

Jan Kubarski

## Characteristic classes of regular Lie algebroids – a sketch

In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1993. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 30. pp. [71]–94.

Persistent URL: <http://dml.cz/dmlcz/701508>

### Terms of use:

© Circolo Matematico di Palermo, 1993

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

# CHARACTERISTIC CLASSES OF REGULAR LIE ALGEBROIDS - A SKETCH

Jan Kubarski

The notion of a Lie algebroid comes from J. Pradines (1967) [21], [22], and was invented in connection with the study of differential groupoids. This notion plays an analogous role as the Lie algebra of a Lie group. Observations concerning characteristic homomorphisms on the ground of principal bundles (such as the Chern-Weil homomorphism, the homomorphism of a flat or a partially flat principal bundle) show that they depend only on the Lie algebroids of these principal bundles [12], [13], [14]. This enables us to build a theory of characteristic classes for Lie algebroids and, next, to apply this technique to the investigation of some geometric structures defined on objects not being principal bundles but possessing Lie algebroids, such as transversally complete foliations [18], [19], nonclosed Lie subgroups [11], [19], Poisson manifolds [2] or complete closed pseudogroups [23].

## FUNDAMENTAL DEFINITIONS AND EXAMPLES

We begin with fundamental definitions.

**Definition 1.** By a Lie algebroid on a manifold  $M$  we mean a system

$$A = (A, [\cdot, \cdot], \gamma)$$

---

*This paper is in final form and no version of it will be submitted for publication elsewhere*

consisting of a vector bundle  $p:A \longrightarrow M$  and mappings

$$[\cdot, \cdot]: \text{Sec} A \times \text{Sec} A \longrightarrow \text{Sec} A, \quad \gamma: A \longrightarrow TM,$$

such that

(1)  $(\text{Sec} A, [\cdot, \cdot])$  is an  $\mathbb{R}$ -Lie algebra,

(2)  $\gamma$  is a homomorphism of vector bundles (called an *anchor*),

(3)  $\text{Sec} \gamma: \text{Sec} A \longrightarrow \mathfrak{X}(M), \xi \longmapsto \gamma \circ \xi$ , is a homomorphism of Lie algebras,

$$(4) [\xi, f \cdot \eta] = f \cdot [\xi, \eta] + (\gamma \circ \xi)(f) \cdot \eta, \quad \xi, \eta \in \text{Sec} A, f \in C^\infty(M).$$

[ $\text{Sec} A$  denotes the vector space of global  $C^\infty$  cross-sections of a vector bundle  $A$ ].

A Lie algebroid  $A$  is said to be *transitive* if  $\gamma$  is an epimorphism of vector bundles, and *regular* if  $\gamma$  is a constant rank. In this last case,  $F := \text{Im} \gamma$  is a  $C^\infty$  constant dimensional and completely integrable distribution and such a Lie algebroid is called a *regular Lie algebroid over  $(M, F)$* .

By a (*strong*) *homomorphism*

$$H: (A, [\cdot, \cdot], \gamma) \longrightarrow (A', [\cdot, \cdot]', \gamma')$$

between two Lie algebroids on the same manifold  $M$  we mean a strong homomorphism  $H: A \longrightarrow A'$  of vector bundles, such that

(a)  $\gamma' \circ H = \gamma$ ,

(b)  $\text{Sec} \gamma: \text{Sec} A \longrightarrow \text{Sec} A', \xi \longmapsto H \circ \xi$ , is a homomorphism of Lie algebras.

**Examples 2.** (1) A *finitely dimensional Lie algebra  $\mathfrak{g}$*  forms a Lie algebroid on a one-point manifold.

(2) The *tangent bundle  $TM$*  to a manifold  $M$  forms a Lie algebroid  $(TM, [\cdot, \cdot], \text{id})$  with the bracket  $[\cdot, \cdot]$  of vector fields.

(3) Any *involutive  $C^\infty$  constant dimensional distribution  $F \subset TM$*  forms a regular Lie algebroid with the bracket as above.

(4) Any  *$G$ -principal bundle  $(P, \pi, M, G, \cdot)$*  determines a transitive Lie algebroid  $A(P) = (A(P), [\cdot, \cdot], \gamma)$  [7], [10], [17], in which

(a)  $A(P) := TP/G$ ,

(b) the bracket is defined in such a way that the canonical isomorphism  $\text{Sec} A(P) \xrightarrow{\cong} \mathfrak{X}^R(P)$  is an isomorphism of

Lie algebras, where  $\mathfrak{X}^R(P)$  is the Lie algebra of right-invariant vector fields on  $P$ ,

(c) the anchor  $\gamma$  is given by  $\gamma([v]) = \pi_*(v)$ .

(5) Any vector bundle  $f$  determine a transitive Lie algebroid  $A(f)$  equal to  $A(Lf)$  - the Lie algebroid of the principal bundle  $Lf$  of repers of  $f$ .

**Proposition 3.** [12] *Let  $f$  be any vector bundle on a manifold  $M$ . For a point  $x \in M$ , there exists a natural isomorphism*

$$A(f)_{|_x} \cong \left\{ l: \text{Sec}f \rightarrow f_{|_x}; l \text{ is linear and} \right. \\ \left. \exists v \in T_x M, \forall f \in \Omega^0(M), \forall v \in \text{Sec}f, l(f \cdot v) = f(x) \cdot l(v) + v(f) \cdot v(x) \right\}$$

Therefore we have a (canonical) isomorphism of  $\Omega^0(M)$ -modules

$$\text{Sec}A(f) \xrightarrow{\cong} \left\{ \begin{array}{l} \text{differential operators } \mathcal{L}: \text{Sec}f \rightarrow \text{Sec}f \\ \text{such that } \mathcal{L}(f \cdot v) = f \cdot \mathcal{L}(v) + X(f) \cdot v \\ \text{for some } X \in \mathfrak{X}(M), \end{array} \right\} . \blacksquare$$

Let, in the sequel,  $\mathcal{L}_\xi$  denote the (covariant) differential operator corresponding to the cross-section  $\xi$  of  $\text{Sec}A(f)$ .

(6) Any transversally complete foliation  $(M, \mathcal{F})$  determines a transitive Lie algebroid  $A(M, \mathcal{F}) = (A(M, \mathcal{F}), [\cdot, \cdot], \gamma)$  in the following way [18], [19]: the closure of the leaves of  $\mathcal{F}$  form another foliation  $\mathcal{F}_b$ , called *basic*, being a simple one determined by some fibration  $\pi_b: M \rightarrow W$  with a Hausdorff manifold  $W$ , called the *basic fibration*. Let  $E$  and  $E_b$  denote the vector bundles tangent to  $\mathcal{F}$  and  $\mathcal{F}_b$ , respectively,  $Q = TM/E \xrightarrow{r} M$  - the transversal bundle of  $\mathcal{F}$ ,  $\mathfrak{l}(M, \mathcal{F})$  - the Lie algebra of the so-called *transversal fields* being cross-sections of  $Q$  determined by foliate vector fields.

**Lemma 4.** *If  $\pi_b(x) = \pi_b(y)$  then there exists a (canonical) isomorphism  $\alpha_x^y: Q_{|_x} \rightarrow Q_{|_y}$  having the property:  $\alpha_x^y(\zeta_x) = \zeta_y$  for any transversal field  $\zeta \in \mathfrak{l}(M, \mathcal{F})$ . ■*

The construction of the Lie algebroid  $A(M, \mathcal{F})$ :

The space  $A(M, \mathcal{F}) := Q/\approx$  where for  $\bar{v}, \bar{w} \in Q$  we define

$$\bar{v} \approx \bar{w} \Leftrightarrow \pi_b(r(\bar{v})) = \pi_b(r(\bar{w})) \ \& \ \alpha_x^y(\bar{v}) = \bar{w}.$$

The bracket  $[\cdot, \cdot]$  in  $\text{Sec}A(M, \mathcal{F})$  is defined in such a way that the canonical isomorphism  $\text{Sec}A(M, \mathcal{F}) \xrightarrow{\cong} \mathfrak{L}(M, \mathcal{F})$  is an isomorphism of Lie algebras.

The anchor  $\gamma: A(M, \mathcal{F}) \rightarrow TM$  is equal to  $\gamma([\bar{v}]) = \pi_{b*}(v)$ .

(7) Any not necessarily closed connected Lie subgroup  $H$  of a Lie group  $G$  determines a transitive Lie algebroid  $A(G; H)$  on  $G/H$  defined as the Lie algebroid of the foliation  $\mathcal{F} = \{gH; g \in G\}$  of left cosets of  $G$  by  $H$  (such a foliation is, of course, transversally complete). If  $H = \bar{H}$ , then the Lie algebroid  $A(G; H)$  is trivial:  $A(G; H) = T(G/H)$ .

Lemma 5. [11], [12] For any  $t \in \bar{H}$ , the mapping  $R_t: TG \rightarrow TG$ , tangent to the right translation by  $t$ , maps  $E$  onto  $E$  giving an isomorphism  $\bar{R}_t: Q \rightarrow Q$ . The mapping  $Q \times \bar{H} \rightarrow Q$ ,  $(\bar{v}, t) \mapsto \bar{R}_t(\bar{v})$ , is a right free action. ■

Lemma 6. [11], [12] (a) A cross-section  $\zeta \in \text{Sec}Q$  is a transversal field if and only if  $\zeta(gt) = \bar{R}_t(\zeta(g))$  for all  $g \in G$  and  $t \in \bar{H}$ .

(b) The natural equivalence relation  $\approx$  in  $Q$  can be equivalently defined as follows: for  $\bar{v}, \bar{w} \in Q$ ,

$$\bar{v} \approx \bar{w} \Leftrightarrow \exists t \in \bar{H}, \bar{R}_t(\bar{v}) = \bar{w}.$$

This means that  $A(G; H)$  can be defined as the space of orbits of the right action of  $\bar{H}$  on  $Q$ . ■

(8) Any transitive Lie algebroid  $(A, [\cdot, \cdot], \gamma)$  on  $M$  and an involutive distribution  $F \subset TM$  form a regular Lie algebroid  $(A^F, [\cdot, \cdot], \gamma^F)$  such that  $A^F := \gamma^{-1}[F] \subset A$  and  $\gamma^F = \gamma|_{A^F}$ . For example, such an object is determined by a vector bundle and an involutive distribution on the base.

(9) Any Lie groupoid  $\mathfrak{G}$  determines a transitive Lie algebroid  $i^*T^{\alpha}\mathfrak{G}$  [6], [17], [22], whereas any differential groupoid  $\mathfrak{G}$  determines a Lie algebroid in the same way, sometimes being regular [16].

**Remark 7.** A transitive Lie algebroid  $A$  is called *integrable* if it is isomorphic to the Lie algebroid  $A(P)$  of some principal bundle. There exist nonintegrable transitive Lie algebroids discovered by Almeida and Molino in 1985 [1].

**Theorem 8** (Almeida-Molino [1]). *Let  $(M, \mathcal{F})$  be any transversally complete foliation. Then the Lie algebroid  $A(M, \mathcal{F})$  is integrable if and only if the foliation  $(M, \mathcal{F})$  is developable in the sense that the lifting to some covering is simple. ■*

It is evident that any TC-foliation with nonclosed leaves on a simply connected manifold is not developable, therefore its Lie algebroid is not integrable. A more concrete example is the foliation of left cosets of any connected and simply connected Lie group by a Lie subgroup connected and dense in some torus.

CONNECTIONS IN REGULAR LIE ALGEBROIDS

Let  $(A, [\cdot, \cdot], \gamma)$  be any regular Lie algebroid over  $(M, F)$ .  $\mathfrak{g} := \text{Ker} \gamma \subset A$  is a vector bundle. Each fibre  $\mathfrak{g}_x$  of  $\mathfrak{g}$  possesses a structure of a Lie algebra, and  $\mathfrak{g}_x$  is isomorphic to  $\mathfrak{g}_y$  if  $x$  and  $y$  lay on the same leaf of the foliation determined by  $F$ . The short sequence

$$0 \longrightarrow \mathfrak{g} \longleftarrow A \xrightarrow{\gamma} F \longrightarrow 0$$

is called the *Atiyah sequence* of  $A$ . Any splitting  $\lambda: F \longrightarrow A$  of this sequence is called a *connection* in  $A$ .  $\lambda$  determines the so-called *connection form*  $\omega: A \longrightarrow \mathfrak{g}$  as follows:  $\omega|_{\mathfrak{g}} = \text{id}$  and  $\omega \circ \text{Im} \lambda = 0$ , and the *curvature form*  $\Omega \in \text{Sec} \Lambda^2 A^* \otimes \mathfrak{g}$  as a  $\mathfrak{g}$ -horizontal form such that  $\Omega(\lambda X, \lambda Y) = \lambda[X, Y] - [\lambda X, \lambda Y]$  for  $X, Y \in \text{Sec} F$ . It is also convenient to define the so-called *curvature tensor*  $\Omega_b \in \text{Sec} \Lambda^2 F^* \otimes \mathfrak{g}$  (being a tangential differential form [20]) in such a way that  $\Omega_b(X, Y) = \Omega(\lambda X, \lambda Y) (= \lambda[X, Y] - [\lambda X, \lambda Y])$ .  $\lambda$  is said to be a *flat connection* if  $\Omega = 0$  (equivalently,  $\Omega_b = 0$ ).

**Theorem 9.** *If  $A = A(P)$ ,  $P$  being a principal bundle, then*

there is a bijection between connections in  $A$  and in  $P$ . ■

**Theorem 10.** If  $A = A(P)^F$ ,  $P$  being a principal bundle on a manifold  $M$  and  $F$  an involutive distribution on  $M$ , then there is a bijection between connections in  $A$  and partial connections in  $P$  over the distribution  $F$ . ■

**Theorem 11.** [12] If  $A = A(M, \mathcal{F})$ ,  $(M, \mathcal{F})$  being a transversally complete foliation, then there is a bijection between connections in  $A$  and  $C^\infty$  distributions  $C \subset TM$  fulfilling the conditions

- (1)  $C + E_b = TM$ ,
- (2)  $C \cap E_b = E$ ,
- (3)  $C_{ix} = \left\{ X(x); X \in \text{Sec } C \cap L(M, \mathcal{F}) \right\}$  for  $x \in M$ .

[in the case of left cosets of  $G$  by  $H$ , see Example 7 above, condition (3) is equivalent to:

- (3')  $C$  is  $\bar{H}$ -right-invariant].

In particular, such a distribution  $C$  always exists.

A connection in  $A$  is flat if and only if the corresponding distribution in  $TM$  is completely integrable. ■

#### THE CHERN-WEIL HOMOMORPHISM OF A REGULAR LIE ALGEBROID

By a representation of a Lie algebroid  $A$  on a vector bundle  $f$  (both over the same manifold  $M$ ) we mean a homomorphism  $T: A \rightarrow A(f)$  of Lie algebroids. A cross-section  $\nu \in \text{Sec } f$  is said to be  $T$ -invariant if, for each  $\xi \in \text{Sec } A$ ,  $\mathcal{L}_{T \circ \xi}(\nu) = 0$  ( $\mathcal{L}_{T \circ \xi}$  is the differential operator in  $f$  corresponding to the cross-section  $T \circ \xi$ , see example 5 of Lie algebroids). Denote by  $(\text{Sec } f)_{f, (T)}$  the space of all  $T$ -invariant cross-sections of  $f$ . A representation  $T$  induces a representation of  $A$  on each vector bundle associated with  $f$ .

**Theorem 12.** If  $A$  is a transitive Lie algebroid, then each  $T$ -invariant cross-section of  $f$  is uniquely determined by its value at one arbitrarily taken point of  $M$  ( $M$  is assumed to be connected). ■

**Example 13.** The *adjoint representation* of a regular Lie algebroid  $A = (A, [\cdot, \cdot], \gamma)$  over  $(M, F)$  is the representation  $ad_A: A \longrightarrow A(\mathfrak{g})$  defined by

$$\mathcal{L}_{ad_A \circ \xi}(\nu) = [[\xi, \nu]].$$

$ad_A$  induces a representation, denoted also by  $ad_A$ , of  $A$  on the symmetric power  $V^k \mathfrak{g}^*$  of the vector bundle dual to  $\mathfrak{g}$ , and we have:

$$\Gamma \in (Sec V^k \mathfrak{g}^*)_{I^0(ad_A)} \Leftrightarrow \forall \xi \in Sec A, \forall \sigma_1, \dots, \sigma_k \in Sec \mathfrak{g},$$

$$(\gamma \circ \xi) \langle \Gamma, \sigma_1 \vee \dots \vee \sigma_k \rangle = \sum_j \langle \Gamma, \sigma_1 \vee \dots \vee [[\xi, \sigma_j] \vee \dots \vee \sigma_k \rangle.$$

The space  ${}^{k \geq 0} \bigoplus (Sec V^k \mathfrak{g}^*)_{I^0(ad_A)}$  forms an algebra.

Put  $\Omega_F(M) = Sec \wedge F^* (= {}^{k \geq 0} \bigoplus Sec \Omega_F^k(M)$  where  $\Omega_F^k(M) = Sec \wedge^k F^*)$ . This is the space of *real tangential differential forms* [20]. In the space  $\Omega_F(M)$  there works a differential  $\delta^F$  defined by the same formula as for usual differential forms. Let  $H_F(M)$  denote the space of cohomology of the complex  $(\Omega_F(M), \delta^F)$ .

**Theorem 14.** [12] *Let  $\lambda$  be any connection in  $A$  and  $\Omega_b$  its curvature tensor. Define the mapping*

$$\beta: {}^{k \geq 0} \bigoplus (Sec V^k \mathfrak{g}^*)_{I^0(ad_A)} \longrightarrow \Omega_F(M)$$

$$\Gamma \longmapsto \frac{1}{k!} \cdot \langle \Gamma, \underbrace{\Omega_b \vee \dots \vee \Omega_b}_k \rangle$$

(being a homomorphism of algebras).

Then

- (1) the tangential forms  $\beta(\Gamma)$  are closed,
- (2) the induced homomorphism of algebras

$$h_A: {}^{k \geq 0} \bigoplus (Sec V^k \mathfrak{g}^*)_{I^0(ad_A)} \longrightarrow H_F(M)$$

$$\Gamma \longmapsto [\beta(\Gamma)]$$

is independent of the choice of a connection. ■

$h_A$  is called the *Chern-Weil homomorphism* of  $A$ , whereas the

subalgebra  $Pont(A) := Im(h_A) \subset H_F(M)$  - the Pontryagin algebra of  $A$ .

**Theorem 15** (*The comparison with the Chern-Weil homomorphism of a principal bundle [3]*). *If  $A=A(P)$ ,  $P$  being any connected principal bundle, then there exists an isomorphism  $\alpha$  of algebras, making the following diagram commute*

$$\begin{array}{ccc}
 \bigoplus_{k \geq 0} (Sec V^k \mathfrak{g}^*)_{I^{\circ}(ad_A)} & \xrightarrow{h_{A(P)}} & H_{dR}(M) \\
 \cong \downarrow \alpha & & \nearrow h_P \\
 (V\mathfrak{g}^*)_I & & 
 \end{array}$$

where  $\mathfrak{g}$  is the Lie algebra of the structure Lie group  $G$ , and  $(V\mathfrak{g}^*)_I$  is the space of  $Ad$ -invariant polynomials. ■

We pay our attention to the fact that this holds although in the Lie algebroid  $A(P)$  there is no direct information about the Lie group  $G$  (which may be disconnected!).

Besides, the Lie algebroid of a principal bundle  $P$  is - in some sense - a simpler structure than  $P$ . Namely, nonisomorphic principal bundles can possess isomorphic Lie algebroids. For example, there exists a nontrivial principal bundle for which the Lie algebroid is trivial (the nontrivial  $Spin(3)$ -structure of the trivial principal bundle  $RP(5) \times SO(3)$  [9], [10]).

**Remark 16.** [12] If  $A=A(P)^F$ ,  $P$  being a principal bundle on a manifold  $M$  and  $F$  an involutive distribution on  $M$ , then the Chern-Weil homomorphism  $h_A$  of the Lie algebroid  $A$  is called the *tangential Chern-Weil homomorphism of a principal bundle  $P$  over a foliated manifold  $(M, \mathcal{F})$*  ( $\mathcal{F}$  being the foliation determined by  $F$ ).

One can notice that:

(1) Always,  $\Omega_b^o(M, \mathcal{F}) \cdot (Sec V^k \mathfrak{g}^*)_{I^{\circ}(ad_{A(P)})} \subset (Sec V^k \mathfrak{g}^*)_{I^{\circ}(ad_A)}$ , which means that  $\sum f^i \Gamma_i$  is  $ad_A$ -invariant when  $f^i$  are  $\mathcal{F}$ -basic and  $\Gamma_i$  are  $ad_{A(P)}$ -invariant.

The occurring inclusion can not be replaced, in general, by

the equality. Such a situation can hold if  $P$  is connected, but its restriction  $P|_L$  to many leaves  $L$  of the foliation  $F$  is not connected.

(2) If  $G$  is connected, then the above inclusion is an equation; therefore, equivalently,

$$h_A : \Omega_b^0(M, \mathcal{F}) \cdot (V\mathfrak{g}^*)_I \longrightarrow H_F(M), \quad \Sigma f^i \Gamma_i \longmapsto f^i \cdot h_P(\Gamma_i).$$

Remark 17. The case of  $P$  being the principal bundle  $L_G f$  of repers of a  $G$ -vector bundle  $f$ ,  $G \in GL(n, \mathbb{R})$  ( $n = \text{rank } f$ ), is important [20]. In this situation, the homomorphism obtained above is called the *tangential Chern-Weil homomorphism of a vector bundle  $f$  over a foliated manifold  $(M, \mathcal{F})$* . It is trivial when in  $f$  there is a flat partial covariant derivative (over  $F$ ). The superposition (under the assumption that  $L_G f$  is connected)

$$\begin{aligned} (V\mathfrak{g}^*)_I \cong \mathbb{k}(\text{Sec } V^k \mathfrak{g}^*)_I \xrightarrow{I^\circ(ad_{A(P)})} \Omega_b^0(M, F) \cdot \mathbb{k}(\text{Sec } V^k \mathfrak{g}^*)_I \xrightarrow{I^\circ(ad_{A(P)})} \subset \\ \subset \mathbb{k}(\text{Sec } V^k \mathfrak{g}^*)_I \xrightarrow{I^\circ(ad_{A(P)F})} H_F(M) \end{aligned}$$

agrees with the homomorphism obtained by Moore and Schochet [20] to investigating such covariant derivatives. However, the holding of the above strong inclusion can be the source of quite new characteristic classes which cannot be obtained by the construction of Moore-Schochet.

The geometric signification of the Chern-Weil homomorphism in the theory of TC-foliations is presented by the following

Theorem 18. If  $A = A(M, \mathcal{F})$ ,  $(M, \mathcal{F})$  being a transversally complete foliation, and the Chern-Weil homomorphism  $h_A$  is nontrivial, then there exists no completely integrable distribution  $C \subset TM$  satisfying the conditions

- (1)  $C + E_b = TM$ ,
- (2)  $C \cap E_b = E$ ,
- (3)  $C|_x = \{X(x); X \in \text{Sec } C \cap L(M, \mathcal{F})\}$  for  $x \in M$ . ■

Now, we are going to give a wide class of TC-foliations for which the Chern-Weil homomorphisms of the corresponding Lie algebroids are nontrivial. It will be some class of foliations of left cosets of Lie groups by nonclosed connected Lie subgroups.

First, we formulate as preparatory the following theorem.

**Theorem 19.** *Let  $H \subset G$  be any connected Lie subgroup of  $G$  and let  $\mathfrak{h}$ ,  $\bar{\mathfrak{h}}$  and  $\mathfrak{g}$  be the Lie algebras of  $H$ , of its closure  $\bar{H}$  and of  $G$ , respectively. Denote by  $h_p: (V\bar{\mathfrak{h}}^*)_1 \rightarrow H(G/\bar{H})$  the Chern-Weil homomorphism of the  $\bar{H}$ -principal bundle  $P = (G \rightarrow G/\bar{H})$ . Then there exists an isomorphism  $\alpha$  of algebras such that the following diagram commutes:*

$$\begin{array}{ccc}
 \bigoplus_{k \geq 0} (\text{Sec } V\mathfrak{g}^*)_1 & \xrightarrow{h_{A(G;H)}} & H(G/\bar{H}) \\
 \cong \downarrow \alpha & & \nearrow h_p \\
 V(\bar{\mathfrak{h}}/\mathfrak{h})^* & \xrightarrow{\quad} & (V\bar{\mathfrak{h}}^*)_1
 \end{array}$$

Since, for any connected, compact and semisimple Lie group  $G$ ,

$$h_p^{(2)}: (\bar{\mathfrak{h}}^*)_1 \xrightarrow{\cong} H^{(2)}(G/\bar{H})$$

is an isomorphism (cf [4]), we obtain

**Theorem 20.** *If  $G$  is a connected, compact and semisimple Lie group and  $H$  is any nonclosed connected Lie subgroup of  $G$  and  $\bar{H}$  is its closure, then*

$$h_{A(G;H)}^{(2)}: (V\bar{\mathfrak{h}}/\mathfrak{h})^* \xrightarrow{\quad} H^{(2)}(G/\bar{H})$$

is a nontrivial monomorphism; therefore  $h_{A(G;H)}$  is nontrivial. This means that then there exists no  $C^\infty$  completely integrable distribution  $C \subset TG$  such that (1)  $C + E_{\bar{h}} = TG$ , (2)  $C \cap E_{\bar{h}} = E$ , (3)  $C$  is  $\bar{H}$ -right-invariant.

**Corollary 21.** *Taking  $G$  as above and, in addition, simply connected, we obtain a nonintegrable transitive Lie algebroid whose Chern-Weil homomorphism is nontrivial. ■*

Now, we give a simple example of a flat connection in the Lie algebroid  $A(G;H)$ .

**Example 22.** If  $c \subset \mathfrak{g}$  is a Lie subalgebra such that

$$c + \bar{\mathfrak{h}} = \mathfrak{g}, \quad c \cap \bar{\mathfrak{h}} = \mathfrak{h},$$

then the  $G$ -left-invariant distribution  $C$  determined by  $c$  is  $C^\infty$  completely integrable and such that (1)  $C + E_{\bar{\mathfrak{h}}} = TG$ , (2)  $C \cap E_{\bar{\mathfrak{h}}} = E$ , (3)  $C$  is  $\bar{H}$ -right-invariant, therefore  $C$  induces a flat connection in  $A(G;H)$ . The existence of such a Lie subalgebra implies then the triviality of the Chern-Weil homomorphism  $\lambda_{A(G;H)}$  of  $A(G;H)$ .

The previous theorem gives

**Corollary 23.** *If  $G$  is a connected, compact and semisimple Lie group and  $H$  is any nonclosed connected Lie subgroup of  $G$ , and  $\mathfrak{h}$ ,  $\bar{\mathfrak{h}}$  and  $\mathfrak{g}$  are the Lie algebras of  $H$ , of its closure  $\bar{H}$  and of  $G$ , respectively, then no Lie subalgebra  $c \subset \mathfrak{g}$  satisfying  $c + \bar{\mathfrak{h}} = \mathfrak{g}$ ,  $c \cap \bar{\mathfrak{h}} = \mathfrak{h}$ , exists. ■*

This theorem is valid if one weakens the assumption on  $\pi_1(G)$  to be finite [11]. One can also prove that the existence of such a Lie subalgebra  $c$  gives the minimal closedness of  $\mathfrak{h}$  in the sense of Malcev [11].

The analysis as in "Bott's phenomenon" [5] gives the following results.

**Theorem 24.** [15] *If  $A$  is any regular Lie algebroid over  $(M,F)$ ,  $\text{Pont}(A) \subset H_F(M)$  is the Pontryagin algebra of  $A$  and  $A$  admits a partially flat connection  $\lambda'$  over some involutive subdistribution  $F_1 \subset F$  of codimension (with respect to  $F$ ) equalling  $q$ , then*

$$\text{Pont}^p(A) = 0 \text{ for } p \geq 2 \cdot (q+1).$$

*If  $\lambda'$  admits a basic connection, then*

$$\text{Pont}^p(A) = 0 \text{ for } p \geq q+1. \quad \blacksquare$$

One can notice that in the Lie algebroid  $A(G;H)$  any Lie

subalgebra  $c \subset \mathfrak{g}$  such that

$$(1) \quad \bar{\mathfrak{h}} \cap c = \mathfrak{h},$$

$$(2) \quad \mathfrak{f} := \bar{\mathfrak{h}} + c \text{ is a Lie subalgebra of } \mathfrak{g},$$

gives a partial flat connection over the involutive distribution on  $G/\bar{H}$  being the  $G$ -left-invariant one determined by  $\mathfrak{f}/\bar{\mathfrak{h}}$  (codimension of this is equal to  $\text{codim} \mathfrak{f}$ ). If  $\mathfrak{f}$  is a Lie algebra of a compact Lie subgroup of  $G$ , then  $c$  admits a basic connection. From the above we obtain the following corollary:

Corollary 25. [15] *If  $A = A(G; H)$  and  $\text{Pont}^P(A) = 0$ , then there exist no Lie subalgebra  $c$  of  $\mathfrak{g}$  such that*

$$(1) \quad \bar{\mathfrak{h}} \cap c = \mathfrak{h},$$

(2)  $\mathfrak{f} := \bar{\mathfrak{h}} + c$  is a Lie subalgebra of  $\mathfrak{g}$  whose codimension is  $\leq (p/2) - 1$ , or is  $\leq p - 1$  provided that  $\mathfrak{f}$  is a Lie subalgebra of a compact Lie subgroup of  $G$ . ■

Since  $\text{Pont}^2(A) \neq 0$  when  $G$  is compact and semisimple, we obtain

Corollary 26. [15] *If  $G$  is compact and semisimple, and  $H$  is not closed, then there exists no Lie subalgebra  $c \subset \mathfrak{g}$  such that*

$$(1) \quad \bar{\mathfrak{h}} \cap c = \mathfrak{h},$$

(2)  $\mathfrak{f} := \bar{\mathfrak{h}} + c$  is a Lie subalgebra of a closed Lie subgroup of  $G$  whose codimension is 1. ■

#### THE CHARACTERISTIC HOMOMORPHISM OF A FLAT REGULAR LIE ALGEBROID

Consider in a given regular Lie algebroid  $(A, [\cdot, \cdot], \gamma)$  over  $(M, F)$  two geometric structures [13]:

$$(1) \quad \text{a flat connection } \lambda: F \longrightarrow A,$$

$$(2) \quad \text{a subalgebroid } B \subset A \text{ over } (M, F), \text{ see the diagram}$$

$$\begin{array}{ccccccc}
 \emptyset & \longrightarrow & \mathfrak{g} & \xleftrightarrow{\quad} & A & \xrightarrow{\gamma} & F \longrightarrow \emptyset \\
 & & \uparrow & \searrow \omega & \uparrow j & \searrow \lambda & \uparrow id \\
 \emptyset & \longrightarrow & \mathfrak{h} & \xleftrightarrow{\quad} & B & \xrightarrow{\gamma_1} & F \longrightarrow \emptyset
 \end{array}
 \quad (*)$$

Notice that  $\mathfrak{h} = \mathfrak{g} \cap B$  ( $\mathfrak{h} := \text{Ker } \gamma_1$ ).

The system  $(A, B, \lambda)$  will then be called an *FS-regular Lie algebroid (over  $(M, F)$ )*.

We construct some characteristic classes of an *FS-regular Lie algebroid  $(A, B, \lambda)$* , measuring the independence of  $\lambda$  and  $B$ , i.e. to what extent  $\text{Im } \lambda$  is not contained in  $B$ . First, we give some examples of such Lie algebroids.

**Examples 27. (1).** Let  $P$  be a flat  $G$ -principal bundle with a flat connection  $\omega$  and  $P'$  a reduction of  $P$ .  $\omega$  determines a flat connection  $\lambda$  in  $A(P)$ , and the system  $(A(P), A(P'), \lambda)$  is an *FS-transitive Lie algebroid*.

(2) (*An important generalization of the above example*). Let  $(P, P', \omega')$  be any foliated  $G$ -principal bundle on a manifold  $M$  [5], with an  $H$ -reduction  $P'$  and a flat partial connection  $\omega'$  over an involutive distribution  $F \subset TM$ .  $\omega$  determines a flat connection  $\lambda$  in the regular Lie algebroid  $A(P)^F$  over  $(M, F)$ , and the system  $(A(P)^F, A(P')^F, \lambda)$  is an *FS-regular Lie algebroid*.

(3) *FS-transitive Lie algebroids on the ground of TC-foliations*. Let  $A = A(M, \mathcal{F})$ ,  $(M, \mathcal{F})$  being an arbitrary *TC-foliation*.

**Proposition 28.** *There exists a 1-1 correspondence between transitive Lie subalgebroids  $B$  of  $A$  and involutive distributions  $\bar{B} \subset TM$  such that*

- (a)  $E \subset \bar{B}$ ,
- (b)  $E_{\bar{b}} + \bar{B} = TM$ ,
- (c)  $\bar{B}|_x = \{X(x); X \in \text{Sec } \bar{B} \cap L(M, \mathcal{F})\}$ ,  $x \in M$ .

For the foliation of left cosets of  $G$  by  $H$ ,

- (c)  $\equiv$  (c'):  $\bar{B}$  is  $C^\infty$  and  $H$ -right-invariant. ■

Some example of a Lie subalgebroid of  $A(G;H)$  is given by

the following theorem.

**Theorem 29.** *If  $\mathfrak{b} \subset \mathfrak{g}$  is a Lie subalgebra such that*

$$\mathfrak{h} \subset \mathfrak{b}, \quad \mathfrak{h} + \mathfrak{b} = \mathfrak{g},$$

*then the  $G$ -left-invariant distribution  $\bar{B}_{\mathfrak{b}}$  determined by  $\mathfrak{b}$  fulfils (a), (b) and (c') from the above proposition, giving at the same time a Lie subalgebroid of  $A(G;H)$ . ■*

To sum up, a system  $(\mathfrak{b}, \mathfrak{c})$  of Lie subalgebras of  $\mathfrak{g}$  such that  $\mathfrak{h} \subset \mathfrak{b}$ ,  $\mathfrak{h} + \mathfrak{b} = \mathfrak{g}$  and  $\mathfrak{h} + \mathfrak{c} = \mathfrak{g}$ ,  $\mathfrak{h} \cap \mathfrak{c} = \mathfrak{h}$ , determines some FS-transitive Lie algebroid on  $G/H$ .

(4) *FS-regular (nontransitive) Lie algebroids on the ground of TC-foliations.* Let  $A = A(M, \mathcal{F})$ ,  $(M, \mathcal{F})$  being an arbitrary TC-foliation.

**Proposition 30.** *An involutive distribution  $\bar{F}$  on  $M$  is a lifting of some involutive distribution  $F$  on the basic manifold  $W$  if and only if*

- (1)  $E_{\mathfrak{b}} \subset \bar{F}$ ,
- (2)  $\bar{F}|_x = \left\{ X(x); X \in \text{Sec } \bar{F} \cap L(M, \mathcal{F}) \right\}, x \in M$ .

*The correspondence  $F \longmapsto \bar{F}$  is 1-1. For the foliation of left cosets of  $G$  by  $H$ ,*

- (2)  $\Leftrightarrow (2')$ :  $\bar{F}$  is  $C^\infty$  and  $H$ -right-invariant. ■

Denote the lifting of  $F \subset TW$  to  $M$  by  $T^F M$ .

**Proposition 31.** *Let  $F \subset TW$  be any foliation of  $W$ . There exists a 1-1 correspondence between partial connections in  $A(M, \mathcal{F})$  over  $F$ , i.e. connections in  $A(M, \mathcal{F})^F$ , and distributions  $\bar{C}$  in  $TM$  such that*

- (a)  $E_{\mathfrak{b}} \cap \bar{C} = E$ ,
- (b)  $E_{\mathfrak{b}} + \bar{C} = T^F M$ ,
- (c)  $\bar{C}|_x = \left\{ X(x); X \in \text{Sec } \bar{C} \cap L(M, \mathcal{F}) \right\}, x \in M$ .

*In particular, such a distribution  $\bar{C}$  always exists. For the foliation of left cosets of  $G$  by  $H$ ,*

(c)  $\equiv$  (c'):  $\bar{c}$  is  $C^\infty$  and  $H$ -right-invariant.

A partial connection in  $A(M, \mathcal{F})$  is flat if and only if the corresponding distribution in  $TM$  is involutive. ■

Some examples of foliations of  $G/R$  and partial connections in  $A(G;H)$  are given by the following theorems.

**Theorem 32.** If  $f \subset \mathfrak{g}$  is a Lie subalgebra such that

$$\bar{h} \subset f,$$

then the  $G$ -left-invariant distribution  $\bar{F}(f) \subset TG$  determined by  $f$  fulfils  $E_b \subset \bar{F}(f)$ , and  $\bar{F}(f)$  is  $H$ -right-invariant, therefore gives some foliation  $F(f)$  of  $G/R$ .

**Theorem 33.** Let  $f$  and  $c$  be Lie subalgebras of  $\mathfrak{g}$  such that

$$\bar{h} \subset f \text{ and } \bar{h} + c = f, \bar{h} \cap c = \bar{h};$$

then the  $G$ -left-invariant distribution  $\bar{c} = \bar{c}(c) \subset TG$  determined by  $c$  is  $C^\infty$ ,  $H$ -right-invariant and fulfils  $E_b \cap \bar{c} = E$ ,  $E_b + \bar{c} = T^F M$ , therefore induces some flat partial connection in  $A(G;H)$  over  $F(f)$ . ■

To sum up, the triple  $(b, f, c)$  of subalgebras of  $\mathfrak{g}$  such that

$$\bar{h} \subset b, \bar{h} + b = \mathfrak{g}, \bar{h} \subset f \text{ and } \bar{h} + c = f, \bar{h} \cap c = \bar{h},$$

determines an FS-regular Lie algebroid.

Return to diagram (\*).

We construct a characteristic homomorphism

$$\Delta_\# : H(\mathfrak{g}, B) \longrightarrow H_F(M)$$

measuring the independence of  $\lambda$  and  $B$  in the sense that  $\Delta_\# = 0$  if  $Im \lambda \subset B$ .

Here  $H(\mathfrak{g}, B) = H((Sec \Lambda(\mathfrak{g}/\mathfrak{h})^*)_{\rho, \delta})$  where

(1)  $(Sec \Lambda(\mathfrak{g}/\mathfrak{h})^*)_{\rho}$  is the space of invariant cross-sections with respect to the canonical representation  $B \longrightarrow A(\Lambda(\mathfrak{g}/\mathfrak{h})^*)$  induced by  $ad|_B : B \longrightarrow A(\mathfrak{g})$ . Precisely,

$$\Psi \in (Sec \Lambda^k(\mathfrak{g}/\mathfrak{h})^*)_{\rho} \Leftrightarrow \forall \xi \in Sec B, \forall \nu_1, \dots, \nu_k \in Sec \mathfrak{g},$$

$$(\gamma_1 \circ \xi) \langle \Psi, [\nu_1] \wedge \dots \wedge [\nu_k] \rangle = \sum_j \langle \Psi, [\nu_1] \wedge \dots \wedge [[\xi, \nu_j]] \wedge \dots \wedge [\nu_k] \rangle,$$

where  $[\nu_j] = s \circ \nu_j \in Sec \mathfrak{g}/\mathfrak{h}$  and  $s : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$  is the canonical

projection.

(2)  $\bar{\delta}$  is a differential in  $(\text{Sec}\Lambda(\mathfrak{g}/\mathfrak{h})^*)_F$ , defined by the formula

$$\langle \bar{\delta}\Psi, [\nu_0] \wedge \dots \wedge [\nu_k] \rangle = - \sum_{i < j} (-1)^{i+j} \langle \Psi, [[\nu_i, \nu_j]] \wedge [\nu_0] \wedge \dots \wedge \hat{i} \wedge \dots \wedge \hat{j} \wedge \dots \wedge [\nu_k] \rangle.$$

First, we construct the homomorphism  $\Delta_{\#}$  on the level of forms.

**Theorem 34.** *There exists a homomorphism of algebras*

$$\Delta_{*} : \text{Sec}\Lambda(\mathfrak{g}/\mathfrak{h})^* \longrightarrow \Omega_F(M)$$

such that

$$\Delta_{*}\Psi(x; w_1, \dots, w_k) = \Psi(x, [\omega(x; \tilde{w}_1)] \wedge \dots \wedge [\omega(x; \tilde{w}_k)])$$

for  $w_j \in F_{|x}$  where  $\tilde{w}_j \in B_{|x}$  satisfy  $\gamma_1(\tilde{w}_j) = w_j$ . ■

**Theorem 35.**  $\Delta_{*}$  restricted to the invariant cross-sections commutes with differentials  $\bar{\delta}$  and  $\delta^F$ , giving a homomorphism  $\Delta_{\#}$  on cohomologies. ■

36. The fundamental properties of  $\Delta_{\#}$  are:

(a)  $\Delta_{*} = 0$  if  $\text{Im}\lambda \subset B$ .

(b) The functoriality of  $\Delta_{\#}$ .

(c) The independence of  $\Delta_{\#}$  of the choice of homotopic subalgebroids  $B$  in the following sense:

If  $B_0$  is homotopic to  $B_1$ , then there exists an isomorphism  $\alpha: H(\mathfrak{g}, B_0) \xrightarrow{\cong} H(\mathfrak{g}, B_1)$  of algebras, such that the diagram

$$\begin{array}{ccc} H(\mathfrak{g}, B_0) & & \\ \downarrow \alpha & \searrow \Delta_{0\#} & \\ H(\mathfrak{g}, B_1) & & H_F(M) \\ & \nearrow \Delta_{1\#} & \end{array}$$

commutes. The relation of homotopy between subalgebroids is naturally defined as follows:  $B_0 \sim B_1 \Leftrightarrow$  there exists a Lie subalgebroid  $B \subset T\mathbb{R} \times A$  (on  $\mathbb{R} \times M$ ) such that  $v \in B_t \Leftrightarrow (\theta_t, v) \in B$ ,  $t=0, 1$  ( $\theta_t$  being the null vector at  $t \in \mathbb{R}$ ). ■

The construction of  $\alpha$  is not trivial. We uses a few times

the existence of global solutions to some systems of differential equations.

**Theorem 37** (*The comparison with a flat principal bundle, see [5]*). *If  $A=A(P)$ ,  $B=A(P')$ , where  $P'$  is a connected  $H$ -reduction, then, for each flat connection in  $P$  and the connection in  $A(P)$  corresponding to it, there exists an isomorphism  $\alpha$  of algebras such that the diagram*

$$\begin{array}{ccc}
 H(\mathfrak{g}, H) & & \\
 \alpha \downarrow \cong & \searrow \Delta_{\#} & \\
 & & H_{dR}(M) \\
 H(\mathfrak{g}, A(P')) & \nearrow \Delta_{\#} &
 \end{array}$$

commutes. ■

Assume that  $A$  and  $B$  mean the same as in Theorem 37 above, but  $\lambda$  is a partially flat connection in  $A$ , say, over an involutive distribution  $F \subset TM$ . Denote by  $\mathcal{F}$  the foliation determined by  $F$ . Equivalently, we have given some foliated principal bundle and an  $H$ -reduction of it. By the general theory, there is a homomorphism of algebras

$$\Delta_{\#} : H(\mathfrak{g}, B^F) \longrightarrow H_F(M).$$

**Theorem 38.** *If  $H$  is connected, then*

$$H(\mathfrak{g}, B^F) \cong \Omega_b^0(M, \mathcal{F}) \cdot H(\mathfrak{g}, H)$$

and  $\Delta_{\#}(\sum f^i \cdot [\psi_i])$ ,  $\psi_i \in (\wedge^k(\mathfrak{g}/\mathfrak{h})^*)$ , is the tangential cohomology class determined by the form

$$\Delta_{\#}(f^i \cdot \psi_i)(x; \omega_1 \wedge \dots \wedge \omega_k) = \sum f^i(x) \cdot \langle \psi_i, [\omega(x; \tilde{\omega}_1)] \wedge \dots \wedge [\omega(x; \tilde{\omega}_k)] \rangle$$

for  $\omega_i \in F_{ix}$  where  $\tilde{\omega}_i \in T_x P'$  satisfy  $\pi'_*(\tilde{\omega}_i) = \omega_i$ . ■

**Problem 39.** Consider an arbitrary leaf  $L$  of  $\mathcal{F}$ . From the functoriality property, under the assumption of the connectivity of  $H$ , we have the following commuting diagram

$$\begin{array}{ccccc}
 f^i \cdot [v_i] & \Omega_b^0(M, \mathcal{F}) \cdot H(\mathfrak{g}, H) & \xrightarrow{\Delta_{\#}} & H_F(M) & \\
 \downarrow & \downarrow & & \downarrow & \\
 [f^i(x) \cdot v_i] & H(\mathfrak{g}, H) & \xrightarrow{\Delta_{L\#}} & H_{\text{dr}}(L) &
 \end{array}$$

Find an example of the situation in which  $\Delta_{\#}$  is not trivial, whereas  $\Delta_{L\#}$  is trivial for each leaf  $L \in \mathcal{F}$ .

An example of the nontrivial  $\Delta_{\#}$  on the ground of TC-foliations.

A) The transitive case. Let  $A = A(M, \mathcal{F})$ ,  $(M, \mathcal{F})$  being an arbitrary TC-foliation. Assume that we are given a Lie subalgebroid  $B \subset A$ , equivalently - a distribution  $\bar{B} \subset TM$  such that

- (a)  $E \subset \bar{B}$ ,
- (b)  $E_b + \bar{B} = TM$ ,
- (c)  $\bar{B}|_x = \{X(x); X \in \text{Sec } \bar{B} \cap L(M, \mathcal{F})\}$  for  $x \in M$ ,

and a flat connection  $\lambda$  in  $A$ , equivalently - a distribution  $C \subset TM$  such that

- (1)  $C + E_b = TM$ ,
- (2)  $C \cap E_b = E$ ,
- (3)  $C|_x = \{X(x); X \in \text{Sec } C \cap L(M, \mathcal{F})\}$  for  $x \in M$ .

From the general theory we obtain:

If the characteristic homomorphism  $\Delta_{\#}$  is not trivial, then  $B$  cannot be homotopically changed to the one which contains  $\text{Im } \lambda$  (equivalently,  $\bar{B} \supset C$ ).

Here we calculate the characteristic homomorphism of the FS-transitive Lie algebroid  $(A(G; H), B, \lambda)$  in which

- (i)  $B = B_b$  is the Lie subalgebroid of  $A(G; H)$  determined by a Lie subalgebra  $\mathfrak{b} \subset \mathfrak{g}$  satisfying (1)  $\mathfrak{h} \subset \mathfrak{b}$ , (2)  $\bar{\mathfrak{h}} + \mathfrak{b} = \mathfrak{g}$ ,
- (ii)  $\lambda$  is the flat connection determined by a Lie subalgebra  $\mathfrak{c} \subset \mathfrak{g}$  satisfying (1)  $\mathfrak{c} + \bar{\mathfrak{h}} = \mathfrak{g}$ , (2)  $\mathfrak{c} \cap \bar{\mathfrak{h}} = \mathfrak{h}$ .

**Theorem 40.** There exist an canonical isomorphism  $\alpha$  of algebras and a homomorphism  $\hat{\Delta}_{\#}$  of algebras, making the

following diagram commute

$$\begin{array}{ccc}
 H(\mathfrak{g}, B) & \xrightarrow{\Delta_{\#}} & H_{dR}(G/R) \\
 \cong \uparrow \alpha & & \uparrow \\
 \Lambda(\bar{\mathfrak{b}}/(\bar{\mathfrak{b}} \cap \mathfrak{b}))^* & \xrightarrow{\hat{\Delta}_{\#}} & H(\Lambda(\mathfrak{g}/\bar{\mathfrak{b}})^*)_I \cong H_1(G/R) .
 \end{array}$$

$(\Lambda(\mathfrak{g}/\bar{\mathfrak{b}})^*)_I$  denotes here the DG-algebra of vectors invariant with respect to the adjoint representation  $Ad^1: \mathbb{R} \longrightarrow GL(\Lambda(\mathfrak{g}/\bar{\mathfrak{b}})^*)$  (cf [5]). The homomorphism  $\hat{\Delta}_{\#}$  on the level of forms is defined by the equality

$$\langle \hat{\Delta}_{\#}(\hat{\Psi}), [\omega_1] \wedge \dots \wedge [\omega_k] \rangle = \langle \hat{\Psi}, \omega_1(\tilde{w}_1) \wedge \dots \wedge \omega_k(\tilde{w}_k) \rangle$$

for  $\hat{\Psi} \in \Lambda^k(\bar{\mathfrak{b}}/(\bar{\mathfrak{b}} \cap \mathfrak{b}))^*$  and  $\omega_i \in \mathfrak{g}$ , where  $\tilde{w}_i \in \mathfrak{b}$  are vectors such that  $[\tilde{w}_i] = [\omega_i]$  ( $\in \mathfrak{g}/\bar{\mathfrak{b}}$ ) where  $\omega_1: \mathfrak{g} \longrightarrow \bar{\mathfrak{b}}/(\bar{\mathfrak{b}} \cap \mathfrak{b})$  is defined as the superposition

$$\omega_1: \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{b} = \bar{\mathfrak{b}}/\mathfrak{b} \oplus \mathfrak{c}/\mathfrak{b} \xrightarrow{pr_1} \bar{\mathfrak{b}}/\mathfrak{b} \longrightarrow \bar{\mathfrak{b}}/(\bar{\mathfrak{b}} \cap \mathfrak{b}) .$$

For a compact  $G$ , the right arrow in the diagram below is an isomorphism. ■

**Theorem 41.**  $\hat{\Delta}_{\#}$  is trivial if and only if  $\mathfrak{c} \subset \mathfrak{b}$ . ■

Each case  $\mathfrak{c} \not\subset \mathfrak{b}$  (for a compact  $G$ ) is the source of the nontrivial characteristic homomorphism of an FS-regular Lie algebroid on the ground of TC-foliations.

**B) The nontransitive case.** Here we calculate the characteristic homomorphism of the FS-regular Lie algebroid  $(A(G;H)^{F(f)}, B_{\mathfrak{b}}^{F(f)}, \lambda_c)$  in which

- (i)  $F(f)$  is the foliation of  $G/R$  determined by a Lie subalgebra  $\mathfrak{f} \subset \mathfrak{g}$  such that  $\bar{\mathfrak{b}} \subset \mathfrak{f}$ ,
- (ii)  $B_{\mathfrak{b}}^{F(f)}$  is the Lie subalgebroid of  $A(G;H)$  where  $B_{\mathfrak{b}}$  is determined by a Lie subalgebra  $\mathfrak{b} \subset \mathfrak{g}$  fulfilling (1)  $\mathfrak{b} \subset \mathfrak{b}$ ,
- (2)  $\bar{\mathfrak{b}} + \mathfrak{b} = \mathfrak{g}$ ,
- (iii)  $\lambda_c$  is the partial flat connection determined by a Lie subalgebra  $\mathfrak{c} \subset \mathfrak{g}$  for which (1)  $\mathfrak{c} \cap \bar{\mathfrak{b}} = \mathfrak{b}$ , (2)  $\mathfrak{c} + \bar{\mathfrak{b}} = \mathfrak{f}$ .

By the general theory, there is a homomorphism

$$\Delta_{\#} = H(\mathfrak{g}, B_{\mathfrak{b}}^{F(f)}) \longrightarrow H_{F(f)}(G/R)$$

of algebras.

Theorem 42. There exist a canonical isomorphism  $\alpha$  of algebras and a homomorphism  $\hat{\Delta}_{\#}$  of algebras, making the following diagram commute

$$\begin{array}{ccc} H(\mathfrak{g}, B_{\mathfrak{b}}^{F(f)}) & \xrightarrow{\Delta_{\#}} & H_{F(f)}(G/R) \\ \uparrow \cong \cdot \alpha & & \uparrow \\ \Omega_b^0(M, \mathcal{F}) \wedge \Lambda(\bar{\mathfrak{b}}/(\bar{\mathfrak{b}} \cap \mathfrak{b}))^* & \xrightarrow{id \cdot \hat{\Delta}_{\#}} & \Omega_b^0(M, \mathcal{F}) \cdot H_{F(f), I}(G/R) \end{array}$$

The homomorphism  $\hat{\Delta}_{\#}$  on the level of forms is defined by the equality

$$\langle \hat{\Delta}_{\#}(\hat{\Psi}), [\omega_1] \wedge \dots \wedge [\omega_k] \rangle = \langle \hat{\Psi}, \omega_1(\tilde{w}_1) \wedge \dots \wedge \omega_k(\tilde{w}_k) \rangle$$

for  $\hat{\Psi} \in \Lambda^k(\bar{\mathfrak{b}}/(\bar{\mathfrak{b}} \cap \mathfrak{b}))^*$  and  $\omega_i \in \mathfrak{f}$ , where  $\tilde{w}_i \in \mathfrak{b} \cap \mathfrak{f}$  are vectors such that  $[\tilde{w}_i] = [\omega_i]$  ( $\in \mathfrak{f}/\bar{\mathfrak{b}}$ ) where  $\omega_i: \mathfrak{f} \longrightarrow \bar{\mathfrak{b}}/(\bar{\mathfrak{b}} \cap \mathfrak{b})$  is defined as the superposition

$$\omega_i: \mathfrak{f} \longrightarrow \mathfrak{f}/\mathfrak{b} = \bar{\mathfrak{b}}/\mathfrak{b} \oplus \mathfrak{c}/\mathfrak{b} \xrightarrow{pr_1} \bar{\mathfrak{b}}/\mathfrak{b} \longrightarrow \bar{\mathfrak{b}}/(\bar{\mathfrak{b}} \cap \mathfrak{b}).$$

For a compact  $G$ , the canonical inclusion  $\Omega_{F(f), I}(G/R) \hookrightarrow \Omega_{F(f)}(G/R)$  induces a monomorphism on cohomologies  $H_{F(f), I}(G/R) \hookrightarrow H_{F(f)}(G/R)$ , therefore the nontriviality of  $\Delta_{\#}$  implies the same for  $\hat{\Delta}_{\#}$ . ■

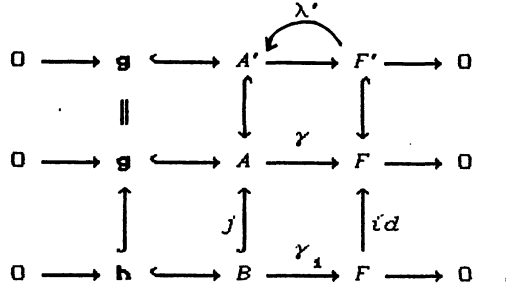
Theorem 43.  $\hat{\Delta}_{\#}$  is trivial if and only if  $\mathfrak{c} \subset \mathfrak{b}$ . ■

Each case  $\mathfrak{c} \not\subset \mathfrak{b}$  (for a compact  $G$ ) is the source of the nontrivial characteristic homomorphism of an FS-regular Lie algebroid on the ground of TC-foliations.

### THE CHARACTERISTIC HOMOMORPHISM OF PARTIALLY FLAT REGULAR LIE ALGEBROIDS

Consider in a given regular Lie algebroid  $(A, [\cdot, \cdot], \gamma)$  over  $(M, \mathcal{F})$  two geometric structures [14]:

- (1) a partial flat connection  $\lambda': F' \longrightarrow A' (:= \gamma^{-1}[F'])$ ,
- (2) a subalgebroid  $B < A$  over  $(M, F)$ , see the diagram



The system  $(A, B, \lambda)$  will then be called a *PFS-regular Lie algebroid (over  $(M, F, F')$ )*.

**Examples 44.** Examples 27(2) and (4) from the previous part:

- a foliated bundle  $(P, P', \omega')$  [5],
  - a triple  $(\mathfrak{b}, \mathfrak{f}, \mathfrak{c})$  of some subalgebras of  $\mathfrak{g}$ ,
- are the source of PFS-regular Lie algebroids:

$$(A(P), A(P'), \lambda'), \text{ and } (A(G, H), B_{\mathfrak{b}}, \lambda_c),$$

respectively.

We construct some characteristic homomorphism of a PFS-regular Lie algebroid  $(A, B, \lambda')$

$$\Delta_{q', \#} : H(\mathcal{W}(\mathfrak{g}, \mathfrak{h})_{q', l', \delta}) \longrightarrow H_F(M),$$

measuring the independence of  $\lambda'$  and  $B$ , where

$$\mathcal{W}(\mathfrak{g}, \mathfrak{h})_{q', l', \delta} = (\text{Sec}(\wedge(\mathfrak{g}/\mathfrak{h})^* \otimes \mathcal{V}^{(q')} \mathfrak{g}^*))_{l', \delta}$$

is the space of invariant cross-sections with respect to the canonical representation of  $B$ , and  $\delta$  is the differential defined point by point, coming from the differential  $d_x$  in the Weil algebra  $W\mathfrak{g}_{|x}$  of the Lie algebra  $\mathfrak{g}_{|x}$ , and  $q' \geq \text{codim} F'$  (and  $q' \geq [q/2]$  for the "basic" case).

The homomorphism  $\Delta_{q', \#}$  on the level of forms is constructed as follows:

Take  $s: \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ , the canonical projection, and

$\Lambda s^* \circ id: \Lambda(\mathfrak{g}/\mathfrak{h})^* \otimes V^{\leq q'} \mathfrak{g}^* \longrightarrow \Lambda \mathfrak{g}^* \otimes V^{\leq q'} \mathfrak{g}^*$ , the induced homomorphism, and

$$k: \mathbb{1} \text{Sec}(\Lambda \mathfrak{g}^* \otimes V^l \mathfrak{g}^*) \longrightarrow \Omega(A)$$

$$\psi \otimes \Gamma \longmapsto \omega^\wedge(\psi) \wedge \Omega^\vee(\Gamma)$$

where  $\omega^\wedge(\psi) = \frac{1}{k!} \langle \psi, \omega \wedge \dots \wedge \omega \rangle$  for  $\psi \in \text{Sec} \Lambda^k(\mathfrak{g}/\mathfrak{h})^*$  and  $\Omega^\vee(\Gamma) = \frac{1}{l!} \langle \Gamma, \Omega \wedge \dots \wedge \Omega \rangle$  for  $\Gamma \in \text{Sec} V^l \mathfrak{g}^*$ , whereas  $\omega$  and  $\Omega$  are the connection form and the curvature form of some adapted connection.

The form  $j^*(k(\Lambda s^* \circ id(\Psi)))$  is  $\mathfrak{h}$ -horizontal, which implies the existence of a tangential differential form  $\Delta \Psi \in \Omega_F(M)$  such that  $\gamma^*(\Delta \Psi) = j^*(k(\Lambda s^* \circ id(\Psi)))$ . Put  $\Delta_{q'} = (\Psi \longmapsto \Delta \Psi)$ .

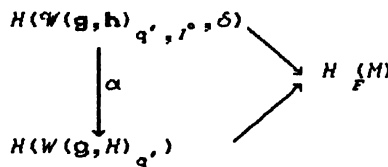
**Theorem 45.** *If  $q' \geq \text{codim} F'$  (and  $q' \geq [q/2]$  for the "basic" case), then  $\Delta_{q'}: \mathcal{V}(\mathfrak{g}, \mathfrak{h})_{q', r, \delta} \longrightarrow \Omega_F(M)$  commutes with suitable differentials, giving a homomorphism on cohomologies. ■*

The properties:

- (1) The functoriality.
- (2) The independence of the choice of an adapted connection.
- (3) For two Lie subalgebroids being homotopic, the corresponding characteristic homomorphisms are equivalent. ■

The comparison with the characteristic homomorphism of foliated bundles [5] is described by the following theorem.

**Theorem 46.** *Let  $A = A(P)$ ,  $B = A(P')$ ,  $P'$  being a  $H$ -reduction. If  $P'$  is connected, then there exists an isomorphism  $\alpha$  of algebras, such that the following diagram*



commutes. ■

## REFERENCES

- [1] ALMEIDA, R. & MOLINO, P., Suites d'Atiyah et feuilletages transversalement complets, *C. R. Acad. Sci. Paris Ser. I Math.*, 300 (1985), 13-15.
- [2] DAZORD, P. & SONDAZ, D. Varietes de Poisson - Algebroides de Lie, *Publ. Dep. Math. Universite de Lyon 1*, 1/B (1988).
- [3] GREUB, W. & HALPERIN, S. & VANSTONE, R., *Connections, Curvature, and Cohomology Vol. II*, Academic Press, New York and London, 1973.
- [4] — *Connections, Curvature, and Cohomology Vol. III*, Academic Press, New York and London, 1976.
- [5] KAMBER, F. & TONDEUR, PH. Foliated Bundles and Characteristic Classes, *Lectures Notes in Mathematics* 493, Springer-Verlag 1975.
- [6] KUBARSKI, J., Exponential mapping for Lie groupoids, *Colloq. Math.* XLVII (1982), 267-282.
- [7] — Lie algebroid of a principal fibre bundle - three equivalent definitions, *Prace Naukowe Politechniki Szczecińskiej*, 11 (1988), 123-145.
- [8] — Some invariants of Lie algebroids of principal fibre bundles, *Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II*, 21(1989), 235-258.
- [9] — Pontryagin algebra of a transitive Lie algebroid, *Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II*, 22(1989), 117-126.
- [10] — Lie algebroid of a principal fibre bundle, *Publ. Dep. Math. Universite de Lyon 1*, (to appear).
- [11] — A criterion for the minimal closedness of the Lie subalgebra corresponding to a connected nonclosed Lie subgroup, *Revista Matematica de la Universidad Complutense de Madrid* (to appear).
- [12] — The Chern-Weil homomorphism of regular Lie algebroids, *In preparation*.
- [13] — The characteristic homomorphism of flat regular Lie algebroids, *In preparation*.
- [14] — The characteristic homomorphism of partially flat

- regular Lie algebroids, *In preparation*.
- [15] — Bott's phenomenon in the theory of nonclosed Lie subgroups, *In preparation*.
- [16] KUMPERA, A. An Introduction to Lie groupoids, *Nucleo de Estudos e Pasquisas Cientificas*, Rio de Janeiro, 1971.
- [17] MACKENZIE, K. *Lie groupoids and Lie algebroids in differential Geometry*, Cambridge, 1987.
- [18] MOLINO, P., Etude des feuilletages transversalement complets et applications, *Ann. Sci. École Norm. Sup.*, 10(3) (1977), 289-307.
- [19] — Riemannian Foliations, *Progress in Mathematics Vol. 73*, Birkhäuser Boston Basel, 1988.
- [20] MOORE, C. C. & SCHOCHET, C., Global Analysis on Foliated Spaces, *Mathematical Sciences Research Institute publications; 9. 1988*, Springer-Verlag New-York Inc.
- [21] PRADINES, J., Theorie de Lie pour les groupoides differentiables dans la categorie des groupoides, Calcul differential dans la categorie des groupoides infinitesimaux", *C. R. Acad. Sci. Ser. A-B, Paris*, 264, (1967), 265-248,
- [22] — Theorie de Lie pour les groupoides differerentiabiles, *Atti Conv Intern Geom 7 Diff. Bologna*, 1967, Bologna-Amsterdam.
- [23] SILVA, A. M., Atiyah sequences and complete closed pseudogroupes preserving a local parallelism. *Holomorphic dynamics, Proc. 2nd Int. Colloq. Dyn. Syst., Mexico City/Mex. 1986, Lecture Notes Math. 1345, 302-316 (1988)*.

JAN KUBARSKI, INSTITUTE OF MATHEMATICS,  
 TECHNICAL UNIVERSITY OF ŁÓDŹ,  
 90-924 ŁÓDŹ, AL. POLITECHNIKI 11,  
 POLAND