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In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1994. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 37. pp. [103]–114.

Persistent URL: http://dml.cz/dmlcz/701549

Q-DEFORMED INVERSE SCATTERING PROBLEM

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Starting from the physical point of view on the Miura transformation as reflectionless potential and its connection with supersymmetry we defined scaling q-deformation of this to obtain q-deformed supersymmetric quantum mechanics (SSQM) and application for solution an inverse problem is shown.

In this paper we present an interesting application of quantum group theory and q-deformed systems which have found interest in such diverse areas of mathematics and theoretical physics as knot theory and topology,conformal field theory,statistical QM and integrable models [Jim90].

This paper is in final form and no version of it will be submitted for publication elsewhere.

Integrable models represented by nonlinear evolution equations with their symmetries play for us the crucial role to obtain q-deformed object, especially the Miura transformation, which will play the main role in obtaining the q-deformed SSQM and factorization.

Knowing the role that symmetries play in physics it is natural to deform a real physical system, as for example a physical solution of nonlinear evolution eq. as a soliton or a potential.

In this spirit let us concern our attention on the well known integrable models described by the KdV and MKdV eqs. [CD82], where the solitonic solution also plays the role of potential.

Let us write the KdV eq. in the form

$$V_{+t} - 6V_+V_{+x} + V_{+xxx} = 0 , \qquad (1)$$

where $V_+ = v^2 + v_x$, $v = v(x,t)$, $v_t = \frac{dv}{dt}$, $v_x = \frac{dv}{dx}$, (2)

corresponds to the Miura transformation coupling KdV with MKdV, which has the form

$$v_t - 6v^2v_x + v_{xxx} = 0 . \qquad (3)$$

The MKdV eq. (3) is invariant under the transformation

$$v \to -v . \qquad (4)$$

From the symmetry of MKdV under (4) follows

$$V_+ \longrightarrow V_- = v^2 - v_x , \qquad (5)$$

which is also the solution of KdV eq.

$$V_{-t} - 6V_-V_{-x} + V_{-xxx} = 0 . \qquad (6)$$

Suppose now to interpret the Miura transformation as a transformation which defines a function v in terms of the $V_+ (V_-)$.

Then v is a solution of Riccati`s eq. and through eq. $v = \psi_x/\psi$ one is led to consider as associated to the KdV eq. (1), the Schrödinger eq.

$$\psi_{xx} - V_+\psi = 0$$

with a potential V_+ that satisfies (1).

From physical point of view on $V_\pm$ as the potentials we can choose the q-deformation of $V_\pm$ as the q-scaling deformation or if we look on $V_\pm$ as the soliton solution of KdV, as the s-shift deformation. At first we shall study the q-scaling deformation, which is physically interesting.

The Riccati eq. $V_+ = v^2 + v_x = \mu^2$
has the solution $v = \mu\, th\, \mu x \quad (V_- = \mu^2,\ v = -\mu\, th\, \mu x)$.

General q-scaling deformation operator can be defined on the continnous functions

$$\begin{aligned} D_q v(x) &= v(qx) \,, \\ D_q v_1(x)\, v_2(x) &= \left[D_q v_1(x)\right]\left[D_q v_2(x)\right] \,, \\ D_q \frac{d}{dx} &= \frac{1}{q}\frac{d}{dx} D_q \end{aligned} \tag{7}$$

with the group properties

$$D_{q_1} . D_{q_2} = D_{q_1 q_2} \,, \quad D_{q^{-1}} = D_q^{-1} \,, \quad D_1 = 1 \,.$$

So the solution of Riccati`s eq.

$$q^2 D_q V_+ = q^2 v^2(qx) + q v_x(qx) = \mu^2 \tag{8}$$

has the self-similar form

$$q v(qx) = \mu\, th\, \mu x \Rightarrow v(x) = q^{-1}\mu\, th\, q^{-1}\mu x \,.$$

Using self-similarity we get

$$\uparrow^2 = q^2 D_q V_+ = q(q v_q^2 + v_{qx}) = q V_{q+} \,, \tag{9}$$

where $v_q = D_q v(x) = v(qx) = q^{-1} v(x)$.

The same is valid for V_- when

$$q \to q^{-1} \,, \quad v(x) \to -q^{-1} v(q^{-1}x) \,.$$

So we define the q-deformed Miura transformations

$$\begin{aligned} V_{q+} &= q^{-1} V_+ = q D_q V_+ = q v_q^2 + v_{qx} \,, \\ V_{q-} &= q V_- = q^{-1} D_{q^{-1}} V_- = q^{-1} v_{q^{-1}}^2 - v_{q^{-1}x} \,, \end{aligned} \tag{10}$$

where q is a real number $q \neq 0$.

Under the transformation (4) $q V_{q+} \to q^{-1} V_{q-}$ because

$$\begin{aligned} q V_{q+} &= v^2 + v_x \,, \\ q^{-1} V_{q-} &= v^2 - v_x \,. \end{aligned} \tag{11}$$

From (11) follows

$$v^2 = \frac{q V_{q+} + q^{-1} V_{q-}}{2} \,, \tag{12}$$

$$v_x = \frac{q V_{q+} - q^{-1} V_{q-}}{2} \,, \tag{13}$$

$$\frac{d}{dx} \sqrt{\frac{q V_{q+} + q^{-1} V_{q-}}{2}} = \frac{q V_{q+} - q^{-1} V_{q-}}{2} \,. \tag{14}$$

In the theory of the spectral transforms and solitons [CD82] there is shown that the Schrödinger factorization is equivalent to the Miura transformation.

The intimate connection between the Miura transformation and supersymmetric "square root" was established and the method for obtaining the superpartner potential in SSQM was discussed in connection with nonlinear eqs. and reflectionless potentials [Hru89,Bag89].

In this direction we can speculate how to obtain q-deformed SSQM on the backround of physical motivation.

It is well known that the one soliton solution of the KdV eq. (1) has the form

$$V_+(x,t) = -\frac{1}{L^2}\,\mathrm{sech}^2\left(\frac{x-\frac{2}{L^2}t}{\sqrt{2}L}\right),$$

where L is the constant and V_+ represents a reflectionless potential (for more details see [Hru89]).

Let we suppose $V_{q\pm}$ like the KdV solutions, as q-deformed potentials in the sub-Hamiltonians $H_{q\pm}$ by the following way

$$\begin{aligned} H_{q+} &\equiv q^{-1}H_+ = \frac{1}{2}q^{-1}(p^2+V_+) \quad , \\ H_{q-} &\equiv q H_- = \frac{1}{2}q\,(p^2+V_-) \quad ; \end{aligned} \tag{15}$$

where

$$\begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = H_{SSQM} = \begin{pmatrix} A^+A & 0 \\ 0 & AA^+ \end{pmatrix} = \frac{1}{2}(p^2+w^2+\sigma_3 w_x)$$

$$[x,p] = i \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The q-deformed SSQM Hamiltonian and corresponding factorization have the form

$$H_q = \{Q_q^+, Q_q\} = \begin{pmatrix} A_q^+A_q & 0 \\ 0 & A_qA_q^+ \end{pmatrix} = \begin{pmatrix} H_{+q} & 0 \\ 0 & H_{-q} \end{pmatrix} \tag{16}$$

and q-deformed supercharges are

$$Q_q = \begin{pmatrix} 0 & 0 \\ A_q & 0 \end{pmatrix} \quad , \quad Q_q^+ = \begin{pmatrix} 0 & A_q^+ \\ 0 & 0 \end{pmatrix}. \tag{17}$$

From definition (15) we can see

$$A_q^+A_q = q^{-1}A^+A \quad , \quad A_qA_q^+ = q\,AA^+ . \tag{18}$$

We can then define the commutator between q-deformed objects as q-commutator

$$[A_q^+, A_q] \equiv [A, A^+]_q = qAA^+ - q^{-1}A^+A \, , \; [A, A^+]_q = -[A^+, A]_{q^{-1}}$$

and analogously q-anticommutator

$$\{B_q^+, B_q\} \equiv \{B^+, B\}_q = qB^+B + q^{-1}BB^+ \, , \; \{B^+, B\}_q = \{B, B^+\}_{q^{-1}}.$$

The operators Q_q^+, Q_q, H_q satisfy the q-deformed version of SSQM algebra:
$$\{Q^+, Q\}_q = H_q \quad ,$$
$$\{Q, Q\}_q = \{Q^+, Q^+\}_q = 0 \quad , [H, Q]_q = [Q^+, H]_q = 0 . \tag{19}$$

In the SSQM we know the form of factorization operators
$$A = \frac{1}{\sqrt{2}}(p + iv) \qquad , \qquad A^+ = \frac{1}{\sqrt{2}}(p - iv) .$$

The question is how to define factorization operators A_q, A_q^+ so as to respect (18).

From physical point of view on $V_\pm$ as the reflectionless potentials we have chosen q-deformation (10) as the q-scaling deformation.

For obtaining the Hermitian conjugate of D_q we use the Hilbert space of functions where for scalar product one has
$$\varphi, \psi \in \mathcal{L}_2 \, , \, (\varphi(x), \psi(qx) = D_q\psi(x)) = (q^{-1}\varphi(q^{-1}x) = D_q^+\varphi(x), \psi(x))$$
and D_q^+ can be found
$$D_q^+ = q^{-1} D_{q^{-1}} \quad , \quad (D_q^+)^+ = D_q . \tag{20}$$
We can define q-scalinq deformed factorization operators
$$A_q^+ = \frac{1}{\sqrt{2}}(p - iv) D_q \quad , \tag{21}$$
$$A_q = \frac{q^{-1} D_{q^{-1}}}{\sqrt{2}} (p + iv) \quad ,$$
respecting (18),because after little algebra using relations (7,20,21) we can get
$$A_q^+ A_q = \frac{1}{2} q^{-1} (p^2 + v^2 + v_x) = q^{-1} A^+ A = q^{-1} H_+ =$$
$$= H_{+q} = \frac{1}{2} q D_q (p^2 + V_+) = q D_q H_+ . \tag{22}$$
$$A_q A_q^+ = \frac{1}{2} q (p^2 + q^{-2} v^2 (q^{-1}x) - q^{-1} v_x (q^{-1}x)) = q A A^+ = q H_- =$$
$$= H_{-q} = \frac{1}{2} q^{-1} D_{q^{-1}} (p^2 + V_-) = q^{-1} D_{q^{-1}} H_- . \tag{23}$$

So starting from the q-scaling deformation of the Miura transformation and the connection between N-soliton solutions of KdV, SSQM and reflectionless potentials we have obtained q-deformed SSQM in correspondence with V.Spiridonov [Spi92].

We can see directly that the spectra of $H_{q\pm}$ sub-Hamiltonians are related via the q^{-2}- scaling what can be seen by the following way:

Let $H_{q\pm}\psi_\pm = E_{q\pm}\psi_\pm$, $A\psi_+ = \psi_-$, $A^+\psi_- = \psi_+$,

then $AH_{q+}\psi_+ = AE_{q+}\psi_+ = E_{q+}\psi_-$,

but $AH_{q+}\psi_+ = q^{-1}AA^+A\psi_+ = q^{-2}H_{q-}\psi_- = q^{-2}E_{q-}\psi_-$

so $E_{q+} = q^{-2}E_{q-}$ and possible exception concerns only the lowest level as in SSQM.

From SSQM let us suppose [Hru89]

$$V_- = v^2 - v_x = \frac{1}{2}L^2 \tag{24}$$

what is a very simple Riccati eq.,whose solution is given by substitution $v = -(\ln\psi_0)_x$ .Here ψ_0 is the solution of the zero-energy Schrödinger eq.

$$\psi_{0xx} - V_-\psi_0 = \psi_{0xx} - \frac{1}{2L^2}\psi_0 = 0 \tag{25}$$

and $\psi_0 = \text{const.} \times \cosh\frac{x}{L\sqrt{2}}$, $v = -\frac{1}{\sqrt{2}L}\tanh\frac{x}{\sqrt{2}L}$.

The superpartner to V_- is V_+ and

$$V_+ = v^2 + v_x = V_- + 2v_x = \frac{1}{2L^2} - \frac{1}{L^2}\operatorname{sech}^2\frac{x}{L\sqrt{2}} .$$

Now if we denote the superpartner potentials

$$\eta_0(x) = v^2 - v_x - \frac{1}{2L^2} = 0 \quad , \quad \eta_1(x) = v^2 + v_x - \frac{1}{2L^2} = -\frac{1}{L^2}\operatorname{sech}^2\frac{x}{L\sqrt{2}}$$

then

$$\eta_{0q} = q^{-1}D_{q^{-1}}\eta_0 = 0 \quad , \quad \eta_{1q} = qD_q\eta_1 = -\frac{q}{L^2}\operatorname{sech}^2\frac{qx}{L\sqrt{2}} = -2q^{-1}\left(\frac{q}{\sqrt{2}L}\operatorname{th}\frac{qx}{\sqrt{2}L}\right)_x =$$
$$= -2q^{-1}\left(\ln\psi_0\left(E_1 = \frac{1}{2L^2}\right)\right)_{xx} .$$

Using the results from SSQM we can construct the symmetric reflectionless $\eta_j(x)$, $j=1,2\ldots,N$ and for arbitrary j we may assume $\eta_{j-1}(x)$ to be known and define v_j by

$$\eta_{j-1}(x) = v_j^2 - v_{jx} - E_j .$$

Then the superpartner has the form

$$\eta_j(x) = v_j^2 + v_{jx} - E_j$$

and this procedure is invariant under the self-similarity transformation: $v_j = q^j v(q^j x)$.

The crucial point for the construction of the whole chain of the corresponding supersymmetric Hamiltonians $H_j = \frac{1}{2}(p^2+\eta_j)$ is that the superpartner can be expressed via the eigenfunction construction of the corresponding Hamiltonian as we can see from the following:

$$H_+ = A^+A = H_- + [A^+, A] = H_- + 2v_x = H_- - 2(\ln\psi_0)_{xx}. \quad (26a)$$

For the q-scaling deformation we have

$$H_{q+} = H_{q-} - [A, A^+]_q = H_{q-} + [A^+, A]_{q^{-1}} = H_{q-} - 2q^{-1}(\ln\psi_0(qx))_{xx} \quad (26b)$$

Using the results from SSQM [Hru89] we can demonstrate that for the q-SSQM the symmetric reflectionless potentials has the form

$$\eta_{Nq} = q D_q \eta_N = -2q^{-1}(\ln\det D_{Nq})_{xx} , \quad (27)$$

where the elements of the matrix D_{Nq} are given by

$$[D_{Nq}]_{jk} = \frac{1}{2}(q\gamma_k)^{j-1}\left[\exp(q\gamma_k x) + (-1)^{j+k}\exp(-q\gamma_k x)\right]. \quad (28)$$

It corresponds to the reparametrization $\gamma_j \to q\gamma_j$.

This results can be generalized for $\eta_j \to q^j\eta_j$ for the U(N)-vector non-linear Schrödinger eq., factorization and the relation with SSQM using the results in ref. [Hru89] .

These formulas in this reference coincide with formulas in the work [Spi92] ,when we take the ansatz:

$$\varkappa_j(x) = q^j \varkappa(q^j x), \quad j = 1,2 \ldots \infty .$$

Physically realy the q-scaling deformation D_q corresponds $E_q \sim D_q p^2 = q^{-2} \frac{d^2}{dx^2} D_q$,i.e. q^{-2} - scaling deformation of the energy what is nothing new.But via the connection SSQM with reflectionless potentials q-scaling deformation is equivalent to the reparametrization $\mu_i \to q\mu_i$ in this potential and it is physically interesting,because any solvable discrete spectrum problem can be represented via the chain of these potential.

The KdV eq. is invariant under the Galilean transformation [CD82]:

$$\begin{aligned} x' &= x - 6\Lambda t , \\ t' &= t , \\ V_+' &= V_+ + \Lambda . \end{aligned} \tag{29}$$

From the physical point of view on V_+ as the solitonic solution we have the travelling wave property

$$V_+(x - 6\Lambda t) = -\frac{1}{2} 6\Lambda \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\Lambda}(x - 6\Lambda t + x_0)\right) . \tag{30}$$

For $\Lambda = \mu^2$ the q-scaling deformation is equivalent to the $\mu \to q\mu$ and $\Lambda_q = q^2\mu^2$.

But if one chooses the deformation as the shift operator S_q: $S_q V_+(x) = V_+(x - 6\Lambda_q t)$, where $S_q = \exp\left(-6\Lambda_q t \frac{d}{dx}\right)$ evidently such deformation does not change the spectrum of V_+ and we obtain ordinary isospectral Hamiltonians in SSQM.

So from the Galilean invariance of KdV the S_q- shift deformation gives physically the old results.

The role of SSQM is well known in the connection between the conserved quantities of the KdV eq. and SG eq.where the spectral problem for SG is a problem of factorization of KdV spectral problem (or SSQM "square root")[Bag89].

The factorization is important also in the case of q-deformation of Bäcklund transformation for SG eq. [CD82].

Let SG eq. has the form

$$\partial_+ \partial_- \varphi = \sin \varphi \quad , \tag{31}$$

where $\partial_\pm = \frac{\partial}{\partial x_\pm}$, $x_\pm = \frac{1}{2}(x \pm t)$ and let it has two independent solutions

$$\begin{aligned} \varphi &= q^{-1} u + q v \\ \bar{\varphi} &= q^{-1} u - q v \end{aligned} \quad , \tag{32}$$

and we shall study the eqs.

$$\begin{aligned} q^{-1} \partial_- u &= q \sin q v \quad , \\ q\, \partial_+ v &= q^{-1} \sin q^{-1} u \quad . \end{aligned} \tag{33}$$

The following relations are valid

$$\begin{aligned} \partial_+ \partial_- q^{-1} u &= \partial_+ (q \sin q v) = \cos q v \sin q^{-1} u \, , \\ \partial_- \partial_+ q v &= \partial_- (q^{-1} \sin q^{-1} u) = \cos q^{-1} u \sin q v \end{aligned} \tag{34}$$

and

$$\partial_+ \partial_- (q^{-1} u + q v) = \sin (q^{-1} u + q v) = \tag{35a}$$
$$= \cos q v \sin q^{-1} u + \cos q^{-1} u \sin q v \, ,$$

$$\partial_- \partial_+ (q^{-1} u - q v) = \sin (q^{-1} u - q v) = \tag{35b}$$
$$= \cos q v \sin q^{-1} u - \cos q^{-1} u \sin q v \, .$$

After derivation the eq. (35a) with $\partial_{q^{-1}u}$ and ∂_{qv} we obtain

$$\frac{\partial^2_{q^{-1}u}\,(q^{-1}\sin q^{-1}u)}{q^{-1}\sin q^{-1}u} = \frac{\partial^2_{qv}\,(q\sin qv)}{q\sin qv} = -1 \qquad (36)$$

and the same for (35b).

But from (32)

$$q^{-1}u = \frac{1}{2}(\varphi + \tilde{\varphi}) \quad , \quad qv = \frac{1}{2}(\varphi - \varphi')$$

and so we get

$$\begin{aligned} \partial_-(q^{-1}u) &= \partial_-\left[\tfrac{1}{2}(\varphi + \tilde{\varphi})\right] = q\sin qv \,, \\ \partial_+(qv) &= \partial_+\left[\tfrac{1}{2}(\varphi - \tilde{\varphi})\right] = q^{-1}\sin q^{-1}u . \end{aligned} \qquad (37)$$

The relation (37) is q-deformed Bäcklund transformation.

This result exactly corresponds to the known result for the Bäcklund transformation when

$$\begin{aligned} u_q &= q^{-1}u \rightarrow u \,, \\ v_q &= qv \rightarrow v \,. \end{aligned}$$

For sinh-Gordon eq. the procedure is the same only in the relation (36) (i.e. analog of the Schrödinger eq.) we get +1 (i.e. analog of the spectral parameter).

This deep coincidence between results from nonlinear evolution eq. and SSQM (factorization,Miura transformation, Bäcklund transf.,Darboux transf.) also in the case of q-deformation gives to speculate about the q-deformed inverse scattering problem.

Generally it can give the connection between q-analysis deformation and quantum algebras and the following physical application for the q-deformed inverse scattering problem:

Let $\{E_n^*\}$ is the spectrum of the unknown potential and

$$E_n^* = f_{D_q}(E_n)$$

where f_{D_q} is a general deformation D_q of the analytical function f,which maps the spectrum E_n of the analytical known potential η_N (given via SSQM) on E_n^*.

Then the unknown potential has the form $f_{D_q}(\eta_N)$.

The role of the q-deformed transformations of the nonlinear evolution eqs. is crucial in such application.

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