfrak{g}$ be a simple complex Lie algebra. Let us fix an invariant scalar product $(\ ,\ )$ on $\mathfrak{h}^*$ where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra. We choose a basis of simple roots $\{\alpha_i\}_{i=1}^{r}$ such that $(\alpha_i,\alpha_i) \in \mathbb{Q}$. Let $U_h(\mathfrak{g})$ be a quantized universal enveloping algebra. I recall that $U_h(\mathfrak{g})$ is a topological Hopf algebra over $\mathbb{C}[[h]]$. It contains 1 and generated by $\{X_i^{\pm}, H_i\}_{i=1}^{r}$ and relations:

$$[H_i, X_j^{\pm}] = \pm(\alpha_i,\alpha_j)X_j^{\pm}, \quad [X_i^{+}, X_j^{-}] = \delta_{ij}\frac{sh(\frac{h}{2}H_i)}{sh(\frac{h}{2})},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} q_i^{-k(1-a_{ij}-k)} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k} \qquad (1)$$

where $((a_{ij}))$ is a Cartan matrix for $\mathfrak{g}$, $q_i = e^{h/2(\alpha_i,\alpha_i)}$, $(n)_t = \frac{t^n-1}{t-1}$, $(n)_t! = (1)_t \dots (n)_t$.

$U_h(\mathfrak{g})$ is a Hopf algebra with comultiplication Δ defined by

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(X_i^{\pm}) = X_i^{\pm} \otimes e^{h/4 H_i} + e^{-h/4 H_i} \otimes X_i^{\pm}. \qquad (2)$$

3. Let G be a simply connected simple Lie group such that $Lie\, G = \mathfrak{g}$. Hopf algebra $\mathbb{C}[G]_h$ of regular functions on quantum group G was introduced in [1] (see also [4-6]). It is the algebra of matrix elements of some finite dimensional representations of $U_h(\mathfrak{g})$. Following [4-6] we introduse structure of Hopf $*$-algebra on $\mathbb{C}[G]_h$ such that

$$\ell^*(a) = \overline{\ell(S(a)^*)} \qquad (3)$$

where $\ell \in \mathbb{C}[G]_h$, $a \in U_h(\mathfrak{g})$, S is antipode, $*$ is an antilinear antiautomorphism of algebra $U_h(\mathfrak{g})$ defined by $H_i^* = H_i, (X_i^{\pm})^* = X_i^{\mp}$.

4.It is known ([4,6]) that for every Schubert cell $X_w \subset G/B$ (G/B is flag manifold) there is the irreducible $*$-representation π_w of $\mathbb{C}[G]_h$ in Hilbert space.Here w is an arbitrary element of Weyl group W .Then we can define ([7] for $G = SL_2(\mathbb{C})$ and [4,5] for general case) quantum Weyl group.I recall that for every $w \in W$ one can to construct some Gelfand-Naimark-Segal state $\bar{w} \in \mathbb{C}[G]_h^*$ ([4,5]).Let s_i be a simple reflection which corresponds to i-th vertex of Dynkin diagram.

THEOREM 1 ([4]) a) If $i \neq j$ then $\underbrace{\bar{s}_i \bar{s}_j \bar{s}_i \ldots}_{m_{ij}} = \underbrace{\bar{s}_j \bar{s}_i \bar{s}_j \ldots}_{m_{ij}}$

where $(s_i s_j)^{m_{ij}} = 1$ in usual Weyl group.

b) $e^{-\frac{h}{2(\alpha_i,\alpha_i)} H_i^2} \bar{s}_i^2$ commutes with algebra $U_h(s\ell(2)_i)$ generated by $X_i^{\pm}, H_i$.

Let w_0 be the element of maximal length, $R \in U_h(\mathfrak{g})^{\otimes 2}$ be the universal quantum R-matrix([1] , § 13).Let us choose an orthonormal basis $\{I_k\}$ in $\mathfrak{h}$ and let $\bar{w}_0' = e^{-h/4 \sum_k I_k^2} \bar{w}_0$.

THEOREM 2.

$$\Delta(\bar{w}_0') = R^{-1} (\bar{w}_0' \otimes \bar{w}_0') .$$

In the case $\mathfrak{g} = s\ell(n)$ theorem 2 is proved in [5].

Let $w_0 = s_{i_1} \ldots s_{i_N}$ be a reduced expansions.It is known that set $\mathcal{D} = \{\alpha_{i_1}, \ldots, s_{i_1} \ldots s_{i_{N-1}} \alpha_{i_N}\}$ coinsides with the set Δ_+ of positive roots.Then we define a total order on $\Delta_+ = \mathcal{D}$ (read $\mathcal{D}$ from right to left).Let $\alpha = s_{i_1} s_{i_2} \ldots s_{i_{p-1}}(\alpha_{i_p})$ .We define

$$E_\alpha, F_\alpha \in U_h(\mathfrak{g}), \quad E_\alpha = T_{i_1} \ldots T_{i_{p-1}}(E_{i_p}), \quad F_\alpha = T_{i_1} \ldots T_{i_{p-1}}(F_{i_p}),$$

$$E_i = X_i^+ \exp(-\tfrac{h}{4} H_i), \quad F_i = X_i^- \exp(\tfrac{h}{4} H_i),$$

$$T_i(x) = \bar{s}_i x \bar{s}_i^{-1} .$$

THEOREM 3.

$$R = \prod_{\alpha \in \Delta_+} \exp_{q_\alpha^{-2}} ((1 - q_\alpha^{-2}) E_\alpha \otimes F_\alpha) e^{\frac{h}{2} \sum_k I_k \otimes I_k}$$

where $q_\alpha = e^{h/4(\alpha,\alpha)}$, $\exp_t(x) = \sum_{n \geq 0} \frac{x^n}{(n)_t!}$

and product is taken according to our order.

In the case $\mathfrak{g} = s\ell(n)$ theorem3 is proved in [3].

Another application of quantum Weyl group is connected with Hecke algebras.Let $\mathfrak{g}$ be a simply-laced Lie algebra such that $(\alpha_i, \alpha_i) = 2$ .I recall that Hecke algebra $H_q(W)$ has generators $\{T_i\}_{i=1}^{r}$ with relations:

$$\underbrace{T_i T_j T_i \dots}_{m_{ij}} = \underbrace{T_j T_i T_j \dots}_{m_{ij}} \quad , \quad (T_i - q^{-2})(T_i + 1) = 0$$

We concider a simple $U_h(\mathfrak{g})$-module $L(\Lambda)$ with the highest weight Λ such that $\Lambda(H_i) = (\alpha_{max}, \alpha_i)$ where α_{max} is a maximal root. Therefore $L(\Lambda)$ is a quantum analogue of the adjoint representations of $\mathfrak{g}$. Let $L(\Lambda)_0 = \{x \in L(\Lambda) \mid ax = 0 \ \forall a \in \mathfrak{f}\}$.

THEOREM 4. For every $i \in [1, r]$ we have:

a) $\bar{s}_i(L(\Lambda)_0) \subset L(\Lambda)_0$;

b) $(\bar{s}_i - q^{-2})(\bar{s}_i + 1)|_{L(\Lambda)_0} = 0$

where $q = e^{h/2}$.

Therefore we can say that quantum Weyl group acts on $L(\Lambda)_0$ as Hecke algebra. This result is also obtained by G. Lusztig ([2]).

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AUTHOR: Ya.S. SOIBELMAN, CHAIR OF ALGEBRA, MECH.-MATHEMATICAL DEPARTMENT, ROSTOV UNIVERSITY, ROSTOV-ON-DON, ENGELSA 105, 344006, USSR.