RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 43 (1996), pp. 111-121

LIFTINGS OF COVARIANT $(0,2)$–TENSOR FIELDS TO THE BUNDLE OF K–DIMENSIONAL 1–VELOCITIES

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Abstract. We introduce and study some liftings of $(0,2)$–tensor fields on a manifold M to the bundle T^1_kM. Then we determine all first order natural $\mathbb{R}$–linear operators transforming $(0,2)$–tensor fields to T^1_kM. Finally we classify first order natural operators transforming symmetric $(0,2)$–tensor fields on M into $(0,2)$–tensor fields on T^1_kM.

1. Introduction

The bundle $T^1_kM = J^1_0(\mathbb{R}^k, M)$ of all k–dimensional 1–velocities plays an important role in differential geometry, especially in the analytical mechanics. In particular, for $k=1$ we obtain the classical tangent bundle $TM = T^1_1M$ and the linear frame bundle $FM = \mathrm{inv}J^1_0(\mathbb{R}^m, M)$, $m = \dim M$, is an open dense subset of T^1_mM.

We shall use the concept of a natural operator, which can be considered as a generalization of the concept of a geometrical construction, [6]. Using such a point of view, Kowalski and Sekizawa determined all first order natural operators transforming Riemannian metrics to the linear frame bundle FM, [7]. Further, Janyška has in [5] classified first order natural operators from Riemannian metrics into 2–forms on the tangent bundle TM. Moreover, the first author determined in [3] all first order natural operators from general $(0,2)$–tensor fields into $(0,2)$–tensor fields on TM.

In this paper we first study the classical linear liftings of $(0,2)$–tensor fields to the bundle T^1_kM, namely the vertical and the complete lifts. Then we prove that if $k>1$, then there is no natural isomorphism between $T^1_kT^*M$ and $T^*T^1_kM$. Further we introduce the antisymmetric lift and then some nonlinear liftings. Moreover, we determine all first order natural $\mathbb{R}$–linear operators transforming $(0,2)$–tensor fields on M into $(0,2)$–tensor fields on T^1_kM. Finally we classify first order natural operators transforming symmetric $(0,2)$–tensor fields on M into $(0,2)$–tensor fields on T^1_kM.

All manifolds and maps are assumed to be infinitely differentiable and all manifolds are paracompact.

2. The fundamental liftings

Let M be an m–dimensional smooth manifold. We denote by $p_M : TM \to M$ the tangent bundle and by $q_M : T^*M \to M$ the cotangent bundle of M. Let $\pi_M : T^1_kM =$

Supported by the GA CR, Grant No. 201/93/2125 and by the Maria Curie Skłodowska University.
This paper is in final form and no version of it will be submitted for publication elsewhere.

$J_0^1(\mathbb{R}^k, M) \to M$ be the bundle of k–dimensional 1–velocities. It is well known that the linear frame bundle $FM = \mathrm{inv}J_0^1(\mathbb{R}^m, M)$ is an open dense subset of $T_m^1 M$. The canonical coordinates (x^i) on M induce the additional coordinates $(y^i = dx^i)$ on TM, (p_i) on T^*M and $(y_\alpha^i, \alpha = 1, \dots, k)$ on $T_k^1 M$. The bundle $T_k^1 M$ can be identified with the Whittney sum $T_k^1 M = TM \oplus \cdots \oplus TM$ of k copies of TM. Further, we have k canonical projections $p_{TM}^\alpha : T_k^1 M \to TM$, $\alpha = 1, \dots, k$, $(x^i, y_1^i, \dots, y_k^i) \mapsto (x^i, y_\alpha^i)$.

Let $f : M \to \mathbb{R}$ be a function on M. The vertical lift f^V of f to $T_k^1 M$ is a function $f^V : T_k^1 M \to \mathbb{R}$ defined by $f^V = f \circ \pi_M$. Further, we define the α–complete lift $f^{C,\alpha} : T_k^1 M \to \mathbb{R}$, $\alpha = 1, \dots, k$ by $f^{C,\alpha}(j_0^1 \gamma) = \frac{\partial (f \circ \gamma)}{\partial t^\alpha}\Big|_0$. Obviously, $f \mapsto f^{C,\alpha}$ is a linear map of $C^\infty(M)$ into $C^\infty(T_k^1 M)$ satisfying $(f \cdot g)^{C,\alpha} = f^{C,\alpha} \cdot g^V + f^V \cdot g^{C,\alpha}$ for all $f, g \in C^\infty(M)$, $\alpha = 1, \dots, k$. Mikulski has recently proved that the $(k+1)$ lifts f^V, $f^{C,1}, \dots, f^{C,k}$ generate all natural liftings of functions to the bundle $T_k^1 M$. By [9], all natural transformations $C^\infty(M) \mapsto C^\infty(T_k^1 M)$ are of the form $\Phi(f^V, f^{C,1}, \dots, f^{C,k})$, where $\Phi : \mathbb{R}^{k+1} \mapsto \mathbb{R}$ is an arbitrary smooth function. Finally, the complete lift of f to $T_k^1 M$ is defined as the sum $f^C = \sum_{\alpha=1}^k f^{C,\alpha}$, [2]. It is interesting to point out that $f^{C,\alpha} = (p_{TM}^\alpha)^* \widetilde{f}^C$, where $\widetilde{f}^C$ is the complete lift of f to TM defined by $\widetilde{f}^C(y) = df_x(y)$, $x = p_M(y)$, in coordinates $\widetilde{f}^C(y) = \frac{\partial f(x)}{\partial x^i} y^i$.

Let X be a vector field on M. We define the α–vertical lift $X^{V,\alpha}$, $\alpha = 1, \dots, k$ of X to $T_k^1 M$ by means of translations in the α–directions in the individual fibres of $T_k^1 M$. If ω is a 1–form on M, then we have k functions $i_\alpha \omega : T_k^1 M \to \mathbb{R}$, $\alpha = 1, \dots, k$ defined by $(i_\alpha \omega)(u) = \omega(p_{TM}^\alpha(u))$. Then the α–vertical lift $X^{V,\alpha}$ can be also defined by $X^{V,\alpha}(i_\beta \omega) = \delta_\beta^\alpha \omega(X)$, [8]. Finally, the complete lift X^C of X to $T_k^1 M$ is defined as the flow prolongation of X, $X^C = \frac{\partial}{\partial t}\big|_0 (T_k^1(\exp tX))$, where $\exp tX$ means the flow of X, [6], [11]. By [10] the α–vertical and the complete lifts of X can be also defined by means of their actions on liftings of functions. We have

Lemma 1. *Let X and Y be arbitrary vector fields on M and let f be an arbitrary function on M. Then*

I. $X^C(f^{C,\alpha}) = (Xf)^{C,\alpha}, \quad \alpha = 1, \dots, k,$
$X^C(f^C) = (Xf)^C, \quad X^C(f^V) = (Xf)^V,$

II. $X^{V,\alpha}(f^{C,\beta}) = \delta_\beta^\alpha (Xf)^V, \quad \alpha, \beta = 1, \dots, k,$
$X^{V,\alpha}(f^V) = 0, \quad X^{V,\alpha}(f^C) = (Xf)^V, \quad \alpha = 1, \dots, k,$

III. $[X^C, Y^{V,\alpha}] = [X, Y]^{V,\alpha}, \quad [X^{V,\alpha}, Y^{V,\beta}] = 0, \quad \alpha, \beta = 1, \dots, k,$
$[X^C, Y^C] = [X, Y]^C.$

In coordinates, if $X = \xi^i \frac{\partial}{\partial x^i}$, then $X^{V,\alpha} = \xi^i \frac{\partial}{\partial y_\alpha^i}$, $X^C = \xi^i \frac{\partial}{\partial x^i} + \frac{\partial \xi^i}{\partial x^j} y_\alpha^j \frac{\partial}{\partial y_\alpha^i}$. Now we define the vertical and the complete lifts of $(0,2)$–tensor fields to $T_k^1 M$. We shall use the following

Lemma 2. *If G and H are $(0,2)$–tensor fields on $T_k^1 M$ such that for all vector fields X_1, X_2 on M we have $G(X_1^C, X_2^C) = H(X_1^C, X_2^C)$, then $G = H$.*

Proof. It suffices to prove that if $G(X_1^C, X_2^C) = 0$ for all vector fields X_1, X_2 on M, then $G = 0$. Suppose that

$$G = A_{ij} dx^i \otimes dx^j + B_{ij}^\alpha dx^i \otimes dy_\alpha^j + C_{ij}^\alpha dy_\alpha^i \otimes dx^j + D_{ij}^{\alpha\beta} dy_\alpha^i \otimes dy_\beta^j. \tag{1}$$

If $X_1 = \frac{\partial}{\partial x^i}$, $X_2 = \frac{\partial}{\partial x^j}$, then $G(X_1^C, X_2^C) = A_{ij} = 0$. Further, let $X_1 = \frac{\partial}{\partial x^i}$, $X_2 = \xi^i \frac{\partial}{\partial x^i}$. Then $G(X_1^C, X_2^C) = B_{ij}^\alpha \frac{\partial \xi^j}{\partial x^k} y_\alpha^k$, which implies $B_{ij}^\alpha = 0$. Quite analogously we prove that $C_{ij}^\alpha = 0$ and $D_{ij}^{\alpha\beta} = 0$. □

Let G be an arbitrary $(0,2)$–tensor field on M.

Definition 1. The *vertical lift* of G to $T_k^1 M$ is a $(0,2)$–tensor field G^V on $T_k^1 M$ defined by $G^V(X_1^C, X_2^C) = (G(X_1, X_2))^V$ for all vector fields X_1, X_2 on M.

Definition 2. The *α–complete lift* of G to $T_k^1 M$ is a $(0,2)$–tensor field $G^{C,\alpha}$ on $T_k^1 M$ defined by $G^{C,\alpha}(X_1^C, X_2^C) = (G(X_1, X_2))^{C,\alpha}$, $\alpha = 1, \dots, k$ for all vector fields X_1, X_2 on M. The *complete lift* G^C of G to $T_k^1 M$ is defined by $G^C(X_1^C, X_2^C) = (G(X_1, X_2))^C$ for all vector fields X_1, X_2 on M.

If G is a 2–form on M, then $G^V = \pi_M^* G$ is exactly the pull-back of G to $T_k^1 M$. Analogously, $G^{C,\alpha} = (p_{TM}^\alpha)^* \widetilde{G}^C$, where $\widetilde{G}^C$ is the complete lift of G to TM, [3]. We have $G^C = \sum_{\alpha=1}^k G^{C,\alpha}$. In coordinates, if $G = g_{ij} dx^i \otimes dx^j$, then

$$G^V = g_{ij} dx^i \otimes dx^j, \tag{2}$$

$$G^{C,\alpha} = \frac{\partial g_{ij}}{\partial x^k} y_\alpha^k dx^i \otimes dx^j + g_{ij} dx^i \otimes dy_\alpha^j + g_{ij} dy_\alpha^i \otimes dx^j. \tag{3}$$

One proves easily

Lemma 3. *Let F and G be $(0,2)$–tensor fields on M. We have*

I. $(aF + bG)^V = aF^V + bG^V$, $(aF + bG)^C = aF^C + bG^C$ *for all* $a, b \in \mathbb{R}$, $(aF + bG)^{C,\alpha} = aF^{C,\alpha} + bG^{C,\alpha}$ *for all* $a, b \in \mathbb{R}$, $\alpha = 1, \dots, k$,

II. $(F \otimes G)^{C,\alpha} = F^{C,\alpha} \otimes G^V + F^V \otimes G^{C,\alpha}$ *for all* $\alpha = 1, \dots, k$, $(F \otimes G)^C = F^C \otimes G^V + F^V \otimes G^C$, $(F \otimes G)^V = F^V \otimes G^V$,

III. *If G is symmetric (or antisymmetric), then G^V, $G^{C,\alpha}$ and G^C are symmetric (or antisymmetric) as well, $\alpha = 1, \dots, k$,*

IV. *If G is a 2–form on M, then G^V, $G^{C,\alpha}$ and G^C are 2–forms on $T_k^1 M$ and we have $(dG)^V = dG^V$, $(dG)^{C,\alpha} = dG^{C,\alpha}$, $(dG)^C = dG^C$, $\alpha = 1, \dots, k$,*

V. *If G has rank r, then $G^{C,\alpha}$ and G^C have rank $2r$ and G^V has rank r,*

VI. *If G is a Riemannian metric on M, then G^V, $G^{C,\alpha}$ and G^C are degenerated metrics on $T_k^1 M$,*

VII. $G^V(X^{V,\alpha}, Y^C) = 0$, $G^V(X^{V,\alpha}, Y^{V,\beta}) = 0$ *for all vector fields X, Y on M, $\alpha, \beta = 1, \dots, k$,*

VIII. $G^{C,\alpha}(X^{V,\beta}, Y^C) = G^{C,\alpha}(X^C, Y^{V,\beta}) = \delta_\beta^\alpha (G(X,Y))^V, G^{C,\alpha}(X^{V,\beta}, Y^{V,\gamma}) = 0$ *for all vector fields X, Y on M, $\alpha, \beta, \gamma = 1, \dots, k$,*

IX. $G^C(X^{V,\alpha}, Y^C) = G^C(X^C, Y^{V,\alpha}) = (G(X,Y))^V, G^C(X^{V,\alpha}, Y^{V,\beta}) = 0$ *for all vector fields X, Y on M, $\alpha, \beta = 1, \dots, k$.*

Denote by $\kappa_M : TT_k^1 M \to T_k^1 TM$ the isomorphism defined by the exchange homomorphism of Weil algebras of functors TT_k^1 and $T_k^1 T$, [6]. This isomorphism can be also defined by $\kappa_M \left(\frac{\partial}{\partial s}\big|_0 (j_0^1 \delta_s) \right) = j_0^1 \left(\frac{\partial}{\partial s}\big|_0 \delta_t \right)$, where $\delta(s,t) : \mathbb{R} \times \mathbb{R}^k \to M$,

$\delta(s,t) = \delta_s(t) = \delta_t(s)$. The complete lift of a vector field X on M to $T_k^1 M$ can be also described by $X^C = \kappa_M^{-1} \circ T_k^1 X$. Now we present a similar geometrical characterization of complete lifts of $(0,2)$–tensor fields to $T_k^1 M$. We first define natural transformations $s_M^\alpha : T_k^1 T^* M \to T^* T_k^1 M$ over the identity $\mathrm{id}_{T_k^1 M}$ of $T_k^1 M$, $\alpha = 1, \ldots, k$. If $X : M \to TM$ is a vector field and $\omega : M \to T^* M$ is a 1–form, then the contraction $\langle \omega, X \rangle : M \to \mathbb{R}$ is a function on M. Then s_M^α, $\alpha = 1, \ldots, k$ are defined by $\langle s_M^\alpha \circ T_k^1 \omega, X^C \rangle = \langle \omega, X \rangle^{C,\alpha}$. Analogously, one can also define a natural transformation $s_M : T_k^1 T^* M \to T^* T_k^1 M$ over $\mathrm{id}_{T_k^1 M}$ by $\langle s_M \circ T_k^1 \omega, X^C \rangle = \langle \omega, X \rangle^C$. Obviously, s_M is the sum of all s_M^α, $\alpha = 1, \ldots, k$ on the vector bundle $T^* T_k^1 M \to T_k^1 M$. If $k = 1$, then s_M is exactly the isomorphism $TT^* M \to T^* TM$ defined by Tulczyjev and Modugno and Stefani, cf. [6]. We shall denote by $(x^i, p_i, x_\alpha^i, p_{i,\alpha})$ or $(x^i, y_\alpha^i, r_i dx^i + s_i^\alpha dy_\alpha^i)$ the local coordinates on $T_k^1 T^* M$ or $T^* T_k^1 M$, respectively. Then the equations of s_M^α are $y_\alpha^i = x_\alpha^i$, $r_i = p_{i,\alpha}$, $s_i^\beta = \delta_\alpha^\beta p_i$ and the local coordinate expression of s_M is $y_\alpha^i = x_\alpha^i$, $r_i = \sum_{\alpha=1}^k p_{i,\alpha}$, $s_i^\alpha = p_i$, $\alpha = 1, \ldots, k$.

Remark 1. The well known isomorphism $TT^* M \to T^* TM$ is a particular case of the isomorphism $T_1^k T^* M \to T^* T_1^k M$, where $T_1^k M = J_0^1(\mathbb{R}, M)$ is the bundle of 1–dimensional k–velocities, [1]. On the other hand, if $k > 1$, then neither s_M^α nor $s_M : T_k^1 T^* M \to T^* T_k^1 M$ are isomorphisms. Moreover, the following assertion enables us to clarify that if $k > 1$, then *there is no natural isomorphism* $T_k^1 T^* M \to T^* T_k^1 M$.

Proposition 1. *All natural transformations of $T_k^1 T^* M$ into $T^* T_k^1 M$ are of the form*

$$
\begin{aligned}
& y_\alpha^i = A_\alpha^1 x_1^i + \cdots + A_\alpha^k x_k^i, \\
(4) \qquad & s_i^\alpha = B^\alpha p_i, \\
& r_i = (A_1^1 B^1 + \cdots + A_k^1 B^k) p_{i,1} + \cdots + (A_1^k B^1 + \cdots + A_k^k B^k) p_{i,k} + C p_i,
\end{aligned}
$$

where A_α^β, B^α and C are arbitrary smooth functions of the invariants $I_\beta = p_i x_\beta^i$, $\beta = 1, \ldots, k$.

Proof. Denote by G_m^r the group of all invertible r–jets of $\mathbb{R}^m$ into $\mathbb{R}^m$ with source and target zero. By the general theory of natural operations in differential geometry developed by Kolář, Michor and Slovák in [6], it suffices to determine all G_m^2–equivariant maps of the corresponding standard fibres,

$$
\begin{aligned}
y_\alpha^i &= y_\alpha^i(x_\beta^i, p_i, p_{i,\gamma}; \beta, \gamma = 1, \ldots, k), \\
s_i^\alpha &= s_i^\alpha(x_\beta^i, p_i, p_{i,\gamma}; \beta, \gamma = 1, \ldots, k), \\
r_i &= r_i(x_\beta^i, p_i, p_{i,\gamma}; \beta, \gamma = 1, \ldots, k).
\end{aligned}
$$

We shall denote by (a_j^i, a_{jk}^i) the canonical coordinates in G_m^2 and by tilde the coordinates of the inverse element. One evaluates easily the following transformation laws, which represent the action of G_m^2 on the standard fibres

$$
\begin{aligned}
&\overline{x}_\alpha^i = a_j^i x_\alpha^j, \quad \overline{p}_i = \widetilde{a}_i^j p_j, \quad \overline{p}_{i,\alpha} = \widetilde{a}_i^j p_{j,\alpha} + \widetilde{a}_{ik}^j a_\ell^k x_\alpha^\ell p_j, \\
&\overline{y}_\alpha^i = a_j^i y_\alpha^j, \quad \overline{s}_i^\alpha = \widetilde{a}_i^j s_j^\alpha, \quad \overline{r}_i = \widetilde{a}_i^j r_j + \widetilde{a}_{\ell i}^j a_k^\ell y_\alpha^k s_j^\alpha.
\end{aligned}
$$

Consider first y^i_α. The equivariance on the kernel of the jet projection $G^2_m \to G^1_m$ gives that y^i_α are independent of $p_{i,\alpha}$. Then the tensor evaluation theorem from [6] yields the first equation of (4). Quite analogously we deduce the second equation of (4). Assume finally r_i in the form $r_i = k^\beta p_{i,\beta} + \widetilde{r}_i(x^i_\alpha, p_i, p_{i,\beta})$. Then the equivariance reads $k^\beta = A^\beta_\alpha B^\alpha$ (the sum through α) and $\widetilde{r}_i$ have the tensorial transformation law. This completes the proof. □

The natural transformation s_M corresponds to $A^i_j = \delta^i_j$, $B^\alpha = 1$, $B^\beta = 0$ for $\beta \neq \alpha$ and $C = 0$.

Each tensor field $G = g_{ij} dx^i \otimes dx^j$ on M can be identified with the linear map $G_L : TM \to T^*M$, $p_i = g_{ij} y^j$, [3]. Let $\overline{G}_L : TT^1_k M \to T^*T^1_k M$ be the linear map over the identity $\mathrm{id}_{T^1_k M}$ of $T^1_k M$ corresponding to the $(0,2)$–tensor field (1) on $T^1_k M$. The coordinate expression of $\overline{G}_L : (x^i, y^i_\alpha, X^i, Y^i_\alpha) \mapsto (x^i, y^i_\alpha, r_i dx^i + s^\alpha_j dy^j_\alpha)$ is $r_i = A_{ij} X^j + B^\alpha_{ij} Y^j_\alpha$, $s^\alpha_i = C^\alpha_{ik} X^k + D^{\alpha\beta}_{ik} Y^k_\beta$. Using the definitions of G_L, $\overline{G}_L$, s^α_M and s_M we deduce

Proposition 2. *Let G be an arbitrary $(0,2)$–tensor field on M. Then*

I. *$G^{C,\alpha}$ is the only $(0,2)$–tensor field $\overline{G}$ on $T^1_k M$ satisfying $\overline{G}_L = s^\alpha_M \circ T^1_k G_L \circ \kappa_M$.*

II. *G^C is the only $(0,2)$–tensor field $\overline{G}$ on $T^1_k M$ satisfying $\overline{G}_L = s_M \circ T^1_k G_L \circ \kappa_M$.*

Each $(0,2)$–tensor field G on M defines k 1–forms τ^α on $T^1_k M$, $\alpha = 1, \dots, k$, $\tau^\alpha(u) = G(-, p^\alpha_{TM}(u))$. In coordinates, $\tau^\alpha = g_{ij} y^j_\alpha dx^i$. In other words, $\tau^\alpha = (G_L \circ p^\alpha_{TM})^* \omega$, where $\omega = p_i dx^i$ is the canonical Liouville 1–form on T^*M.

Definition 3. The *α–antisymmetric lift* of a $(0,2)$–tensor field G on M to $T^1_k M$ is the 2–form $G^{A,\alpha}$ on $T^1_k M$ defined by $G^{A,\alpha} = d\tau^\alpha$.

Obviously, $G^{A,\alpha} = (G_L \circ p^\alpha_{TM})^* \Omega$ is the pull-back of the canonical symplectic form $\Omega = d\omega$. In coordinates,

$$G^{A,\alpha} = \frac{\partial g_{jm}}{\partial x^i} y^m_\alpha dx^i \wedge dx^j - g_{ij} dx^i \wedge dy^j_\alpha. \tag{5}$$

3. Classification of linear natural liftings $T^* \otimes T^* \rightsquigarrow (T^* \otimes T^*) T^1_k$

In this section we determine all first order linear natural operators transforming $(0,2)$–tensor fields on M to the bundle of k–dimensional 1–velocities $T^1_k M$. We first prove the following auxiliary assertion.

Lemma 4. *All G^1_m–equivariant smooth maps $\underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{k} \times \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \to \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$, $E_{ij} = E_{ij}(y^i_\alpha, g_{ij}; \alpha = 1, \dots, k)$ are of the form*

$$E_{ij} = \varphi_1 g_{ij} + \varphi_2 g_{ji} + \varphi_3^{\gamma\delta} g_{ik} y^k_\gamma g_{js} y^s_\delta + \varphi_4^{\gamma\delta} g_{ik} y^k_\gamma g_{sj} y^s_\delta + \varphi_5^{\gamma\delta} g_{ki} y^k_\gamma g_{sj} y^s_\delta + \varphi_6^{\gamma\delta} g_{ki} y^k_\gamma g_{js} y^s_\delta$$

where $\varphi_i = \varphi_i(g_{ij} y^i_\alpha y^j_\beta; \alpha, \beta = 1, \dots, k)$.

Proof. Introduce new variables $u^i, v^i \in \mathbb{R}^m$, $\overline{u}^i = a^i_j u^j$, $\overline{v}^i = a^i_j v^j$ and consider the sum $E_{ij} u^i v^j$. This is a G^1_m–invariant smooth function $\psi = \psi(y^i_\alpha, g_{ij}, u^i, v^i; \alpha =$

$1,\dots,k)$. By the tensor evaluation theorem, [6], we have

$$\psi=\varphi(g_{ij}y^i_\alpha y^j_\beta, g_{ij}u^iu^j, g_{ij}v^iv^j, g_{ij}u^iv^j, g_{ij}v^iu^j, \\ g_{ij}y^i_\alpha u^j, g_{ij}u^iy^j_\alpha, g_{ij}y^i_\alpha v^j, g_{ij}v^iy^j_\alpha; \alpha,\beta=1,\dots,k).$$

Differentiating with respect to u^i and putting $u^i=0$ we obtain $E_{ij}v^j=\psi_1 g_{ij}v^j+\psi_2 g_{ji}v^j+\psi_3^\gamma g_{ij}y^j_\gamma+\psi_4 g_{ji}y^j_\gamma$, where $\psi_i=\psi_i(g_{ij}y^i_\alpha y^j_\beta, g_{ij}v^iv^j, g_{ij}v^iy^j_\alpha, g_{ij}y^i_\alpha v^j)$. Finally, differentiating with respect to v^i and setting $v^i=0$ we get the assertion. □

As a direct consequence we have

Lemma 5. *All G^1_m–equivariant smooth maps $\underbrace{\mathbb{R}^m\times\dots\times\mathbb{R}^m}_{k}\times\mathbb{R}^{m*}\odot\mathbb{R}^{m*}\to\mathbb{R}^{m*}\otimes\mathbb{R}^{m*}$, $E_{ij}=E_{ij}(y^i_\alpha, g_{ij};\alpha=1,\dots,k)$, where g_{ij} are symmetric in i and j, are of the form $E_{ij}=\varphi_1 g_{ij}+\varphi_2^{\gamma\delta}g_{ik}y^k_\gamma g_{js}y^s_\delta$, where $\varphi_i=\varphi_i(g_{ij}y^i_\alpha y^j_\beta;\alpha,\beta=1,\dots,k)$.*

Quite analogously to Lemma 4 one can prove

Lemma 6. *All G^1_m–equivariant smooth maps $\underbrace{\mathbb{R}^m\times\dots\times\mathbb{R}^m}_{k}\times\mathbb{R}^{m*}\otimes\mathbb{R}^{m*}\times\mathbb{R}^{m*}\otimes\mathbb{R}^{m*}\otimes\mathbb{R}^{m*}\to\mathbb{R}^{m*}\otimes\mathbb{R}^{m*}$, $E_{ij}=E_{ij}(y^i_\alpha, g_{ij}, g_{ij,k};\alpha=1,\dots,k)$, which are linear in g_{ij} and $g_{ij,k}$, are of the form $E_{ij}=\varphi_1 g_{ij}+\varphi_2 g_{ji}+\varphi_3^\alpha g_{ij,k}y^k_\alpha+\varphi_4^\alpha g_{ji,k}y^k_\alpha+\varphi_5^\alpha g_{ki,j}y^k_\alpha+\varphi_6^\alpha g_{kj,i}y^k_\alpha+\varphi_7^\alpha g_{ik,j}y^k_\alpha+\varphi_8^\alpha g_{jk,i}y^k_\alpha$, $\varphi_i\in\mathbb{R}$.*

Let G^V or $G^{C,\alpha}$ or $G^{A,\alpha}$ be the vertical or α–complete or α–antisymmetric lifts of a $(0,2)$–tensor field G on M to the bundle T^1_kM defined in (2), (3) and (5), respectively. In what follows we shall denote by G' the $(0,2)$–tensor field on M given by $\langle G, X\otimes Y\rangle=\langle G', Y\otimes X\rangle$ for all vector fields X and Y on M, in coordinates

$$G'=g_{ji}dx^i\otimes dx^j. \tag{6}$$

Now we deduce

Proposition 3. *All first order natural $\mathbb{R}$-linear operators $T^*\otimes T^*\rightsquigarrow(T^*\otimes T^*)T^1_k$ transforming $(0,2)$–tensor fields on M into $(0,2)$–tensor fields on T^1_kM are of the form*

$$G\mapsto A_1G^V+A_2(G')^V+\sum_{\alpha=1}^k A_3^\alpha G^{C,\alpha}+ \\ \sum_{\alpha=1}^k A_4^\alpha (G')^{C,\alpha}+\sum_{\alpha=1}^k A_5^\alpha G^{A,\alpha}+\sum_{\alpha=1}^k A_6^\alpha (G')^{A,\alpha}, \tag{7}$$

where all $A's$ are arbitrary real numbers.

Proof. Each $(0,2)$–tensor field on T^1_kM is of the form (1). By [6] we have to determine all G^2_m–equivariant maps $J^1_0(T^*\otimes T^*)\mathbb{R}^m\oplus T^1_k\mathbb{R}^m\to(T^*\otimes T^*)T^1_k\mathbb{R}^m$, in local coordinates $(g_{ij}, g_{ij,k}, y^i_\alpha)\mapsto(A_{ij}, B^\alpha_{ij}, C^\alpha_{ij}, D^{\alpha\beta}_{ij})$, which are linear in g_{ij} and $g_{ij,k}$. Using standard evaluations we determine the following transformation formulas

$$
\begin{aligned}
\overline{g}_{ij} &= \widetilde{a}_i^k \widetilde{a}_j^\ell g_{k\ell}, \\
\overline{g}_{ij,k} &= \widetilde{a}_i^m \widetilde{a}_j^n \widetilde{a}_k^p g_{mn,p} + (\widetilde{a}_{ik}^m \widetilde{a}_j^n + \widetilde{a}_i^m \widetilde{a}_{jk}^n) g_{mn}, \\
\overline{y}_\alpha^i &= a_j^i y_\alpha^j, \\
\overline{A}_{ij} &= \widetilde{a}_i^m \widetilde{a}_j^n A_{mn} + \widetilde{a}_i^m \widetilde{a}_{rj}^n \overline{y}_\beta^r B_{mn}^\beta + \widetilde{a}_{ri}^m \widetilde{a}_j^n \overline{y}_\alpha^r C_{mn}^\alpha + \widetilde{a}_{ri}^m \widetilde{a}_{sj}^n \overline{y}_\alpha^r \overline{y}_\beta^s D_{mn}^{\alpha\beta}, \\
\overline{B}_{ij}^\beta &= \widetilde{a}_i^m \widetilde{a}_j^n B_{mn}^\beta + \widetilde{a}_{ri}^m \widetilde{a}_j^n \overline{y}_\alpha^r D_{mn}^{\alpha\beta}, \\
\overline{C}_{ij}^\alpha &= \widetilde{a}_i^m \widetilde{a}_j^n C_{mn}^\alpha + \widetilde{a}_i^m \widetilde{a}_{rj}^n \overline{y}_\beta^r D_{mn}^{\alpha\beta}, \\
\overline{D}^{\alpha\beta} &= \widetilde{a}_i^m \widetilde{a}_j^n D_{mn}^{\alpha\beta}.
\end{aligned} \tag{8}
$$

Consider first $D_{ij}^{\alpha\beta}(y_\alpha^i, g_{ij}, g_{ij,k}; \alpha = 1, \dots, k)$. By Lemma 6,

$$
\begin{aligned}
D_{ij}^{\alpha\beta} &= \varphi_1^{\alpha\beta} g_{ij} + \varphi_2^{\alpha\beta} g_{ji} + \varphi_3^{\alpha\beta\gamma} g_{ij,k} y_\gamma^k + \varphi_4^{\alpha\beta\gamma} g_{ji,k} y_\gamma^k \\
&\quad + \varphi_5^{\alpha\beta\gamma} g_{ki,j} y_\gamma^k + \varphi_6^{\alpha\beta\gamma} g_{kj,i} y_\gamma^k + \varphi_7^{\alpha\beta\gamma} g_{ik,j} y_\gamma^k + \varphi_8^{\alpha\beta\gamma} g_{jk,i} y_\gamma^k.
\end{aligned} \tag{9}
$$

Then the equivariance on the kernel of the jet projection $G_m^2 \to G_m^1$ yields

$$
D_{ij}^{\alpha\beta} = d_1^{\alpha\beta} g_{ij} + d_2^{\alpha\beta} g_{ji} + d_3^{\alpha\beta\gamma} (g_{ij,k} - g_{ji,k} + g_{ki,j} - g_{kj,i} + g_{jk,i} - g_{ik,j}) y_\gamma^k.
$$

Moreover, we shall assume that C_{ij}^α, B_{ij}^α and A_{ij} are of the form analogous to that of (9). Using equivariance on the kernel of the jet projection $G_m^2 \to G_m^1$ and then the full equivariance we obtain

$$
\begin{aligned}
D_{ij}^{\alpha\beta} &= 0, \\
C_{ij}^\alpha &= (A_3^\alpha + A_6^\alpha) g_{ij} + (A_4^\alpha + A_5^\alpha) g_{ji}, \\
B_{ij}^\alpha &= (A_3^\alpha - A_5^\alpha) g_{ij} + (A_4^\alpha - A_6^\alpha) g_{ji}, \\
A_{ij} &= A_1 g_{ij} + A_2 g_{ji} + A_3^\alpha g_{ij,k} y_\alpha^k + A_4^\alpha g_{ji,k} y_\alpha^k \\
&\quad + A_5^\alpha (g_{jk,i} - g_{ik,j}) y_\alpha^k + A_6^\alpha (g_{kj,i} - g_{ki,j}) y_\alpha^k,
\end{aligned}
$$

which is nothing else but the coordinate form of (7). □

4. First order natural operators $T^* \odot T^* \rightsquigarrow (T^* \otimes T^*) T_k^1$

Notice that all the liftings of a $(0,2)$– tensor field to the bundle $T_k^1 M$ defined up till now are linear. Now we shall define some nonlinear liftings. Denote by $f^{\alpha\beta} = g_{ij} y_\alpha^i y_\beta^j$ the function on $T_k^1 M$ given by the full contraction and let τ^α be the 1–forms defined in the second section. We can define the following $(0,2)$–tensor fields on $T_k^1 M$

$$
\tau^\alpha \otimes \tau^\beta, \tau^\alpha \otimes df^{\beta\gamma}, df^{\alpha\beta} \otimes \tau^\gamma, df^{\alpha\beta} \otimes df^{\gamma\delta}. \tag{9}
$$

The aim of this section is to prove

Proposition 4. *All first order natural operators $T^*\odot T^* \rightsquigarrow (T^*\otimes T^*)T^1_k$ transforming symmetric $(0,2)$– tensor fields on M into $(0,2)$–tensor fields on $T^1_k M$ are of the form*

$$G \mapsto \sum_{\alpha=1}^{k} A_1^\alpha G^{C,\alpha} + \sum_{\alpha=1}^{k} A_2^\alpha G^{A,\alpha} + A_3 G^V + \sum_{\alpha,\beta=1}^{k} A_4^{\alpha\beta}\tau^\alpha\otimes\tau^\beta +$$
$$\sum_{\alpha,\beta,\gamma,\delta=1}^{k} A_5^{\alpha\beta\gamma\delta} df^{\alpha\beta}\otimes df^{\gamma\delta} + \sum_{\alpha,\beta,\gamma=1}^{k} A_6^{\alpha\beta\gamma}\tau^\alpha\otimes df^{\beta\gamma} + \sum_{\alpha,\beta,\gamma=1}^{k} A_7^{\alpha\beta\gamma} df^{\alpha\beta}\otimes\tau^\gamma$$

where all $A's$ are arbitrary smooth functions of the invariants $I_{\alpha\beta} = g_{ij}y^i_\alpha y^j_\beta$, $\alpha,\beta = 1,\dots,k$.

Proof. It suffices to determine all G^2_m–equivariant maps $J^1_0(T^*\odot T^*)\mathbb{R}^m \oplus T^1_k\mathbb{R}^m \to (T^*\otimes T^*)T^1_k\mathbb{R}^m$ of the form $(g_{ij}, g_{ij,k}, y^i_\alpha) \mapsto (A_{ij}, B^\alpha_{ij}, C^\alpha_{ij}, D^{\alpha\beta}_{ij})$. Consider first $D^{\alpha\beta}_{ij}(y^i_\alpha, g_{ij}, g_{ij,k}; \alpha = 1,\dots,k)$. By [7], if G is symmetric, then

$$g_{ap}\left(\frac{\partial D^{\alpha\beta}_{ij}}{\partial g_{aq,r}} + \frac{\partial D^{\alpha\beta}_{ij}}{\partial g_{ar,q}}\right) = 0. \tag{10}$$

Furthermore, if we suppose G to be regular, then we can contract (10) with the inverse matrix g^{pk}. We get

$$\frac{\partial D^{\alpha\beta}_{ij}}{\partial g_{aq,r}} + \frac{\partial D^{\alpha\beta}_{ij}}{\partial g_{ar,q}} = 0.$$

Using symmetry of G and the cyclic permutation in the indices (a,q,r) we prove analogously to [7] that

$$\frac{\partial D^{\alpha\beta}_{ij}}{\partial g_{pq,r}} = 0. \tag{11}$$

Regular $(0,2)$–tensor fields form an open dense subset among all $(0,2)$–tensor fields. We have proved that (11) holds on the open dense subset, so that (11) holds everywhere. Hence $D^{\alpha\beta}_{ij}$ are independent of $g_{ij,k}$. By Lemma 5,

$$D^{\alpha\beta}_{ij} = \varphi_1^{\alpha\beta} g_{ij} + \varphi_2^{\alpha\beta\gamma\delta} g_{ik}y^k_\gamma g_{js}y^s_\delta.$$

Moreover, we assume that C^α_{ij} are of the form

$$\begin{aligned} C^\alpha_{ij} = {} & B_1^{\alpha\beta} g_{ij,k}y^k_\beta + B_2^{\alpha\beta} g_{ki,j}y^k_\beta + B_3^{\alpha\beta} g_{kj,i}y^k_\beta + C_1^{\alpha\beta\gamma\delta\varepsilon} g_{im,n}y^m_\beta y^n_\gamma g_{jp,q}y^p_\delta y^q_\varepsilon + \\ & C_2^{\alpha\beta\gamma\delta\varepsilon} g_{im,n}y^m_\beta y^n_\gamma g_{pq,j}y^p_\delta y^q_\varepsilon + C_3^{\alpha\beta\gamma\delta\varepsilon} g_{mn,i}y^m_\beta y^n_\gamma g_{jp,q}y^p_\delta y^q_\varepsilon + \\ & C_4^{\alpha\beta\gamma\delta\varepsilon} g_{mn,i}y^m_\beta y^n_\gamma g_{pq,j}y^p_\delta y^q_\varepsilon + D_1^{\alpha\beta\gamma\delta} g_{ik}y^k_\beta g_{jp,q}y^p_\gamma y^q_\delta + \\ & D_2^{\alpha\beta\gamma\delta} g_{ik}y^k_\beta g_{pq,j}y^p_\gamma y^q_\delta + D_3^{\alpha\beta\gamma\delta} g_{jk}y^k_\beta g_{ip,q}y^p_\gamma y^q_\delta + \\ & D_4^{\alpha\beta\gamma\delta} g_{jk}y^k_\beta g_{pq,i}y^p_\gamma y^q_\delta + \widetilde{C}^\alpha_{ij}(y^i_\alpha, g_{ij}, g_{ij,k}; \alpha = 1,\dots,k) \end{aligned} \tag{12}$$

with undetermined coefficients B_1, B_2, B_3, $C_1, \dots, C_4, D_1, \dots, D_4$. Using equivariance on the kernel of the jet projection $G_m^2 \to G_m^1$ and then the full equivariance we get $C_{ij}^\alpha = \frac{1}{2}\varphi_1^{\alpha\beta}(g_{ij,k}+g_{ki,j}-g_{kj,i})y_\beta^k+\frac{1}{2}\varphi_2^{\alpha\beta\gamma\delta}g_{ik}y_\beta^k g_{pq,j}y_\gamma^p y_\delta^q+C_1^\alpha g_{ij}+C_2^{\alpha\beta\gamma}g_{ik}y_\beta^k g_{js}y_\gamma^s$. Using the same procedure for B_{ij}^α and for A_{ij} we complete the proof. □

Remark 2. Kowalski and Sekizawa have in [7] determined all first order natural operators transforming Riemannian metrics to the frame bundle FM. Their construction essentially employs the regularity of a Riemannian metric and the corresponding Levi-Civita connection which can be canonically associated to each regular symmetric $(0,2)$–tensor field. On the other hand, in the case of a general symmetric $(0,2)$–tensor field (not necessarily regular) we have no canonical connection at our disposal. Hence the result of Kowalski and Sekizawa is not a particular case of Proposition 4. On the contrary, owing to the regularity of a metric, the set of natural operators of Kowalski and Sekizawa is even wider than the set of natural operators from our assertion.

Remark 3. Each $(0,2)$–tensor field $G = g_{ij}dx^i \otimes dx^j$ on M defines a 2–form $\omega = (g_{ij} - g_{ji})dx^i \otimes dx^j$ on M, so that the pull-back $R = \pi_M^*(d\omega)$ is a 3–form on T_k^1M, $R = R_{ijk}dx^i \otimes dx^j \otimes dx^k$. If G is a general $(0,2)$–tensor field, then we have further $\binom{k}{3}$ invariants $I_{\alpha\beta\gamma} = R_{ijk}y_\alpha^i y_\beta^j y_\gamma^k$, $\alpha, \beta, \gamma = 1, \dots, k$. Notice that if G is symmetric, then all $I_{\alpha\beta\gamma}$ vanish (cf. Proposition 4).

5. Correction

The first author should like to make an apology for an error in the proof of Theorem in [3]. This section is devoted to the correction of this mistake. Let G be an arbitrary $(0,2)$–tensor field on M and G' be given by (6). Let $\beta = g_{ij}y^j dx^i$ be the 1–form on TM defined by $\langle \beta, X \rangle = \langle G, -, X \rangle$ for all vector fields X on M, cf. [3], p. 217. Analogously, we shall denote by β' the 1–form on TM defined by $\langle \beta', X \rangle = \langle G, X, - \rangle$, $\beta' = g_{ij}y^i dx^j$. Finally, let $f : TM \to \mathbb{R}$ be a function defined by the contraction, $f = g_{ij}y^i y^j$. Then the exterior differential df is further 1–form on TM. Evaluating tensor products of 1–forms β, β' and df, we obtain 9 nonlinear liftings, which were not included in Theorem in [3]. The correct form of Theorem from [3], p. 222 is the following

Theorem. *For $m \geqq 3$, all first order natural operators $T^* \otimes T^* \rightsquigarrow (T^* \otimes T^*)T$ transforming $(0,2)$–tensor fields on M into $(0,2)$–tensor fields on TM are of the form*

$$\begin{aligned}(13)\quad G \mapsto K_1(G')^C &+ K_2G^C + K_3(G')^V + K_4G^V + K_5(G')^A + K_6G^A \\ &+ K_7\beta \otimes \beta + K_8\beta' \otimes \beta' + K_9\beta \otimes \beta' + K_{10}\beta' \otimes \beta + K_{11}\beta \otimes df \\ &+ K_{12}\beta' \otimes df + K_{13}df \otimes \beta + K_{14}df \otimes \beta' + K_{15}df \otimes df\end{aligned}$$

where $K_i = K_i(g_{ij}y^i y^j)$ are arbitrary smooth functions of the invariant I_1 and G^C, G^V and G^A denote the canonical liftings.

Correction of the proof. On the right hand side of s_i in (8) in [3] the following term

$$\begin{aligned}&+ (\alpha_1 g_{jn}X^j y^n + \alpha_2 g_{mj}y^m X^j \\ &+ \alpha_3(g_{mn,j}y^m y^n X^j + g_{mj}y^m Y^j + g_{jm}y^m Y^j))(g_{si}y^s + g_{is}y^s)\end{aligned}$$

is missing and analogously on the right hand side of r_i in (8) in [3] we have to add

$$
\begin{aligned}
&+ (\alpha_1 g_{jn} X^j y^n + \alpha_2 g_{mj} y^m X^j \\
&+ \alpha_3 (g_{mn,j} y^m y^n X^j + g_{mj} y^m Y^j + g_{jm} y^m Y^j)) g_{pq,i} y^p y^q \\
&+ (\beta_1 g_{jn} X^j y^n + \beta_2 g_{mj} y^m X^j \\
&+ \beta_3 (g_{mn,j} y^m y^n X^j + g_{mj} y^m Y^j + g_{jm} y^m Y^j)) g_{si} y^s \\
&+ (\gamma_1 g_{jn} X^j y^n + \gamma_2 g_{mj} y^m X^j \\
&+ \gamma_3 (g_{mn,j} y^m y^n X^j + g_{mj} y^m Y^j + g_{jm} y^m Y^j)) g_{is} y^s .
\end{aligned}
$$

This corresponds to (13), where $K_7 = \gamma_1$, $K_8 = \beta_2$, $K_9 = \gamma_2$, $K_{10} = \beta_1$, $K_{11} = \gamma_3$, $K_{12} = \beta_3$, $K_{13} = \alpha_1$, $K_{14} = \alpha_2$ and $K_{15} = \alpha_3$. □

Then the correct form of Corollary 1 and Corollary 2 in [3], p. 223 is:

Corollary 1. *For $m \geqq 3$, all first order natural operators transforming symmetric or antisymmetric $(0,2)$–tensor fields on M into $(0,2)$–tensor fields on TM are of the form*

$$G \mapsto K_1 G^C + K_2 G^V + K_3 G^A + K_4 \beta \otimes \beta + K_5 \beta \otimes df + K_6 df \otimes \beta + K_7 df \otimes df$$

where $K_i = K_i(I_1)$ are arbitrary smooth functions of the invariant I_1.

Corollary 2. *For $m \geqq 3$, all first order natural $\mathbb{R}$–linear operators $T^* \otimes T^* \rightsquigarrow (T^* \otimes T^*)T$ are of the form*

$$G \mapsto K_1 (G')^C + K_2 G^C + K_3 (G')^V + K_4 G^V + K_5 (G')^A + K_6 G^A,$$

where K_i are arbitrary real numbers.

References

1. Cantrijn, F., Crampin, M., Sarlet W. and Saunders D., *The canonical isomorphism between T^kT^*M and T^*T^kM*, C. R. A. S. Paris **309** (1989), 1509–1514.
2. Cordero L. A., Dodson C. T. J. and de Leon M., *Differential Geometry of Frame Bundles*, Kluwer Academic Publishers, 1988.
3. Doupovec M., *Natural liftings of $(0,2)$–tensor fields to the tangent bundle*, Arch. Math.(Brno) **30** (1994), 215–225.
4. Gancarzewicz J., *Complete lifts of tensor fields of type $(1,k)$ to natural bundles*, Zeszyty naukowe UJ, Prace Matematyczne **23** (1982), 51–84.
5. Janyška J., *Natural 2–forms on the tangent bundle of a Riemannian manifold*, Rend. Circ. Matem. Palermo **32** (1993), 165–174.
6. Kolář I., Michor P. W., and Slovák J., *Natural Operations in Differential Geometry*, Springer, 1993.
7. Kowalski O. and Sekizawa M., *Natural transformations of Riemannian metrics on manifolds to metrics on linear frame bundle – a classification*, Differential Geometry and its Applications, Proceedings, D. Reidel Publishing Company (1987), 149–178.
8. Kurek J., *On the first order natural operators transforming 1–forms on manifolds to linear frame bundle*, Demonstratio Math. **XXVI,2** (1993), 287–293.
9. Mikulski W. M., *Natural transformations transforming functions and vector fields to functions on some natural bundles*, Math. Bohemica **2** (1992), 217–223.

10. Morimoto A., *Liftings of some types of tensor fields and connections to tangent bundles of p^r velocities*, Nagoya Math. J. **40** (1970), 13–31.
11. Zajtz A., *Foundations of Differential Geometry of Natural Bundles* (to appear).

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