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NATURAL LIFTINGS OF FOLIATIONS TO THE r-TANGENT BUNDLE

Włodzimierz M. Mikulski

In this paper a classification of natural liftings of foliations to the r-tangent bundle is given.

0. All manifolds are assumed to be finite dimensional, Hausdorff, without boundary and C^∞. Mappings are assumed to be of class C^∞ and foliations are assumed to be C^∞ and without singularities. A foliation is identified with its class of cocycles or its tangent distribution. For equivalent definitions of foliations see [5].

From now on we fix two natural numbers n, p such that $p < n$. Let $r \geq 0$ be an integer. Suppose that for any n-dimensional manifold M and any p-dimensional foliation F defined on M there exists a foliation $L(F)$ on T^rM projectable (by the r-tangent bundle projection $\pi : T^rM \to M$) onto F. According to the general theory of natural transformations, see [2], we introduce the following definition.

DEFINITION 0.1. A system $L = \{L(F)\}$ as above is called a *natural lifting of foliations to the r-tangent bundle* if the following naturality condition is satisfied: for any n-manifold M, any foliation F of dimension p on M and any diffeomorphism φ from an n-manifold N onto an open subset of M we have $L(\varphi^{-1}F) = (T^r\varphi)^{-1}L(F)$, where $\varphi^{-1}F$ is the inverse image of F.

Let us remark that for any manifold M the r-tangent bundle T^rM is the fibre bundle of r-jets at 0 of curves $\mathbb{R} \to M$ with the target projection $\pi : T^rM \to M$,

[0]) This paper is in final form and no version of it will be submitted for publication elsewhere.

$\pi(j_0^r\gamma) = \gamma(0)$. If (U, x^i) is a chart on M, then the induced chart $(\pi^{-1}(U),\ x^{i,\lambda})$ on T^rM is given by

$$x^{i,\lambda}(j_0^r\gamma) = \frac{1}{\lambda!}\frac{d^\lambda}{dt^\lambda}(x^i \circ \gamma)(0),$$

where $i = 1, ..., dim(M)$, $\lambda = 0, ..., r$. If $f : M \to N$ is a mapping then the induced mapping $T^rf : T^rM \to T^rN$ is given by $T^rf(j_0^r\gamma) = j_0^r(f \circ \gamma)$, see [4], [1].

We have the following examples of natural liftings of foliations to the r-tangent bundle.

Example 0.1. Let F be a p-dimensional foliation on an n-manifold M. Let $q = 0, ..., r$. It is well-known that T^qM admits a (canonically dependent on F) foliation T^qF of dimension $p(q+1)$ projecting onto F. This foliation is defined by a cocycle $(T^qU_i,\ T^qf_i,\ T^qg_{ij})$, where $(U_i,\ f_i,\ g_{ij})$ is a cocycle defining F. We put

$$L_q^r(F) = (\pi_q^r)^{-1}(T^qF)\ ,$$

where $\pi_q^r : T^rM \to T^qM$ is the projection given by $j_0^r\gamma \to j_0^q\gamma$. Then $L_q^r(F)$ is a foliation on T^rM of dimension $p(q+1)+(r-q)n$ projectable onto F and the system $L_q^r = \{L_q^r(F)\}$ is a natural lifting of foliations to the r-tangent bundle.

The main result of this paper is the following theorem.

Theorem 0.1. *Any natural lifting of foliations to the r-tangent bundle is equal to one of liftings $L_0^r, ..., L_r^r$ described in Example 0.1*

This theorem extends the result of [3] to the case when $r > 1$.

1. From now on the usual coordinates on $\mathbb{R}^n$ are denoted by x^i, the standard p-dimensional foliation on $\mathbb{R}^n$ generated by $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^p}$ is denoted by F^p and the induced coordinates on $T^r\mathbb{R}^n$ are denoted by $x^{i,\lambda}$, where $i = 1, ..., n$, $\lambda = 0, ..., r$. We shall use the following simple properties of liftings of foliations to the r-tangent bundle.

Applying the naturality condition to the inclusion maps we obtain the following locality condition.

Lemma 1.1. *Let L be a natural lifting of foliations to the r-tangent bundle. Suppose that F_1, F_2 are two p-dimensional foliations on an n-manifold M such that $F_1 = F_2$ on some open subset $U \subset M$. Then $L(F_1) = L(F_2)$ on $\pi^{-1}(U) \subset T^rM$.*

Using the locality condition and the naturality condition to the adapted charts we deduce the following fact.

LEM 1.2. *Let L', L'' be natural liftings of foliations to the r-tangent bundle. Suppose that $L'(F^p) = L''(F^p)$. Then $L' = L''$.*

From the definitions of L^r_q and π^r_q given in Example 0.1 it follows the following lemma.

LEM 1.3. *For any $q \in \{0, ..., r\}$ the foliation $L^r_q(F^p)$ is generated by*

$$\frac{\partial}{\partial x^{i,\lambda}}; \qquad (i,\lambda) \in \{1,...,p\} \times \{0,...,q\} \bigcup \{1,...,n\} \times \{q+1,...,r\}.$$

The foliation $ker(d\pi^r_q) \subset TT^r\mathbb{R}^n$ is generated by

$$\frac{\partial}{\partial x^{i,\lambda}}; \qquad (i,\lambda) \in \{1,...,n\} \times \{q+1,...,r\}.$$

In order to prove the first part of Lemma 1.3 we observe that F^p is defined by the cocycle $\{(\mathbb{R}^n, f, id_{\mathbb{R}^{n-p}})\}$, where $f : \mathbb{R}^n \to \mathbb{R}^{n-p}$ is the projection onto last $n-p$ factors. Therefore $T^qF^p = ker(dT^qf)$ is generated by $\frac{\partial}{\partial x^{i,\lambda}}$, where $(i,\lambda) \in \{1,...,p\} \times \{0,...,q\}$.

2. We will prove Theorem 0.1 by induction with respect to r. For $r = 0$ this theorem is trivial. The inductive step is based on the following lemmas.

LEM 2.1. *Let L be a natural lifting of foliations to the r-tangent bundle and let $j^r_0\xi \in T^r\mathbb{R}^n$. If $Y_1, ..., Y_s$ are linearly independent constant (with respect to the induced coordinates $x^{i,\lambda}$) vector fields on $T^r\mathbb{R}^n$ such that $L(F^p)_{j^r_0\xi}$, the tangent distribution of $L(F^p)$ at $j^r_0\xi$, is generated by $Y_1(j^r_0\xi), ..., Y_s(j^r_0\xi)$, then $L(F^p)$ is generated by $Y_1, ..., Y_s$.*

In particular, if Z is a constant vector field on $T^r\mathbb{R}^n$ such that $Z(j^r_0\xi) \in L(F^p)_{j^r_0\xi}$, then Z is an $L(F^p)$-vector field.

PROOF: Let $j^r_0\eta \in T^r\mathbb{R}^n$ and $Z_1, ..., Z_s$ be linearly independent constant vector fields on $T^r\mathbb{R}^n$ such that $L(F^p)_{j^r_0\eta}$ is generated by $Z_1(j^r_0\eta), ..., Z_s(j^r_0\eta)$. Using the naturality condition with respect to the homotheties $tid_{\mathbb{R}^n}$, $t \neq 0$, we obtain

$$L(F^p)_{j^r_0(t\eta)} = span\{Z_1(j^r_0(t\eta)), ..., Z_s(j^r_0(t\eta))\},$$

because $tid_{\mathbb{R}^n}$ preserves F^p and $t\frac{\partial}{\partial x^{i,\lambda}}(j^r_0(t\eta)) = T^r(tid_{\mathbb{R}^n})(\frac{\partial}{\partial x^{i,\lambda}}(j^r_0\eta))$. If $t \to 0$ then

$$L(F^p)_{j^r_0 0} = span\{Z_1(j^r_0 0), ..., Z_s(j^r_0 0)\}.$$

In particular
$$L(F^p)_{j_0^r 0} = span\{Y_1(j_0^r 0), ..., Y_s(j_0^r 0)\} .$$
Then
$$L(F^p)_{j_0^r \eta} = span\{Y_1(j_0^r \eta), ..., Y_s(j_0^r \eta)\} ,$$
because $Y_1, ..., Y_s, Z_1, ..., Z_s$ are constant. The first part of the lemma is proved.

Now, let $Y_1 = Z, Y_2, ..., Y_s$ be linearly independent constant vector fields on $T^r\mathbb{R}^n$ such that $L(F^p)_{j_0^r \xi}$ is generated by $Y_1(j_0^r \xi), ..., Y_s(j_0^r \xi)$. Then $L(F^p)$ is generated by $Y_1, ..., Y_s$. Hence Z is an $L(F^p)$-vector field. □

LEM 2.2. *Let L be a natural lifting of foliations to the r-tangent bundle. Then*
$$ker(d\pi^r_{r-1}) \bigcap L^r_r(F^p) \subset L(F^p).$$
If additionally $L(F^p) - L^r_r(F^p) \neq \emptyset$, then
$$ker(d\pi^r_{r-1}) \subset L(F^p).$$

PROOF: By the definition of liftings we have
$$d\pi(L(F^p)_\sigma) = F^p_0 = span(\frac{\partial}{\partial x^1}(0), \frac{\partial}{\partial x^p}(0)),$$
where $\sigma = j_0^r(\tau \to \tau e_n) \in T^r\mathbb{R}^n$, $e_j = (0, ..., 1, ..., 0) \in \mathbb{R}^n$, 1 in j position. Of course, $d\pi(\frac{\partial}{\partial x^{j,0}}(\sigma)) = \frac{\partial}{\partial x^j}(0)$. Then for any $i \in \{1, ..., p\}$ there exists a vector $Y_i \in ker(d_\sigma \pi) \subset T_\sigma T^r\mathbb{R}^n$ such that
$$\frac{\partial}{\partial x^{i,0}}(\sigma) + Y_i \in L(F^p)_\sigma.$$
By the naturality of L with respect to
$$\varphi = (x^1, ..., x^{i-1}, x^i + (x^n)^r x^i, x^{i+1}, ..., x^n)$$
preserving $germ_0(F^p)$ and σ we deduce that
$$dT^r\varphi(\frac{\partial}{\partial x^{i,0}}(\sigma) + Y_i) \in L(F^p)_\sigma.$$
We see that
$$\begin{aligned} dT^r\varphi(\frac{\partial}{\partial x^{i,0}}(\sigma)) &= dT^r\varphi(\frac{d}{d\tau}_{|\tau=0}[\tau \to j_0^r(t \to \tau e_i + te_n)]) \\ &= \frac{d}{d\tau}_{|\tau=0}[\tau \to j_0^r(t \to \varphi(\tau e_i + te_n))] \\ &= \frac{d}{d\tau}_{|\tau=0}[\tau \to j_0^r(t \to (\tau t^r + \tau)e_i + te_n)] \\ &= \frac{\partial}{\partial x^{i,0}}(\sigma) + \frac{\partial}{\partial x^{i,r}}(\sigma). \end{aligned}$$

On the other hand, since $j_0^r\varphi = id$, then $T_0^r\varphi = id$, and then $dT^r\varphi(Y_i) = Y_i$ as Y_i is generated by a curve in $T_0^r\mathbb{R}^n$. Hence $\frac{\partial}{\partial x^{i,r}}(\sigma) \in L(F^p)_\sigma$. Now, by Lemma 2.2 $\frac{\partial}{\partial x^{i,r}}$ is an $L(F^p)$-vector field. The first part of the lemma is proved because of Lemma 1.3.

It remains to prove the second part of the lemma. By Lemmas 2.1, 1.3 and the first part of Lemma 2.2 there exist $\lambda \in \{1, ..., r\}$, $i \in \{p+1, ..., n\}$ and a constant $L(F^p)$-vector field Y on $T^r\mathbb{R}^n$ of the form

$$Y = \frac{\partial}{\partial x^{i,\lambda}} + \sum_{(j,\mu)\in A} a_{j\mu}\frac{\partial}{\partial x^{j,\mu}}$$

where $A = \{p+1, ..., n\} \times \{\lambda, ..., r\} - \{(i,\lambda)\}$ and $a_{j,\mu} \in \mathbb{R}$. Consider the following two cases.

(I) $n-p \geq 2$. Then without loss of generality we can assume that $i \neq n$. Consider $q \in \{1, ..., n\}$. By the naturality condition with respect to

$$\psi = (x^1, ..., x^{q-1}, x^q + x^i(x^n)^{r-\lambda}, x^{q+1}, ..., x^n)$$

preserving $germ_0(F^p)$ and σ we get

$$dT^r\psi(Y(\sigma)) \in L(F^p)_\sigma.$$

We see that

$$\begin{aligned} dT^r\psi(\frac{\partial}{\partial x^{j,\mu}}(\sigma)) =& dT^r\psi(\frac{d}{d\tau}_{|\tau=0}[\tau \to j_0^r(t \to \tau t^\mu e_j + te_n)]) \\ =& \frac{d}{d\tau}_{|\tau=0}[\tau \to j_0^r(t \to \psi(\tau t^\mu e_j + te_n))] \\ =& \frac{d}{d\tau}_{|\tau=0}[\tau \to j_0^r(t \to \delta_j^i \tau t^\mu t^{r-\lambda} e_q + te_n + \tau t^\mu e_j)] \\ =& \begin{cases} \frac{\partial}{\partial x^{j,\mu}}(\sigma), & \text{if} \quad (j,\mu) \in A \\ \frac{\partial}{\partial x^{i,\lambda}}(\sigma) + \frac{\partial}{\partial x^{q,r}}(\sigma), & \text{if} \quad (j,\mu) = (i,\lambda) \end{cases} \end{aligned}$$

i.e. $dT^r\psi(Y(\sigma)) = Y(\sigma) + \frac{\partial}{\partial x^{q,r}}(\sigma)$. Hence $\frac{\partial}{\partial x^{q,r}}(\sigma) \in L(F^p)_\sigma$. Now, by Lemma 2.1 $\frac{\partial}{\partial x^{q,r}}$ is an $L(F^p)$-vector field. The lemma in this case is proved because of Lemma 1.3.

(II) $n-p=1$. By the naturality condition with respect to

$$\eta = (x^1, ..., x^{n-1}, x^n + (x^n)^{r-\lambda+1})$$

preserving $germ_0(F^p)$ we get

$$dT^r\eta(Y(\sigma)) \in L(F^p)_{T^r\eta(\sigma)}.$$

We see that

$$\begin{aligned}
dT^r\eta(\frac{\partial}{\partial x^{n,\mu}}(\sigma)) =& dT^r\eta(\frac{d}{d\tau}_{|\tau=0}[\tau \to j_0^r(t \to \tau t^\mu e_n + te_n)]) \\
=& \frac{d}{d\tau}_{|\tau=0}[\tau \to j_0^r(t \to \eta(\tau t^\mu e_n + te_n))] \\
=& \frac{d}{d\tau}_{|\tau=0}[\tau \to j_0^r(t \to \tau t^\mu e_n + te_n + (\tau t^\mu + t)^{r-\lambda+1}e_n)] \\
=& \frac{d}{d\tau}_{|\tau=0}[\tau \to j_0^r(t \to \tau t^\mu e_n + te_n + t^{r-\lambda+1}e_n + \\
& (r-\lambda+1)t^{r-\lambda}\tau t^\mu e_n + ...)] \\
=& \frac{d}{d\tau}_{|\tau=0}[\tau \to j_0^r(t \to \tau t^\mu e_n + te_n + t^{r-\lambda+1}e_n + \\
& (r-\lambda+1)t^{r-\lambda}\tau t^\mu e_n)] \\
=& \begin{cases} \frac{\partial}{\partial x^{n,\mu}}(T^r\eta(\sigma)), & \text{if} \quad \mu > \lambda \\ (r-\lambda+1)\frac{\partial}{\partial x^{n,r}}(T^r\eta(\sigma)) + \frac{\partial}{\partial x^{n,\lambda}}(T^r\eta(\sigma)), & \text{if} \quad \mu = \lambda \end{cases}
\end{aligned}$$

i.e. $dT^r\eta(Y(\sigma)) = Y(T^r\eta(\sigma)) + (r-\lambda+1)\frac{\partial}{\partial x^{n,r}}(T^r\eta(\sigma))$. Hence

$$\frac{\partial}{\partial x^{n,r}}(T^r\eta(\sigma)) \in L(F^p)_{T^r\eta(\sigma)}.$$

Now, by Lemma 2.1 $\frac{\partial}{\partial x^{n,r}}$ is an $L(F^p)$-vector field. Reminding the first part of this lemma and Lemma 1.3 we can finish the proof. □

3. We are now in position to make the inductive step. Let L be a natural lifting of foliations to the r-tangent bundle. From Lemma 2.1 we deduce that there exist linearly independent constant (with respect to the induced coordinates) vector fields $Y_1, ..., Y_s$ on $T^r\mathbb{R}^n$ such that $L(F^p)$ is generated by $Y_1, ..., Y_s$.

Suppose that $r \geq 1$. Since any constant vector field on $T^r\mathbb{R}^n$ is π^r_{r-1}-conjugate with a constant vector field on $T^{r-1}\mathbb{R}^n$, then $L(F^p)$ is π^r_{r-1}-conjugate with a foliation $\tilde{L}(F^p)$ on $T^{r-1}\mathbb{R}^n$, i.e. $d\pi^r_{r-1}(L(F^p)) = \tilde{L}(F^p)$. Hence, by the naturality condition applied to the adapted charts $L(F)$ is π^r_{r-1}-conjugate with a foliation $\tilde{L}(F)$ on $T^{r-1}M$ for any p-foliation F on an n-manifold M. Of course, the system $\tilde{L} = \{\tilde{L}(F)\}$ is a natural lifting of foliations to the $r-1$-tangent bundle.

By the inductive assumption $\tilde{L}$ is equal to one of liftings $L_0^{r-1}, ..., L_{r-1}^{r-1}$. If $\tilde{L} = L_q^{r-1}$ with $q < r-1$, then by Lemmas 2.2 and 1.3 we get $L(F^p) = L_q^r(F^p)$. Similarly, if $\tilde{L} = L_{r-1}^{r-1}$, then either $L(F^p) = L_{r-1}^r(F^p)$ or $L(F^p) = L_r^r(F^p)$ because of Lemmas 2.2 and 1.3. Now by Lemma 1.2 we complete the proof. □

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