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THE CYCLIC HOMOLOGY OF $P(G)$

BOHUMIL CENKL AND MICHELINE VIGUÉ-POIRRIER

Let $P(G) = \bigoplus_{n \geq 0} P^n(G)$ be the free associative differential graded algebra, over a commutative ring K , associated with the data of a finitely generated torsion free nilpotent group G as in [1]. More precisely $P(G) = (T(V), d^*)$, where V is a free \mathbb{Z} -module, graded in degree one, i.e. $V = V^{-1}$, and $d^* : T(V)^n \rightarrow T(V)^{n+1}$. To such a cochain algebra corresponds a negatively graded chain algebra $P_{-*} = P_{-*}(G) = T(V_*)$ with the differential $d_* = d^*$ and $V_{-1} = V^1$. Recall that the total Hochschild complex C_* of $(P_{-*}(G), d_*)$ is negatively graded. By definition $HH_*(P(G), d^*) = HH_*(P_{-*}(G), d_*) = H_*(C_*, d + b)$ and $C_* = \bigoplus_{n \geq 0} C_{-n}$. The definition of the Connes boundary $B : C_{-n} \rightarrow C_{-n+1}$ can be found in [2] and [3]. Thus we have a bicomplex:

$$\begin{array}{ccccccccccc}
 & & 0 & \xleftarrow{B} & C_0 & \xleftarrow{B} & \dots & \xleftarrow{B} & C_{-n+1} & \xleftarrow{B} & C_{-n} & \xleftarrow{B} & \dots \\
 & & \downarrow & & \downarrow^{b+d} & & & & \downarrow^{b+d} & & \downarrow^{b+d} & & \\
 0 & \xleftarrow{\quad} & C_0 & \xleftarrow{B} & C_{-1} & \xleftarrow{B} & \dots & \xleftarrow{B} & C_{-n} & \xleftarrow{B} & C_{-n-1} & \xleftarrow{B} & \dots
 \end{array}$$

for $n \geq 0$.

The total complex $(\text{Tot } C)_* = \bigoplus_{n \geq 0} (\text{Tot } C)_{-n}$ is negatively graded. We have $(\text{Tot } C)_{-n} = C_{-n} \oplus C_{-n+2} \oplus C_{-n+4} \oplus \dots$. This complex will be denoted by $K[u] \otimes_B C_*$, where $K[u]$ is the polynomial algebra on the generator u of degree -2 . The differential is the operator $b + d + uB$.

Definition. The homology $HC_*(P^*, d^*)$ of the complex $(K[u] \otimes_B C_*, b + d + uB)$, is called the cyclic homology of $(P(G), d^*)$.

The homology $HC_*(P^*, d^*)$ is negatively graded. Using the terminology introduced by Jones, it is called the negative cyclic homology. Then it is denoted by $HC_*^-(P_{-*}(G), d_*)$.

Let $\bar{V} = V_{-1} = P^1(G)$. Then according to the Theorem 1.5 in [2]

$$HH_*(P_{-*}, d_*) = H_*(P_{-*} \oplus P_{-*} \otimes \bar{V}, \delta),$$

where $\delta = d + \delta' + \delta''$, $d = d_*$ on P , $\delta'(a \otimes \bar{v}) = (-1)^{|a|}(av - (-1)^{|a|}va)$, $\delta''(a \otimes \bar{v}) = da \otimes \bar{v} - S(a, dv)$ for $a \in P, v \in V$. δ' and δ'' are both zero maps on P , (see [1], page 6). When the complex $(T(V), d)$ is negatively graded, we get similar results as those stated in [2], Theorem 2.4. Let

$$K_* = (K[u] \otimes (P_{-*} \oplus P_{-*} \otimes \bar{V}), D),$$

where $|u| = -2, D = 0$ on $K[u]$, and

$$D(u^n \otimes (a + b\bar{v})) = u^n \otimes \delta(a + b\bar{v}) + u^{n+2} \beta(a)$$

when $a \in P_{-\ast}, b \in P_{-\ast}, \bar{v} \in \bar{V}$, and

$$\beta(v_1 \dots v_p) = v_1 \dots v_{p-1} \otimes \bar{v}_p + \sum_{i=1}^{p-1} (-1)^{|v_{i+1}| + \dots + |v_p|} (|v_1| + \dots + |v_i|) v_{i+1} \dots v_p v_1 \dots v_{i-1} \otimes \bar{v}_i.$$

Using the norm $\|\cdot\|$ on P (see [1], page 3), we define a filtration on the complex K_\ast by setting

$$F_i = \{c = u^n \otimes (a + b\bar{v}) | \max(\|a\|, \|b\| + \|v\|) \leq i\}.$$

It is obvious that the filtration is an ascending filtration $F_i \subset F_{i+1}$. Then from the construction of the differential D on K it follows that

$$DF_i \subset F_i$$

Let $\{E^r, d^r\}$ be the spectral sequence corresponding to the filtration $\{F_i\}$. Let $F_i = F'_i \oplus F''_i$, where

$$F'_i = \{w \otimes a \in K[u] \otimes P | \|w \otimes a\| \leq i\},$$

$$F''_i = \{w \otimes (b \otimes \bar{v}) \in K[u] \otimes (P \otimes \bar{V}) | \|w \otimes (b \otimes \bar{v})\| \leq i\},$$

and let $p' : F'_i \rightarrow E_i^0, p'' : F''_i \rightarrow E_i^0, p = p' + p'' : F_i \rightarrow E_i^0$ be the projections.

Next consider the maps d, δ' and δ'' (page 5 of [1])

$$d = 1 \otimes d : K[u] \otimes P \rightarrow K[u] \otimes P,$$

$$\delta' = 1 \otimes \delta' : K[u] \otimes (P \otimes \bar{V}) \rightarrow K[u] \otimes P,$$

$$\delta'' = 1 \otimes \delta'' : K[u] \otimes (P \otimes \bar{V}) \rightarrow K[u] \otimes (P \otimes \bar{V}),$$

$$\delta''' : K[u] \otimes P \rightarrow K[u] \otimes (P \otimes \bar{V}),$$

where

$$\delta'''(u^n \otimes (v_1 \dots v_p)) = u^{n+1} \otimes (v_1 \dots v_{p-1} \otimes \bar{v}_p + \sum_{i=1}^{p-1} (-1)^{\epsilon_i} v_{i+1} \dots v_p v_1 \dots v_{i-1} \otimes \bar{v}_i),$$

$$\epsilon_i = (|v_{i+1}| + \dots + |v_p|)(|v_1| + \dots + |v_i|).$$

Then from (pages 6 and 7 of [1]) it follows that on the elements $u^n \otimes (a \oplus (b \otimes \bar{v})) \in K[u] \otimes (P^1 \oplus (P^1 \otimes \bar{V}))$ of norm equal to i ,

$$d(u^n \otimes a) = -u^n \otimes \sum a^t \cdot t + \dots,$$

$$\delta'(u^n \otimes b \otimes \bar{v}) = u^n \otimes (bv + vb),$$

$$\delta''(u^n \otimes b \otimes \bar{v}) = -u^n \otimes \sum b^t \cdot t \otimes \bar{v} + u^n \otimes S(b, v^t \cdot t) + \dots,$$

$$\delta'''(u^n \otimes a) = u^{n+1} \otimes \bar{a}.$$

Here \dots stands for the terms of filtration $\leq i - 1$. This proves

Lemma 1. Let $u^n \otimes (a \oplus (b \otimes \bar{v}))$ be an element of $P^1 \oplus (P^1 \otimes \bar{V})$ of norm i then

$$p\mathcal{L}(u^s \otimes (a \oplus (b \otimes \bar{v}))) = p' \{u^s \otimes (bv + vb - \sum a^t \cdot t)\} \\ + p'' \{u^s \otimes (S(b, v^t \cdot t) - \sum b^t \cdot t \otimes \bar{v}) + u^{n+1} \otimes \bar{a}\}.$$

Suppose that the group G , which is the fundamental group of a k - dimensional nilmanifold, is a free abelian group. Then a simple verification of the computation preceding Lemma 1 gives

Lemma 2. The E^1 -term of the spectral sequence $\{E^r, d^r\}$ is isomorphic to the cyclic homology

$$HC_*(P(H)),$$

where H is a free abelian group on k generators.

Lemma 3. If H is a free abelian group on k generators, then the cochain algebra $(P^* = P(H), d^*)$ is quasi-isomorphic to the exterior algebra on the free \mathbb{Z} -module generated by k elements of degree one, with zero differential.

Proof. See [1].

Lemma 4. Let K be a commutative ring containing \mathbb{Q} , let $\wedge(f_1, \dots, f_k)$ be the exterior algebra on the K -free module $\oplus_{i=1}^k K f_i$, with $|f_i| = -1$, and let $K[e_1, \dots, e_k]$ be the polynomial algebra on the K -free module $\oplus_{i=1}^k K e_i$ with $|e_i| = 0$. Then ,

$$HC_{-n}^-(\wedge(f_1, \dots, f_k), d = 0) = HC_{-n}^-(K) \oplus \beta(\wedge^{n+1}(f_1, \dots, f_k)) \cdot K[e_1, \dots, e_k]$$

for $n \geq 0$; β is the algebra derivation defined by $\beta(f_i) = e_i$, $\beta(e_i) = 0$, and $\wedge^i(f_1, \dots, f_k)$ denotes the K -vector space generated by words of length i in the variables f_1, \dots, f_k .

Remark. Since $HC_{-n}^-(K) = K$ for n even and $HC_{-n}^- = 0$ for n odd, it follows that for $n \geq k$, $HC_{-n}(\wedge(f_1, \dots, f_k)) = 0$ for n odd and $HC_{-n}(\wedge(f_1, \dots, f_k)) = K$ for n even.

Proof of Lemma 4. A modification of the proof of the Theorem 2.4 in [3] shows that the map

$$\theta : (C_*, b, B) \rightarrow (\wedge(f_i) \otimes K[e_i], 0, \beta), \\ \theta(a_0 \otimes a_1 \otimes \dots \otimes a_p) = \frac{(-1)^{\epsilon_p(a)}}{p!} a_0 \beta(a_1) \dots \beta(a_p)$$

if $a_0 \in \wedge(f_i), a_i \in \wedge^+(f_i), 1 \leq i \leq p$ satisfies

- 1 . $\theta \circ b = 0, \theta \circ B = \beta \circ \theta,$
- 2 . $H_*(C_*, b) = \wedge(f_i) \otimes K[e_i],$
- 3 . $HC_*(C_*, b, B) = HC_*(\wedge(f_i) \otimes K[e_i], 0, \beta).$ Here $HC_*(\wedge(f_i) \otimes K[e_i], 0, \beta)$ is the homology of the complex $L_* = \oplus_{n \geq 0} L_{-n},$

$$L_{-n} = (\wedge(f_i) \otimes K[e_i])_{-n} \oplus (\wedge(f_i) \otimes K[e_i])_{-n+2} \oplus \dots$$

with differential β^L ,

$$\beta^L(a_{-n}, a_{-n+2}, \dots) = (0, \beta(a_{-n}), \beta(a_{-n+2}), \dots).$$

Since $H_*(\wedge(f_i) \otimes K[e_i], \beta)$ is equal to zero in non-zero degrees, and is equal to K in degree zero, it follows that

$$H_{-n}(L_*) = \begin{cases} K \oplus \beta(\wedge^{n+1}(f_i)) \cdot K[e_i] & \text{if } n \text{ is even,} \\ \beta(\wedge^{n+1}(f_i)) \cdot K[e_i] & \text{if } n \text{ is odd} \end{cases}$$

Summarizing our results we get

Theorem. Let G be a finitely generated torsion free nilpotent group, and let $k = \dim K(G, 1)$. Let $P(G)$ be the polynomial cochain algebra of G endowed with the norm $\|\cdot\|$ as defined in [1]. Let K be a commutative ring containing the rationals. Then the norm $\|\cdot\|$ induces filtrations on the Hochschild and Connes complexes such that

1. There is a spectral sequence $E_{p,-q}^r$ converging to $HH_{-n}(P_{-n}(G))$ with the E^1 -term isomorphic to $HH_{-n}(P_{-n}(H))$, where H is the free abelian group on k generators. In fact

$$\bigoplus_{p-q=n} E_{p,-q}^1 = HH_{-n}(P_{-n}(H)) \simeq \wedge^n(f_1, \dots, f_k) \otimes K[e_1, \dots, e_k],$$

where $K[e_1, \dots, e_k]$ is the polynomial algebra on k generators in degree 0, and $\wedge^n(f_1, \dots, f_k)$ is the vector space spanned by words of length n in the exterior algebra on the K -free module generated by k generators in degree one.

2. There is a spectral sequence ${}^c E_{p,-q}^r$ converging to $HC_{-n}^-(P_{-n}(G))$ with ${}^c E^1$ -term isomorphic to $HC_{-n}^-(P_{-n}(H))$. We have

$$\bigoplus_{p-q=n \geq 0} {}^c E_{p,-q}^1 = HC_{-n}^-(P_{-n}(H)) = HC_{-n}^-(K) \oplus V_n$$

with

- a) $HC_{-n}^-(K) = 0$ if n is odd, and $HC_{-n}^-(K) = K$ if n is even,
- b) $V_n = 0$ if $n \geq k$,
- c) if $n < k$, then V_n is the $K[e_1, \dots, e_k]$ -module generated by elements of the form:

$$\sum_{j=1}^{n+1} (-1)^{j-1} e_{i_j} de_{i_1} \wedge \dots \wedge de_{i_{j-1}} \wedge de_{i_{j+1}} \wedge \dots \wedge de_{i_{n+1}}$$

for all $\{i_1 < i_2 < \dots < i_{n+1}\} \in [1, \dots, k]$. de_i stands for the differential form of the i -th variable e_i of the polynomial algebra $K[e_1, \dots, e_k]$.

3. If K is of characteristic zero, then the Connes long exact sequence

$$\dots \rightarrow HC_{-n+2}^- \rightarrow HC_{-n}^- \rightarrow HH_{-n}^- \rightarrow HC_{-n+1}^- \rightarrow \dots$$

induces the short exact sequences:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & HC_{-n}^-(P_{-*}(H))/HC_{-n}^-(K) & \longrightarrow & HH_{-n}(P_{-*}(H)) & & \\
 & & \simeq \downarrow & & \simeq \downarrow & & \\
 0 & \longrightarrow & \text{Ker} \cap (\wedge^n(f_i) \cdot K[e_i]) & \longrightarrow & \wedge^n(f_1, \dots, f_k) \otimes K[e_1, \dots, e_k] & & \\
 & & \longrightarrow & HC_{-n+1}^-(P_{-*}(H))/HC_{-n+1}^-(K) & \longrightarrow & 0 & \\
 & & & \simeq \downarrow & & & \\
 & \xrightarrow{\beta} & \beta[\wedge^n(f_1, \dots, f_k)] \cdot K[e_1, \dots, e_k] & \longrightarrow & 0 & &
 \end{array}$$

for $n \geq 1$.

Corollary. Let G be a finitely generated torsion free nilpotent group and let $k = \dim K(G, 1)$. Let $P(G)$ be the cochain algebra of G with coefficients in a field of zero characteristic. Then

- 1 . $HH_{-n}(P_{-*}(G)) = 0$ if $n > k$,
- 2 . $\tilde{H}C_{-n}^-(P_{-*}(G)) = 0$ if $n \geq k$, and where $\tilde{H}C$ is the quotient of the cyclic homology of P_{-*} over the cyclic homology of the ground field.

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