LIE ALGEBRA OF A FAMILY OF ONE-PARAMETER SUBGROUPS

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1. Introduction

The theory of Lie groups may be viewed essentially as a skill of deriving properties of a Lie group G from the structure of the family $\Lambda(G)$ of all continuous one parameter subgroups of G.

The first part of this procedure is introducing the appropriate real Lie algebra structure on $\Lambda(G)$. Classicaly this is achived by identifying $\Lambda(G)$ with the family of all left invariant vector fields on G and it is possible due to the existence of a C^{∞} Banach manifold structure on G. In a more general situation, when such a manifold structure on G is not available the problem how to associate the proper Lie algebra structure to $\Lambda(G)$ has no obvious solution.

In this note we presente a new, mainly algebraic construction which associates to an arbitrary family Λ of continuous one parameter subgroups of a topological group G a real Lie algebra with gradation. In the first etape Λ is embeded into the group $P(\Lambda)$ formed by all the functions from the real line $\mathbb{R}$ into G which are finite pointwise products of the functions from Λ. The remaining part of the procedure rest only on algebraic and topological properties of $P(\Lambda)$ and may be treated in fact as the construction of the corresponding Lie algebra for an object of the category of "exponential $\mathbb{R}$-groups" - the abstract generalization of $P(\Lambda)$.

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2 Topological $\mathbb{R}$-groups

Let G be a Hausdorff topological group. Let $C_o(\mathbb{R},G)$ denote the family of all continuous G-valued functions on the real line $\mathbb{R}$, satisfying the condition $f(0) = e$. $C_o(\mathbb{R},G)$ with the pointwise multiplication and the compact-open topology is a Hausdorff topological group. It admits also multiplication by real numbers:

(1) $$\mathbb{R} \times C_o(R,G) \ni (s,f) \longrightarrow sf \in C_o(\mathbb{R},G)$$

where

$$(sf)(t)=\begin{cases} f(st) & \text{for } s\geq 0 \\ f(-st)^{-1} & \text{for } s\geq 0 \end{cases}$$

Cleary this multiplication is jointly continuous and it satisfies the following conditions (cf. [8]).

For $s_1,s_2 \in \mathbb{R}$ and $f_1,f_2,f \in C_o(\mathbb{R},G)$

(a) $$s_1(s_2f) = (s_1s_2)f$$

(2) (b) $$s(f_1 \cdot f_2) = (sf_1)(sf_2)$$

(c) $$(-1)f = f^{-1} \ , \ 0f = e$$

where e denotes the constant function equal for each t to the unit element of the group G.

In the abstract setting a topological group H equipped additionally with a binary operation

$$\mathbb{R} \times H \ni (s,t) \longrightarrow sf \in H$$

which is jointly cointinuous and satisfies (2)(a)-(c) (where e denotes the unit element of H) will be called a *topological* $\mathbb{R}$*-group*.

In the obvious way one introduces notions of $\mathbb{R}$*-subgroup*, $\mathbb{R}$*-normal subgroup* and $\mathbb{R}$*-quotient group*.

Let Λ be a family of continuous one parameter subgroups of G. Clearly $\Lambda \subset C_o(\mathbb{R},G)$, and we may consider the $\mathbb{R}$-subgroup $P(\Lambda)$ of $C_o(\mathbb{R},G)$ generated by Λ.

Observe that the elements of Λ are distinguished among those of $P(\Lambda)$ by the property

(3) $$nf = f^n \qquad \text{for } n \in \mathbb{Z}$$

(here after $\mathbb{Z}$ will denote the set of all integers and $\mathbb{N}$ the set of all positive integers).

In the abstract setting we shall call an element f of a topological $\mathbb{R}$-group H *exponential*, if it satisfies the condition (3). We shall call a topological $\mathbb{R}$-group H *exponential* if the set $E(H))$ of all exponential elements of H generates H. Thus $P(\Lambda)$ is an example of an exponential topological $\mathbb{R}$-group. The next proposition suggests that the notion of an exponential $\mathbb{R}$-group may by viewed as a noncommutative generalization of the notion of a topological vector space (cf. [8]).

Proposition 1 *Let H be a topological* $\mathbb{R}$*-group*

(a) $h \in E(H)$ *iff for each* $s_1,s_2 \in \mathbb{R}$

(4) $$(s_1+s_2)h = (s_1h)(s_2h)$$

i.e. iff the function $\mathbb{R} \ni s \longrightarrow sh \in H$ *is a one parameter subgroup of H.*

(b) *H is a linear space iff* $H = E(H)$.

The proof is simple and will be omitted.

3. Lie ring structure associated with a filtred group

Applying standard notation, for arbitrary subsets A and B of a group by $\langle A,B\rangle$ we shall denote the subgroup generated by all the elements $\langle a,b\rangle = a^{-1}b^{-1}a\,b$ with $a \in A$ and $b \in B$

Let H be a topological group. Consider the closed central descending series of H

$$\overline{H}_1 \supset \overline{H}_2 \supset \overline{H}_3 \supset \dots \tag{5}$$

where $H_1 = H$, $H_n = \langle H,H_{n-1}\rangle$ for $n \geq 2$, and $\overline{H}_n$ denotes the closure of H_n. Clearly each $\overline{H}_n$ is a normal subgroup of H and moreover

$$\langle \overline{H}_n,\overline{H}_m\rangle \subset \overline{H}_{n+m}$$

i.e. the sequence (5) is a discrete filtration of H.

It is well known (cf [1], [7] chap. II) that for

$$M_n = \overline{H}_n / \overline{H}_{n+1} \quad \text{and} \quad L(H) = \bigoplus_{n=1}^{\infty} M_n$$

the group $L(H)$ – *the graded group associated to the filtration* (5)– has a structure of Lie ring. This structure is composed of two compatible binary operations: the abelian group multiplication of $L(H)$ and Lie bracket defined as the commun extension of the family of biadditive maps

$$[\cdot,\cdot]_{m,n} : M_m \times M_n \longrightarrow M_{m+n} \qquad m,n = 1,2\dots$$

where $[a,b]_{m,n}$ is defined as the quotient class of $\langle a,b\rangle$ in M_{m+n} for $a \in \overline{H}_m$ and $b \in \overline{H}_n$.

The main point of the present note is the observation that for H a topological exponential $\mathbb{R}$-grpup the Lie ring structure of $L(H)$ may be compatiblly completed by one more operation: a multiplication by real numbers, yielding thus structure of a real Lie algebra on $L(H)$. It should be noted, however, that the $\mathbb{R}$-group structure of H is almost irrelevant here, what realy matters are the exponential properties of H.

4. The Lie algebra of an exponential $\mathbb{R}$-group.

Let H be new a topological exponential $\mathbb{R}$-group. Consider the closed central filtration (5) of H. Since $\overline{H}_n$ are closed, each M_n is a topological $\mathbb{R}$-group, so is $L(H)$. We shall modify its structure, defining the new multiplication "*" by real numbers in M_n by the formula

$$s * (a\bar{H}_{n+1}) = \varkappa_n(s)\cdot(a\bar{H}_{n+1}) \tag{6}$$

where the multiplication on the right hand side of the above equality is the quotient multiplication in M_n and

$$\varkappa_n(s) = (\operatorname{sgn} s)\cdot\sqrt[n]{|s|}$$

for $s \in \mathbb{R}$. Since $\varkappa_n(0) = 0$ and the restriction of $\varkappa_n$ to $\mathbb{R}\setminus\{0\}$ is a continuous automorphism of the multiplicative group $\mathbb{R}\setminus\{0\}$, the multiplication (6) satisfies the conditions (2)(a)-(c). Define finally the new multiplication in $L(H)$ modifying coordinatewise the old one acording to (6).

Theorem 2 *The new multiplication is compatible with the Lie ring structure of* $L(H)$.

The proof is based on some properties of exponential $\mathbb{R}$-groups (cf. [8], Proposition 7).

Proposition 3 *Let H be an exponential $\mathbb{R}$-group and let* $h \in \bar{H}_n$. *Then for each* $k \in \mathbb{N}$

$$kh = h^{k^n} \mod \bar{H}_{n+1} \tag{7}$$

Proof. Observe first, that since the both sides of (7) depend continuously on h and $\bar{H}_{n+1}$ is closed, with no loss of generality we may assume that $h \in H_n$.

Note also that for $a,b \in H_n$ and $q \in \mathbb{N}$

$$a^q b^q = (ab)^q \mod H_{n+1} \tag{8}$$

hence proving (7) we may restrict our attention to the elements of the form $h_m = \{a_1,\dots,a_m\}$, i.e.

$$h_m = \{a_1,\{a_2,\{\dots,\{a_{m-1},a_m\}\dots\} \tag{9}$$

with $a_i \in E(H)$ $i = 1,\dots,m$.

We shall prove (7) for the elements (9) by induction on m.

If $m = 1$, then $h = a_1$ so $h \in E(H)$ and $kh = h^k$. Assume that $h_n = \{a_1,h_{n-1}\}$ where $h_{n-1} \in H_{n-1}$ and h_{n-1} has the form (9) with $m = n-1$. By induction hypothesis

$$kh_{n-1} = h_{n-1}^{k^{n-1}}\cdot r_n$$

where $r_n \in \bar{H}_n$. Observe that for elements b,c,d such that at least one of the three belongs to H_{n-1} we have

$$\{b,cd\} = \{b,c\}\{b,d\} \mod H_{n+1}$$

and thus

$$kh_n = (ka_1, kh_{n-1}) = \left\{a_1^k, h_{n-1}^{k^{n-1}} \cdot r_n\right\} = \left\{a_1^k, h_{n-1}^{k^{n-1}}\right\} =$$
$$= (a_1^k, h_{n-1})^{k^{n-1}} = \left[(a_1, h_{n-1})^k\right]^{k^{n-1}} = h_n^{k^n}$$

where $r_n \in \overline{H}_n$ and all the equalities are understood modulo $\overline{H}_{n+1}$.■

Proposition 4 *Let* h *be an element of a topological* $\mathbb{R}$*-group such that for a fixed* $n \in \mathbb{N}$ *and each* $k \in \mathbb{N}$

$$k^n h = h^{k^n} \tag{10}$$

Then h *is exponential.*

Proof. We shall prove first that for h satisfying (10) and sequences $\{p_k\}_{k=1}^{\infty}$, $\{q_k\}_{k=1}^{\infty}$ of positive integers with

$$\lim_{k\to\infty} q_k^{-1} p_k = 0$$

it holds

$$(11)\quad \begin{aligned} &(a) \quad \lim_{k\to\infty} q_k^{-1}\cdot h^{p_k} = e \\ &(b) \quad \lim_{k\to\infty} q_k^{-1}\cdot h^{q_k} = h \end{aligned}$$

To prove (11)(a) observe that by theorem of Waring (cf. [4], [5]) there exists $s \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ the numbers $r_{k,1},\dots,r_{k,s} \in \mathbb{N} \cup \{0\}$ may be chosen in such a way that

$$p_k = r_{k,1}^n + r_{k,2}^n + \dots + r_{k,s}^n .$$

Thus

$$(12)\quad \begin{aligned} q_k^{-1}\cdot h^{p_k} &= q_k^{-1}\cdot\left[h^{r_{k,1}^n}\cdot h^{r_{k,2}^n}\cdot \dots \cdot h^{r_{k,s}^n}\right] = \\ &= \prod_{i=1}^{s}(q_k^{-1}\cdot r_{k,i}^n)\cdot h \end{aligned}$$

Cleary $\lim_{k\to\infty} q_k^{-1}\cdot r_{k,i} = 0$ for $i = 1,\dots,s$ hence each factor of the right hand side of (12) tends to e.

To prove (11)(b) for a given sequence $\{q_k\}_{k=1}^{\infty}$ of positive integers satisfying $\lim_{k\to\infty} q_k = +\infty$, let the sequence $\{r_k\}_{k=1}^{\infty}$ of positive integers be defined by the inequalities

$$r_k^n \le q_k < (r_k+1)^n \qquad k = 1,2,\dots$$

Then

$$\lim_{k\to\infty} q_k^{-1}(r_k+1)^n = 1$$

and

$$\lim_{k\to\infty} q_k^{-1}((r_k+1)^n - q_k) = 0$$

Hence applying (11)(a) we get

$$\lim_{k\to\infty} q_k^{-1}\cdot h^{q_k} = \lim_{k\to\infty} q_k^{-1}\cdot h^{q_k} \cdot \lim_{k\to\infty} q_k^{-1}\cdot h^{(r_k+1)^n-q_k} =$$

$$= \lim_{k\to\infty} q_k^{-1}\cdot h^{(r_k+1)^n} = \lim_{k\to\infty} q_k^{-1}(r_k+1)^n\cdot h = h$$

This proves (11)(b).

Let now $m \in \mathbb{N}$. By (11)(b)

$$mh = m\cdot \lim_{k\to\infty} (km)^{-1}h^{km} = \lim_{k\to\infty} k^{-1}h^{km} =$$

$$= \left[\lim_{k\to\infty} k^{-1}h^{k}\right]^{m} = h^{m}$$

What was to be proved. ■

Corollary 5 *$(M_n,*)$ is a linear topological space.*

Proof. Clearly $(M_n,*)$ is a topological $\mathbb{R}$-group, and (7) implies that each element of M_n satisfies the condition (10).Hence by Proposition 4 and Proposition 1 $(M_n,*)$ is a linear topological space.■

Proof of Theorem 2. Since $L(H)$ is a direct sum of $\mathbb{R}$-groups and each coordinate of this sum is by Corollary 5 a linear topological space also $L(H)$ is.

The Lie bracket being additive with respect to this linear structure, it satisfies

(13) $$[\ \lambda*a\ ,\ b\ ] = \lambda * [a,b]$$

for each rational λ. The continuity of the multiplication $*$ and the Lie product $[\cdot,\cdot]$ yield (13) for each $\lambda \in \mathbb{R}$. The homogeneity with respect to the second argument results from antisymmetry of the Lie product. This completes the proof.■

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