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UNIVERSAL ENVELOPING ALGEBRAS AND QUANTIZATION

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The basic algebraic structures of classical mechanics are the algebra $V=C^{\infty}(N)$ of smooth functions on the phase space N under ordinary multiplication and the Lie structure on V induced by the Poisson bracket $\{\ ,\ \}$ defined by the symplectic form ω on N .
In canonical coordinates $(q_1,\dots,q_n,p_1,\dots,p_n)$ we have

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i \quad \text{and} \quad \{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right) \quad .$$

In [1] F.Bayen,M.Flato et al. have attempted to study quantization as a deformation of classical mechanics.One then does not define quantum mechanics in terms of operators but in terms of deformations of the usual multiplication and the Poisson bracket of functions on the phase space.The appropriate deformation of the associative algebra structure on V is called a *star-product* .
Since one considers nowadays more general Poisson structures than those on symplectic manifolds (e.g. A.A.Kirollov [7] or A.Lichnerowicz [8]), the natural question is to describe star-products for a given Poisson structure.

Our aim in this note is to show that the universal enveloping algebra can be obtained in this way.
Let us start with some basic notions.

(1.1)**Definition.** *A Poisson structure* on a manifold N is a Lie bracket $\{\ ,\ \}$ on the algebra $V=C^{\infty}(N)$ given by a bilinear derivative,i.e. satisfying

$$\{f,gh\}=\{f,g\}h+\{f,h\}g \ .$$

(1.2)**Remark.** Every Poisson structure is given by a tensor field $P\in\Gamma(\Lambda^2 TN)$ satisfying the identity $[P,P]=0$ for $[\ ,\ ]$ being so called Schouten-Nijenhuis bracket.The Poisson bracket is then defined by $\{f,g\}=P(f,g)$.

Observe now that every such a Poisson structure defines a formal deformation of the algebra V of rank 1 .
To be precise consider the space V_ε of all formal power series $\sum_{k=0}^{\infty} \varepsilon^k x_k$ in ε with coefficients in V .
The two-$\mathbb{k}[[\varepsilon]]$-linear operator $m: V_\varepsilon \times V_\varepsilon \longrightarrow V_\varepsilon$ defined on V (naturally embedded in V_ε) by

$$m(x,y) = xy + \varepsilon P(x,y) \quad , \ x,y\in V \ ,$$

is associative up to the rank 1 ,i.e.

$$m(m(x,y),z) = m(x,m(y,z)) \quad (\mathrm{mod}(\varepsilon^2)) \ ,$$

since P is a two-linear derivation.

(1.3)**Definition.** A *star-product* for a Poisson structure P on an associative commutative algebra (V,m_o) with unit $\underline{1}$ is a formal deformation $(V_\varepsilon,m_\varepsilon)$ of (V,m_o) (i.e. m_ε is a two-$\mathbb{k}[[\varepsilon]]$-linear associative product on V_ε) such that

1) $\quad m_\varepsilon(x,y) = m_o(x,y) + \varepsilon P(x,y) + \sum_{k=2}^{\infty} \varepsilon^k A_k(x,y)$ for $x,y\in V$,

2) $\quad A_k$ is a bilinear differential operator on V ,

3) $\quad A_k(x,y) = (-1)^k A_k(x,y)$ for $x,y\in V$,

4) $\quad A_k$ vanishes on constants

for $k=2,3,\ldots$.

Note that the assumption 4) in the above definition assures that $\underline{1}$ remains the unit in the algebra $(V_\varepsilon,m_\varepsilon)$.
The investigation of the existence of star-products for a given Poisson structure usually leads to difficult questions concerning the Hochschild and Chevalley cohomology (see e.g. [2],[3]).
A significant example of a Poisson structure which does not come from a symplectic one is the canonical Kirillov-Souriau Poisson structure on the dual $\mathfrak{L}^*$ of a Lie algebra $\mathfrak{L}$.

Regarding elements from $\mathfrak{L}$ as functionals on $\mathfrak{L}^*$ we can write

$$P(x,y) = [x,y] \quad ,$$

where [,] is the Lie bracket in $\mathfrak{L}$.
Since any Poisson structure on $C^\infty(\mathfrak{L}^*)$ is a bilinear derivation, it is completely described by the action on functionals.
In local coordinates $x_1,\dots,x_n \in \mathfrak{L}$ on $\mathfrak{L}^*$ we can write

$$2P = \sum_{i,j} [x_i,x_j]\partial_i \wedge \partial_j \quad ,$$

where $\partial_i = \partial/\partial_{x_i}$.

S.Gutt [6] observed that a star-product for P is actually given by the multiplication in the universal enveloping algebra $\mathfrak{U}=\mathfrak{U}(\mathfrak{L})$, or in other words, that the universal enveloping algebra is in fact a star-product (quantization) of P .
V.G.Drinfel'd used in [5] a direct formula for this star-product without mentioned it explicitly. Since the formula seems not to be widespread ,we would like to present it together with a very short proof .

Let $x_1,\dots,x_m$ be a basis of a Lie algebra $\mathfrak{L}$ .The symmetric algebra $S=S(\mathfrak{L})$ can be then naturally identified with the algebra of polynomials on $\mathfrak{L}^*$ and thus regarded as embedded in $C^\infty(\mathfrak{L}^*)$.
On the other hand, S and $\mathfrak{U}$ are naturally isomorphic as vector spaces via the symmetrization mapping.
Let us add a free formal parameter ε putting the Lie bracket in $\mathfrak{L}_\varepsilon$ to be ε[,] .We have the multiplication * on $\mathfrak{U}_\varepsilon$ which can be understood as an associative structure on S_ε (c.f.[4], Ch.2.8).

(1.4)**Theorem** (Gutt,Drinfel'd).*$(S_\varepsilon, *)$ is a star-product for the canonical Poisson structure on the dual $\mathfrak{L}^*$.*
*The multiplication * can be written explicitly in the form*

$$f*g = fg +$$

$$+\sum_{n=1}^{\infty} 1/n! \sum_{|\alpha_i|+|\beta_i|>1} c_{\alpha_1\beta_1}\cdots c_{\alpha_n\beta_n}\,\varepsilon^{\Sigma(|\alpha_i|+|\beta_i|)-n}\,\partial^{\Sigma\alpha_i}(f)\,\partial^{\Sigma\beta_i}(g),$$

where for multiindices $\alpha_i=\alpha=(\alpha^1,\ldots,\alpha^m),\beta_i=\beta=(\beta^1,\ldots,\beta^m)$ *the functional* $c_{\alpha\beta}$ *as an element from* $\mathfrak{L}$ *is the coefficient in the Campbell-Baker-Hausdorff series*

$CH(\sum t_k x_k,\sum s_j x_j)=\sum t_k x_k + \sum s_j x_j + (\varepsilon/2)\sum t_k s_j [x_k,x_j] + \ldots$

standing by $t^\alpha s^\beta = t_1^{\alpha^1}\ldots t_m^{\alpha^m} s_1^{\beta^1}\ldots s_m^{\beta^m}$ *and* ∂^α *denotes* $\partial_1^{\alpha^1}\ldots\partial_m^{\alpha^m}$.

In a more transparent form

$$f*g = \exp((CH(a,b)-a-b)/\varepsilon)(f\otimes g)\ ,$$

where $a=\sum x_k\otimes\partial_k\otimes id$, $b=\sum x_j\otimes id\otimes\partial_j$ *,Lie brackets,and the exponential are from the associative algebra* $\mathfrak{U}_\varepsilon\otimes Diff\otimes Diff$ *for* $Diff$ *being the associative algebra of differential operators on* $\mathfrak{L}^*$ *with constant coefficients and*

$$u\otimes A\otimes B \in \mathfrak{U}\otimes Diff\otimes Diff \approx S\otimes Diff\otimes Diff \subset C^\infty(\mathfrak{L}^*)\otimes Diff\otimes Diff$$

acts on $f\otimes g$ *by* $u\otimes A\otimes B(f\otimes g)=uA(f)B(g)$.

<u>Proof.</u> Let x and y be elements of a (e.g. free) Lie algebra $\mathfrak{L}_\varepsilon$ with the bracket $\varepsilon[\ ,\ ]$.In the universal enveloping algebra $(\mathfrak{U}_\varepsilon,*)$ we can write

$$e^{tx}*e^{sy}=e^{CH(tx,sy)}\ ,$$

where $CH(tx,sy)=\sum t^\alpha s^\beta c_{\alpha\beta}$ is the Campbell-Baker-Hausdorff series.

We have $e^{tx}*e^{sy}=\sum(1/l!h!)t^l s^h x^{*l}*y^{*h}$,and since $x^{*l}=x^l$, $y^{*h}=y^h$ are symmetric we can get the symmetric form of x^l*y^h (i.e.on the level of the symmetric algebra S_ε) just looking at the coefficient by $t^l s^h$ on the right-hand side (which is clearly symmetric).

In result

$$x^l*y^h=l!h!\sum_{n=1}^{\infty}(1/n!)\sum_{\substack{\Sigma\alpha_k=l\\ \Sigma\beta_k=h}} c_{\alpha_1\beta_1}\cdots c_{\alpha_n\beta_n}\varepsilon^{l+h-n}\ .$$

Since $c_{10}=x$ and $c_{01}=y$,it is easy to check that the right-hand term equals

$$x^l y^h + \sum_{n=1}^{\infty} (1/n!) \sum_{\substack{|\alpha_k|>0 \\ |\beta_k|>0}} c_{\alpha_1\beta_1} \cdots c_{\alpha_n\beta_n} \varepsilon^{\Sigma(\alpha_k+\beta_k)-n} \partial_x^{\Sigma\alpha_k}(x^l) \partial^{\Sigma\beta_k}(y^h) \ .$$

Putting $\Sigma t_k x_k$ instead of tx and $\Sigma s_j x_j$ instead of sy and passing to multiindices we get the desired formula. ■

The first terms in the formula are the following

$$f*g=fg+$$
$$(\varepsilon/2)\sum[x_i,x_j]\partial_i(f)\partial_j(g)+(\varepsilon^2/8)\sum[x_i,x_j][x_k,x_l]\partial_i\partial_k(f)\partial_j\partial_l(g) \quad +$$
$$(\varepsilon^2/12)\sum[x_k,[x_j,x_i]](\partial_k\partial_j(f)\partial_i(g)+\partial_i(f)\partial_k\partial_j(g)) \quad +$$
$$o(\varepsilon^2)=$$
$$=fg+(\varepsilon/2)\sum c^n_{ij}x_n\partial_i(f)\partial_j(g)+(\varepsilon^2/8)\sum c^n_{ij}c^h_{kl}x_nx_h\partial_i\partial_k(f)\partial_j\partial_l(g)+$$
$$+ \ (\varepsilon^2/12)\sum c^n_{kl}c^l_{ji}x_n(\partial_k\partial_j(f)\partial_i(g)+\partial_i(f)\partial_k\partial_j(g))+ \ o(\varepsilon^2) \ ,$$

where c^n_{ij} are the structure constants of $\mathfrak{L}$.

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