A K-THEORETIC APPROACH TO CHERN-CHEEGER-SIMONS INVARIANTS

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Nous construisons une application de la K-théorie multiplicative définie par Karoubi vers la cohomologie impaire à coefficients $\mathbf{C}^*$ *sur une variété différentielle ce qui permet d'associer à tout fibré vectoriel complexe plat là-dessus des classes caractéristiques analogues aux classes étudiées par Chern, Cheeger et Simons.*

1. Preliminaries. This paper is an extended version of [7] where all the proofs were suppressed. We construct a natural mapping from the multiplicative K-theory due to Karoubi [4] to the odd cohomology with coefficients $\mathbf{C}^*$ on a differentiable manifold X which allows us to associate to any flat complex vector bundle E on X characteristic classes $\check{C}_k(E) \in H_{dR}^{2k-1}(X; \mathbf{C}^*)$ analogous to the classes studied by Chern, Cheeger, and Simons [1, 2].

Let X be a differentiable manifold, E a complex vector bundle on X, D a connection on E, and R the associated curvature. The differentiable Chern characters

$$\mathrm{ch}_k^{(d)}(E, D) = \left(\frac{i}{2\pi}\right)^k \frac{1}{k!} \ \mathrm{Trace}(R^k)$$

define de Rham cohomology classes $\mathrm{Ch}_k^{(d)}(E) \in H_{dR}^{2k}(X)$, $k = 1, 2, 3, \ldots$, which coincide by the de Rham isomorphism with the "topological" [6] Chern characters $\mathrm{Ch}_k^{(t)}(E) \in H_s^{2k}(X; \mathbf{Q})$ in the singular cohomology. Moreover, the integral Chern classes $C_k(E) = [c_k(E)]$ can be expressed as universal polynomials M_k (inverses of the Newton polynomials) with rational coefficients of the Chern characters.

We now briefly recall the definition of the *multiplicative K-theory* $\mathcal{K}(X)$ of X (associated to the trivial filtration of the de Rham complex) as defined by Karoubi in [4]. A multiplicative fibre bundle is a triplet $\xi = (E, D, \omega)$ where ω is a graded odd differential form, $\omega \in \Omega^{\mathrm{odd}}(X)$, whose boundary is the reduced geometric Chern character, $d\omega = \mathrm{ch}(E, D) = \sum_{k=1}^{\infty} \mathrm{ch}_k(E, D)$. Two multiplicative fibre bundles $\xi = (E, D, \omega)$ and $\xi' = (E', D', \omega')$ are said to be equivalent if there exists an isomorphism $\sigma : E \to E'$ such that

$$\omega' - \omega = \text{C-S}(D, D')$$

where C-S stands for the canonical graded odd Chern-Simons transgression form [2].

Multiplicative K-theory inserts into the exact sequence

$$\text{(1)} \qquad K_1^{\mathrm{top}}(X) \xrightarrow{\sigma_1} \bigoplus_{r=1}^{\infty} H_{dR}^{2r-1}(X) \xrightarrow{\partial} \mathcal{K}(X) \xrightarrow{u} K^{\mathrm{top}}(X) \xrightarrow{\sigma} \bigoplus_{r=1}^{\infty} H_{dR}^{2r}(X).$$

Here $K_1^{\mathrm{top}}(X) = [X, GL(\mathbf{C})]$ or the group of homotopy classes of differentiable maps from X to $GL(\mathbf{C})$, and $K^{\mathrm{top}}(X)$ is the Grothendieck - Atiyah - Hirzebruch group of X [3].

In the exact sequence (1), σ is induced by the differentiable Chern character and u is the forgetful homomorphism. The homomorphism ∂ is defined by associating to an odd closed differential form ω the difference of two multiplicative vector bundles $\partial[\omega] = [T, d, \omega] - [T, d, 0]$ where T denotes a trivial vector bundle endowed with the trivial connection d. Finally, if $\alpha : X \to GL(\mathbf{C})$ is differentiable, $\sigma_1(\alpha)$ is represented by the closed differential form

$$\sum_{r=1}^{\infty} \frac{i^{3r-2}}{(2\pi)^r} \frac{(r-1)!}{(2r-1)!} \operatorname{Trace}(\alpha^{-1} d\alpha)^{2r-1}.$$

2. Chern - Cheeger - Simons invariant. Our aim is to combine the exact sequence (1) with the Bockstein exact sequence associated to the exponential exact sequence $0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}^* \to 0$ in order to find a commutative diagram

$$\begin{array}{ccccccccc}
K_1^{\mathrm{top}}(X) & \xrightarrow{\sigma_1} & \bigoplus_{r=1}^{\infty} H_{dR}^{2r-1}(X) & \xrightarrow{\partial} & \mathcal{K}(X) & \xrightarrow{u} & K^{\mathrm{top}}(X) & \xrightarrow{\sigma} & \bigoplus_{r=1}^{\infty} H_{dR}^{2r}(X) \\
\downarrow{\scriptstyle q_k} & & \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle \check{C}_k} & & \downarrow{\scriptstyle C_k^{(t)}} & & \downarrow{\scriptstyle M_k} \\
H_s^{2k-1}(X;\mathbf{Z}) & \longrightarrow & H_s^{2k-1}(X) & \longrightarrow & H_s^{2k-1}(X;\mathbf{C}^*) & \xrightarrow{\beta_k} & H_s^{2k}(X;\mathbf{Z}) & \longrightarrow & H_s^{2k}(X)
\end{array}$$

Here q_k is minus the suspension of $C_k^{(t)}$, and p_k is the obvious projection multiplied by the coefficient of the homogenous term of M_k, that is to say $(-1)^{k-1}(k-1)!$. The natural map $\check{C}_k$ has the property that one recovers $C_k^{(t)}$ when composing it with the Bockstein homomorphism β_k.

The definition of $\check{C}_k$ necessitates a universal construction. Any vector bundle E of rank n over X with connection D can be pulled back via some connection preserving map $h : X \longrightarrow \hat{X}$ from the tautological bundle with universal connection $\hat{D}$ over the Grassmannian manifold $\hat{X} = G_n(\mathbf{C}^m)$ where m is large enough. The map h is unique up to homotopy. The differentiable and topological Chern classes are the same; in particular, on $\hat{X}$ there exists some singular $(2k-1)$-cochain θ_{2k-1} such that

$$\int_{\gamma} c_k^{(d)}(\hat{E},\hat{D}) - c_k^{(t)}(\hat{E})(\gamma) = \theta_{2k-1}(\partial\gamma) \tag{2}$$

for every singular chain $\gamma \in \Sigma_{2k-1}(\hat{X})$. We now pull back each term of (2) via h. By the naturality of Chern classes, we find

$$h^*(c_k^{(d)}(\hat{E},\hat{D})) = c_k^{(d)}(E,D) = M_k(d\omega_1, d\omega_2, d\omega_3, ..., d\omega_{2k-1}) = d\psi_{2k-1}$$

where the $(2k-1)$-form ψ_{2k-1} on X is defined modulo an exact form. On the other hand, topological Chern classes have integral periods so that a *cocycle* $\check{c}_k(E,D,\omega)$ will be associated to each multiplicative fibre bundle (E,D,ω) by the following definition. For a singular $(2k-1)$-chain λ on X, $\lambda \in \Sigma_{2k-1}(X)$ set

$$\check{c}_k(E,D,\omega) = \int_{\lambda} \psi_{2k-1} - \theta_{2k-1}(h \circ \lambda) \mod \mathbf{Z}.$$

Indeed, $\check{c}_k$ is co-closed:

$$\begin{aligned} \delta\check{c}_k(E,D,\omega)(\lambda) &= \int_{\lambda} d\psi_{2k-1} - \delta\theta_{2k-1}(h \circ \lambda) \mod \mathbf{Z} \\ &= c_k^{(t)}(\hat{E})(h \circ \lambda) \mod \mathbf{Z} \\ &= 0 \mod \mathbf{Z}. \end{aligned}$$

The cohomology class of $\check{c}_k(E, D, \omega)$, to be denoted by $\check{C}_k(E, D, \omega)$, is independent of the choice of θ_{2k-1} as, by Bott periodicity, the universal Grassmannian has no odd cohomology. By standard homotopy arguments, $\check{C}_k(E, D, \omega)$ is also seen to be independent of the choices of h and ψ_{2k-1}. A similar homotopy argument also applies to the proof that $\check{C}_k(E, D, \omega)$ is independent of the choice of the representative of the multiplicative K-theory class, once one recalls from [4] the following alternative characterization of $\mathcal{K}(X)$: two multiplicative vector bundles $\xi_i = (E^i, D^i, \omega^i)$, $i = 0, 1$, are equivalent if and only if there exists a homotopy (D_t, ω_t) such that $D_0 = D^0$, $\omega_0 = \omega^0$, $D_1 = \alpha^*(D^1)$, $\omega_1 = \omega^1$ for an isomorphism $\alpha : E^0 \to E^1$.

Hence we have a natural well-defined map

$$\check{C}_k : \mathcal{K}(X) \to H_s^{2k-1}(X; \mathbf{C}^*).$$

It is appropriate to call the resulting characteristic class the *Chern-Cheeger-Simons invariant* as our construction is analogous to theirs [1], [2].

3. The commutative diagram. We now establish the commutativity of the above diagram. This was the main result announced in [7]. Let us number the squares of the diagram by I-IV from left to right.

Square I: In the de Rham cohomology the arrow $K_1^{\text{top}}(X) \to H^{2k-1}(X)$ is described by integration with respect to the suspension parameter $-1 \leq t \leq 1$. But to compute in terms of differential forms, one needs to deal with the differentiable Chern class and, first of all, to endow with a connection the vector bundle $\pi : E \longrightarrow \Sigma X$ determined by $\alpha : X \longrightarrow GL(\mathbf{C})$ over the suspension ΣX.

For this, let $\Upsilon = \{U, V\}$ be a trivializing open cover of ΣX such that U (resp. V) is the contractible open set obtained by puncturing the suspension double cone at the south pole p_- $(t = -1)$, resp. north pole p_+ $(t = 1)$, i.e. $U = \Sigma X \backslash \{p_-\}$, $V = \Sigma X \backslash \{p_+\}$. Let $\{\mu, \nu\}$ be a partition of the unity subordinate to $\Upsilon = \{U, V\}$ such that $\mu(p_+) = 1$, $\nu|_{[-1,0]\times X} \equiv 1$. Construct a connection D of the vector bundle $\pi : E \longrightarrow \Sigma X$ by choosing for the local connection 1-forms associated to the trivialization Υ

$$\omega_U(x) = \nu(x)\, g_{VU}^{-1}(x)\, dg_{VU}(x) = \nu(x)\, \alpha^{-1}(x)\, d\alpha(x)\ ,\ x \in U$$

$$\omega_V(x) = \mu(x)\, g_{UV}^{-1}(x)\, dg_{UV}(x) = -\mu(x)\, d\alpha(x)\, \alpha^{-1}(x)\ ,\ x \in V$$

Then

$$\begin{aligned} g_{UV}^{-1}.dg_{UV} + g_{UV}^{-1}.\omega_U.g_{UV} &= \alpha.d\alpha^{-1} + \alpha.\nu.\alpha^{-1} d\alpha.\alpha^{-1} \\ &= (-1+\nu)\, d\alpha.\alpha^{-1} \\ &= -\ \mu.d\alpha.\alpha^{-1} \\ &= \omega_V \end{aligned}$$

as wanted. The associated curvature 2-forms are

$$\begin{aligned} \Omega_U &= d\omega_U + \omega_U \wedge \omega_U \\ &= d\nu.\alpha^{-1} d\alpha - \nu(\alpha^{-1} d\alpha)^2 + \nu^2(\alpha^{-1} d\alpha)^2 \\ &= d\nu.\alpha^{-1} d\alpha - \mu\nu(\alpha^{-1} d\alpha)^2 \end{aligned}$$

and

$$\Omega_V = -d\mu.d\alpha.\alpha^{-1} - \mu\nu(d\alpha.\alpha^{-1})^2.$$

To integrate $c_k^{(d)}(E,D)$ from -1 to $+1$ with respect to t we note that the terms of bidegree $(m, 2k-m)$, $m = 2,3,4,...,2k$, of the k'th power of the curvature trivially vanish; the term of bidegree $(0,2k)$ or $(-1)^k(\mu\nu)^k(\alpha^{-1}d\alpha)^{2k-1}$ is traceless, and there only remains the term of bidegree $(1, 2k-1)$ or $(-1)^{k-1}k(\mu\nu)^{k-1}d\nu(\alpha^{-1}d\alpha)^{2k-1}$. Consequently, all the products of Chern characters vanish in the universal polynomials M_k, and there only remains the term in $\mathrm{Ch}_k^{(d)}(E,D)$ whose coefficient is $(-1)^{k-1}(k-1)!$. We thus compute that

$$\begin{aligned}
-\int_{-1}^{1} \mathrm{ch}_k^{(d)}(E,D) &= -\left(\frac{i}{2\pi}\right)^k \frac{1}{k!}\int_{-1}^{1} \mathrm{Tr}(\Omega^k) \\
&= -\left(\frac{i}{2\pi}\right)^k \frac{1}{k!}\,\mathrm{Tr}\,(\alpha^{-1}d\alpha)^{2k-1}(-1)^{k-1}k\int_{-1}^{1}(\mu\nu)^{k-1}d\mu \\
&= \frac{i^{3k-2}}{(2\pi)^k}\frac{1}{(k-1)!}\,\mathrm{Tr}(\alpha^{-1}d\alpha)^{2k-1}\int_0^1 (\mu(t)-\mu(t)^2)^{k-1}\,d\mu(t) \\
&= \frac{i^{3k-2}}{(2\pi)^k}\frac{1}{(k-1)!}\,\mathrm{Tr}(\alpha^{-1}d\alpha)^{2k-1}\int_0^1 (t-t^2)^{k-1}\,dt \\
&= \frac{i^{3k-2}}{(2\pi)^k}\frac{(k-1)!}{(2k-1)!}\,\mathrm{Tr}(\alpha^{-1}d\alpha)^{2k-1}
\end{aligned}$$

and

$$-\int_{-1}^{1} c_k^{(d)}(E,D) = (-1)^{k-1}(k-1)!\left(-\int_{-1}^{1}\mathrm{ch}_k^{(d)}(E,D)\right);$$

that is, the representative of $(p_k \circ \sigma_1)[\alpha]$.

Square II: For $[\omega] \in \bigoplus_{r=1}^{\infty} H_{dR}^{2r-1}(X)$ we find

$$(\check{C}_k \circ \partial)[\omega] = \check{C}_k[T,d,\omega] - \check{C}_k[T,d,0] = [f] \in H_s^{2k-1}(X;\mathbb{C}^*)$$

where modulo $\mathbb{Z}$

$$f(\lambda) = \int_\lambda \psi_{2k-1} - \theta_{2k-1}(h \circ \gamma) + \theta_{2k-1}(h \circ \gamma) = \int_\lambda \psi_{2k-1}.$$

Now, for ω closed, we see that only the homogenous term will survive in the definition of ψ_{2k-1}, that is,

$$\psi_{2k-1} = (-1)^{k-1}(k-1)!\,\omega_{2k-1}.$$

But

$$f(\lambda) = (-1)^{k-1}k! \int_\lambda \omega_{2k-1}, \quad \lambda \in \Sigma_{2k-1}(X)$$

is exactly the cochain needed for the square II to be commutative.

Square III: One only needs to recall the definition of the Bockstein homomorphism.

Square IV: It is trivial.

We have thus established the main theorem of [7].

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