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In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Topology". Circolo Matematico di Palermo, Palermo, 1993. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 32. pp. [31]–38.

Persistent URL: http://dml.cz/dmlcz/701524

GENERALIZED CHERN-SIMONS THEORY AND SKEIN RELATIONS IN ARBITRARY DIMENSIONS

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1. Introduction.

Methods of quantum field theory have been recently successfully used for a description of topological invariants of *low-dimensional* manifolds [13, 14]. In particular, it appears that (three-dimensional) *Chern-Simons* theory can yield skein relations for polynomial invariants of knots and links [3, 8, 14, 15]. In this talk we will introduce, after Myers and Periwal [10], a new generalization of Chern-Simons theory, which is applicable to higher-dimensional manifolds (for alternative possibilities, see [2, 4, 5, 9, 11]). The Chern-Simons form is constructed analogously to the standard one but this time the connection one-form is substituted for an *inhomogeneous* differential form of odd degree. Thus, the form can be defined on an arbitrary odd-dimensional manifold and it can serve the description of higher-dimensional links. To introduce higher-dimensional links into the play one should supplement the theory with *matter* forms defined on the submanifolds corresponding to the links. Since on the one hand we will use the very powerful (although mathematically not well-defined) method of the Feynman path integration, and on the other hand our primary integrand is not formally integrable due to the existence of three highly *reducible* gauge symmetries one should invoke the Batalin-Vilkovisky *antifield-antibracket* quantization formalism [1, 7, 12]. Quantization of the theory means, in particular, that we are able to construct the Feynman path integral Z (13).

The generalized Chern-Simons functional is [10]

$$S^{cl}_{gCS} = \frac{k}{4\pi} \int_{S^d} \mathrm{Tr} \left(\phi \mathrm{d}\phi + \frac{2}{3}\phi^3 \right), \tag{1}$$

* Humboldt fellow.

where the *generalized connection form* $\phi = \sum_{i=0}^{N} \phi_{2i+1}$ $(N = 0, 1, \ldots)$ is a formal sum of Lie-algebra valued $(\phi_{2i+1} = \sum_{a=1}^{d_G} \phi_{2i+1}^a T^a)$ differential forms of odd degree $(\deg \phi_{2i+1} = 2i+1)$ on the trivial bundle B (on a d-dimensional sphere $\mathcal{S}^d$, $d = 2n+1$, $N+1 \leq n \leq 2N+1$). T^a are normalized generators of the simple d_G-dimensional compact Lie group G in the adjoint representation $\mathcal{R}_{Adj}(G)$ $(\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab})$. For $d \geq 5$ $(N \geq 1)$ the functional (1) is globally gauge invariant, and $k \in \mathbf{R}_+$, whereas for $d = 3$ $(N = 0)$ the functional is gauge invariant only modulo 2π and hence $k \in \mathbf{Z}_+$.

The *matter* functional is

$$S_m^{cl} = \int_{\mathcal{L}} \overline{\psi} \mathrm{d}_\phi \psi, \tag{2}$$

where $\psi = \sum_{i=0}^{n-1} \psi_{2i}$ and $\overline{\psi} = \sum_{i=0}^{n-1} \overline{\psi}_{2i}$ are formal sums of Lie-algebra valued $(\phi_{2i} = \sum_{a=1}^{d_G} \phi_{2i}^a t^a)$ differential forms of even degree, t^a are generators of G in irrep $\mathcal{R}(G)$, $d_\phi \equiv d + \phi$, and $\mathcal{L} = \bigcup_{k=1}^{n} \mathcal{L}^{2k-1}$ is a ***higher-dimensional link*** of odd dimension (a disjoint sum of $\mathcal{L}^{2k-1}$, closed submanifolds of $\dim \mathcal{L}^{2k-1} = 2k-1$). On the basis of dimensional analysis we expect that only the *complementary pairs* $\{\mathcal{L}^{2k-1}, \mathcal{L}^{2n-2k+1}\}_{k=1}^{n}$ can form non-trivial links.

The functional defining the whole theory is then

$$S^{cl} = S_{gCS}^{cl} + S_m^{cl}, \tag{3}$$

whereas critical points of this functional are given by

$$F_\phi^a + \frac{4\pi}{k} \overline{\psi} t^a \psi = 0, \tag{4a}$$

$$\mathrm{d}_\phi \psi = 0, \qquad \mathrm{d}_\phi \overline{\psi} = 0, \tag{4b}$$

where the *generalized curvature form* $F_\phi \equiv \mathrm{d}_\phi^2 \equiv \mathrm{d}\phi + \phi^2$ $(\mathrm{d}_\phi \overline{\psi} \equiv \mathrm{d}\overline{\psi} - \overline{\psi}\phi)$.

It appears that (3) possesses the following three (local) gauge symmetries:

i) *Ordinary gauge* symmetry (G)

$$\delta_G \phi = -\mathrm{d}_\phi \gamma \equiv -\mathrm{d}\gamma - [\phi, \gamma], \tag{5a}$$

$$\delta_G \psi = \gamma \psi, \qquad \delta_G \overline{\psi} = -\overline{\psi}\gamma, \tag{5b}$$

where $\gamma = \sum_{i=0}^{N} \gamma_{2i}$ is an inhomogeneous differential form of even degree. The symmetry is, so-called, *on-shell reducible* because (5a) is gauge invariant at a critical point with respect to a new transformation given by

$$\delta_{G'}\gamma = -\mathrm{d}_\phi \gamma', \tag{5'}$$

where $\gamma' = \sum_{i=0}^{N-1} \gamma'_{2i+1}$. We can easily verify that taking $F_\phi = 0$ as a critical point (4) we obtain $\delta_{G'}\delta_G\phi = (-\mathrm{d}_\phi)^2\gamma' = F_\phi\gamma' = 0$. In turn it follows that (5′) is also on-shell gauge invariant with respect to the transformation

$$\delta_{G''}\gamma' = -\mathrm{d}_\phi \gamma'', \tag{5''}$$

where $\gamma'' = \sum_{i=0}^{N-1} \gamma''_{2i}$, and so on. So we conclude that according to the Batalin-Vilkovisky classification scheme [1, 7] we deal with $2N$-stage on-shell reducible gauge symmetry.

ii) *Matter gauge* symmetry (M)

$$\delta_M \phi^a = \frac{4\pi}{k}\overline{\psi} t^a \mu, \tag{6a}$$

$$\delta_M \psi = -\mathrm{d}_\phi \mu \equiv -\mathrm{d}\mu - \phi\mu, \qquad \delta_M \overline{\psi} = 0, \tag{6b}$$

where $\mu = \sum_{i=0}^{n-2} \mu_{2i+1}$. In turn

$$\delta_{M'}\mu = -\mathrm{d}_\phi \mu', \tag{6'}$$

where $\mu' = \sum_{i=0}^{n-2} \mu'_{2i}$, is an on-shell symmetry transformation of (6b). Literally repeating the arguments contained in i) we conclude that the symmetry is $2n-3$-stage on-shell reducible.

iii) $\overline{\textit{Matter}}$ *gauge* symmetry $(\overline{M})$

$$\delta_{\overline{M}} \phi^a = \frac{4\pi}{k}\overline{\mu} t^a \psi, \tag{7a}$$

$$\delta_{\overline{M}} \psi = 0, \qquad \delta_{\overline{M}} \overline{\psi} = \mathrm{d}_\phi \overline{\mu} \equiv \mathrm{d}\overline{\mu} - \overline{\mu}\phi, \tag{7b}$$

where $\overline{\mu} = \sum_{i=0}^{n-2} \overline{\mu}_{2i+1}$, is also $2n-3$-stage on-shell reducible.

2. Path integral.

According to the Batalin-Vilkovisky prescription, to quantize gauge theory one should find, first of all, a solution S (*extended classical action*) of the master equation, $(S,S)=0$, satisfying suitable boundary conditions [1, 7]. We claim that such a solution, of the form analogous to (3), is given by

$$S = S_{gCS} + S_m, \tag{8}$$

where

$$S_{gCS} = \frac{k}{4\pi}\int_{S^4} \mathrm{Tr}\left(\Phi \mathrm{d}\Phi + \frac{2}{3}\Phi^3\right), \tag{8a}$$

$$S_m = \int_{\mathcal{L}} \overline{\Psi}\mathrm{d}_{\Phi}\Psi \equiv \int_{\mathcal{L}} \overline{\Psi}\mathrm{d}_{\phi}\Psi, \tag{8b}$$

and

$$\Phi = \phi + \sum_{i=0}^{N}\left(\sum_{j=-2i-2}^{-1} \Phi^{*j\,0\,0}_{2i+1-j} + \sum_{j=1}^{2i+1} \Phi^{j\,0\,0}_{2i+1-j}\right), \tag{9a}$$

$$\Psi = \psi + \sum_{i=1}^{n-1}\left(\sum_{j=-2i-1}^{-1} \overline{\Psi}^{*\,0\,0\,j}_{2i-j} + \sum_{j=1}^{2i} \Psi^{0\,j\,0}_{2i-j}\right), \tag{9b}$$

$$\overline{\Psi} = \overline{\psi} + \sum_{i=1}^{n-1}\left(\sum_{j=-2i-1}^{-1} \Psi^{*\,0\,j\,0}_{2i-j} + \sum_{j=1}^{2i} \overline{\Psi}^{0\,0\,j}_{2i-j}\right). \tag{9$\overline{b}$}$$

The lower index denotes the form degree (deg), the upper indices denote the ghost numbers (gh_ϕ, gh_ψ and $gh_{\overline{\psi}}$), corresponding to the gauge symmetries (G, M and $\overline{M}$ respectively). The capital components without asterisks are ***ghost fields***, whereas the components with asterisks are corresponding ***antifields***. It is customary and convenient to introduce the (nilpotent) BRST operator s. In the antifield-antibracket language

$$sX = (X,S), \tag{10}$$

where X is any smooth function of fields and antifields. In our case, we can put

$$s\Phi^a = F^a_{\Phi} + \frac{4\pi}{k}\overline{\Psi}t^a\Psi|^0, \tag{11a}$$

$$s\Psi = \mathrm{d}_\phi \Psi, \qquad s\overline{\Psi} = \mathrm{d}_\phi \overline{\Psi}, \tag{11b}$$

where $F_\Phi \equiv \mathrm{d}_\Phi^2 \equiv \mathrm{d}\Phi + \Phi^2$, and the symbol " $|^0$ " indicates that the expression is constrained to the subspace of the zero ghost numbers. The explicit formulas for particular components can be obtained by the expansion of both sides of (11) with respect to (9). Using the (generalized) Bianchi identity, $\mathrm{d}_\Phi F_\Phi = 0$, and the anticommutativity, $\{s, \mathrm{d}_\Phi\} = 0$, we can verify that $s^2 = 0$.

Since $sA_I^* = \delta_r S/\delta A^I$ and $sA^I = \delta_l S/\delta A_I^*$, where A^I (A_I^*) denotes any (anti)field component, and $r(l)$ denotes the right (left) derivative, the master equation is equivalent to the *closedness* of the extended classical action S with respect to the BRST operator s. We verify that by virtue of (11)

$$\begin{aligned} sS &= \frac{k}{2\pi}\int_{S^d} \mathrm{Tr}(s\Phi F_\Phi) + \int_{\mathcal{L}} \mathrm{d}(\overline{\Psi}\mathrm{d}_\phi \Psi) - \int_{\mathcal{L}} \overline{\Psi} F_\phi \Psi \\ &= \frac{k}{2\pi}\int_{S^d} \mathrm{d}\mathrm{Tr}\left(\Phi \mathrm{d}\Phi + \frac{2}{3}\Phi^3\right) + \int_{\mathcal{L}} \mathrm{d}(\overline{\Psi}\mathrm{d}_\Phi \Psi) = 0. \end{aligned} \tag{12}$$

The vanishing of the integrals in (12) follows from the closedness of the manifolds ($\partial S^d = \emptyset$, $\partial \mathcal{L} = \emptyset$), and from the triviality of the bundle B.

Thus the Feynman path integral of the theory assumes the form

$$Z = \int \exp(\mathrm{i}S)|_\Sigma, \tag{13}$$

where the symbol " $|_\Sigma$ " indicates that we should eliminate antifields introducing the functional Γ, and putting

$$A_I^* = \frac{\delta_r \Gamma}{\delta A^I}.$$

3. Link invariants.

To approach the problem of topological invariants one usually introduces a suitable family of invariant functionals called "topological observables". For instance, in the case of Chern-Simons theory one introduces the *Wilson loops* (the trace of holonomy operators). Since in higher dimensions there is no corresponding notion of "Wilson surfaces" one should solve the problem in another way. In our case, the *matter* functional (2) is supposed to play the role of an *observable.*

Consider then the complementary pair of submanifolds $\{\mathcal{L}^{2k-1}, \mathcal{L}^{2n-2k+1}\}$, putting for convenience $n \geq k \geq \left[\frac{n}{2}\right] + 1$. Obviously, in general position the submanifolds do not intersect, but deforming a part of $\mathcal{L}^{2k-1}$ inside a small ball $\mathcal{B}^d$ into $\mathcal{L}'^{2k-1}$ we can meet some obstructions formed by the second submanifold ($\mathcal{L}^{2n-2k+1}$). General position and dimensional analysis indicates that the obstructions can form a collection of points $\{\mathcal{P}_l\}_{l=1}^m$. It appears that just these points are a source of "non-abelian linking numbers", and they give a measurable contribution to Z. Let us concentrate on the change of the integrand in (13) caused by the deformation of $\mathcal{L}^{2k-1}$. Analytically, the deformation can be expressed by the use of the Stokes theorem and it yields

$$\Delta S_m = \int_{\mathcal{M}} \mathrm{d}(\overline{\Psi}\mathrm{d}_\Phi \Psi) = \int_{\mathcal{M}} (\mathrm{d}_\Phi \overline{\Psi} \mathrm{d}_\Phi \Psi + \overline{\Psi} F_\Phi \Psi), \tag{14}$$

where $\partial\mathcal{M} = \mathcal{L}^{2k-1} \sqcup \overline{\mathcal{L}'^{2k-1}}$, and hence $\mathcal{M} \cap \mathcal{L}^{2n-2k+1} = \bigcup_{l=1}^m \mathcal{P}_l$. Now we substitute the curvature F_Φ on the RHS of (14) for the functional derivative operator

$$F_\Phi^a \longrightarrow \frac{4\pi}{\mathrm{i}k\sqrt{g}} * \frac{\delta}{\delta\phi^a}, \tag{15}$$

where " $*$ " denotes the Hodge star operator. The substitution (15) is an identity provided the ordering of the terms is such that the functional derivative can act on S_{gCS} producing the curvature F_Φ in correspondence to (4a). Essentially, (15) is a translation operator in a function space. Integrating by parts in (13) yields the "monodromy operator"

$$M = \exp\left[\frac{4\pi}{\mathrm{i}k} \sum_{a=1}^{d_G} (\overline{\psi}' t^a \psi')(\overline{\psi}'' t^a \psi'')(\mathcal{P}_l)\right] \tag{16}$$

for each intersection $\mathcal{P}_l$, where the prime (the double prime) means that the fields belong to $\mathcal{M}$ ($\mathcal{L}^{2n-2k+1}$). Since the form degree of the exponent as well as the ghost numbers should vanish only non-ghost zero-forms can enter (16).

To calculate the *matrix elements* of (16) one should introduce the following "holomorphic" scalar product

$$(f,g) = \frac{1}{2\pi\mathrm{i}} \int fg \exp(\mathrm{i}\overline{\psi}\psi)\mathrm{d}\overline{\psi}\mathrm{d}\psi. \tag{17}$$

Expanding (16) in a power series, multiplying with respect to the primed and double primed fields independently, and next resumming we get the *monodromy matrix*

$$\mathbf{M} = \left(\overline{\psi}'\overline{\psi}'', M\psi''\psi'\right) = \exp\left(\frac{4\pi}{ik}\sum_{a=1}^{d_G} t^a \otimes t^a\right). \tag{18}$$

Collecting the terms (18) coming from the all points $\{\mathcal{P}_l\}_{l=1}^{m}$ we finally obtain

$$\mathbf{M}_m = \mathbf{M}^m = \exp\left(\frac{4\pi m}{ik}\sum_{a=1}^{d_G} t^a \otimes t^a\right). \tag{19}$$

To obtain the corresponding skein relation we should arrange a collection of links differing with respect to the number of the points in the ball $\mathcal{B}^d$. The links entering the collection should possess an increasing number of $\mathcal{P}_l$, i. e., $m = 0, 1, \ldots, m_{max}$. For some m_{max} (it depends on $\mathcal{R}(G)$) $\mathbf{M}_m$'s satisfy the condition of linear dependence and hence a corresponding skein relation emerges. The exact recipe and some concrete examples are given in [6]. As a final remark, it is interesting to note that the form of the monodromy matrix and consequently skein relations are independent of the dimension and they are algebraic identical to the standard three-dimensional case.

The author is indebted to Prof. H. D. Doebner for his kind hospitality in Clausthal, and the organizers of the School for his kind invitation to Srní. The work was supported by the Alexander von Humboldt Foundation and the University of Łódź Grant.

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