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SPECTRAL THEORY OF INVARIANT OPERATORS, SHARP INEQUALITIES, AND REPRESENTATION THEORY

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Lecture 1: Conformal covariants

Let (M, g) be a pseudo-Riemannian manifold of dimension $n \geq 3$. Philosophically speaking, a *conformal covariant* D should be a natural, metric dependent operator on sections of a natural vector bundle with the following property: if the metric g is subjected to a pointwise scaling $\bar{g} = \Omega^2 g$, $0 < \Omega \in C^\infty(M)$, and any relevant auxiliary structures are altered compatibly, then

$$\bar{D} = \Omega^{-b} D[\Omega^a]. \tag{1}$$

Here a and b are real numbers, $\bar{D}$ is computed in the $\bar{g}$-geometry, and $[\Omega^a]$ is multiplication by the function Ω^a. (a, b) is sometimes called the *conformal biweight* of D. The numbers a, b can actually be adjusted as desired by tensoring with *density* bundles. In particular, a and b can be made to vanish for section densities of the correct weights; the result is a *conformally invariant* operator on section densities.

Among possible "auxiliary structures" in the above might be a volume form, a spin structure, or an embedding of M as the boundary of a larger manifold $\mathcal{M}$. One hears most often about conformally covariant *differential* operators, but it makes perfect sense to speak of nonlocal, for example *pseudo-differential*, conformal covariants. Indeed, the Knapp-Stein intertwining operators for representations of the conformal group of conformally compactified flat space are generically nonlocal. The extent to which these nonlocal operators have conformally covariant generalizations to arbitrarily curved manifolds is an open question.

Example 1.1 The best-known conformal covariant is the *conformal Laplacian*

$$Y := \Delta + \frac{n-2}{4(n-1)} K, \qquad K = \text{scalar curvature.}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

Here Δ is the Laplacian δd on functions, where $\delta = d^*$. (We have not assumed that g is positive definite, so Δ may be hyperbolic or ultrahyperbolic.) The weights in (1) are $a = (n-2)/2$ and $b = (n+2)/2$. Though Y is best known for its role in the *Yamabe problem* of prescribing scalar curvature on a Riemannian manifold, it first appeared in the early part of this century in connection with *conformal relativity.* Here too the goal was deformation to constant scalar curvature; the Ω effecting this was viewed as an extra physical field.

Examples 1.2 Suppose g is Riemannian, let M have a boundary ∂M, and consider the boundary problems $(Y, \mathcal{D})$ and $(Y, \mathcal{R})$ with Y as above, $\mathcal{D}$ the Dirichlet operator $u \mapsto u|_{\partial M}$, and $\mathcal{R}$ the *Robin* (conformal Neumann) operator

$$u \mapsto \left(\left(N - \frac{n-2}{2(n-1)} H \right) u \right) \Big|_{\partial M},$$

where N is the inward unit normal, and H is the trace of the second fundamental form. Because

$$\bar{\mathcal{D}} = \Omega^{-a} \mathcal{D}[\Omega^a] \text{ for any } a, \qquad \bar{\mathcal{R}} = \Omega^{-n/2} \mathcal{R}[\Omega^{(n-2)/2}] \tag{2}$$

under $\bar{g} = \Omega^2 g$, the resulting operators $Y_{\mathcal{D}}$ and $Y_{\mathcal{R}}$ are conformally covariant.

More generally, a boundary problem (D, B) produces a conformally covariant operator if (1) holds, and $\bar{B} = \Omega^{-c} B[\Omega^a]$ for some c. That is, B should be conformally covariant, with the same *initial* weight as D.

Example 1.3 Because of (2), the *Dirichlet-to-Robin* operator is a conformally covariant pseudo-differential operator on ∂M. Let φ be a smooth function on ∂M, and extend to $E\varphi \in C^\infty(M)$ with $YE\varphi = 0$. The Dirichlet-to-Robin operator $T = \mathcal{R}E$ thus carries Dirichlet data to Robin data. The conformal covariance relation for Y shows that $[\Omega^{-(n-2)/2}]$ is a bijection $\mathcal{N}(Y) \to \mathcal{N}(\bar{Y})$, and thus

$$\bar{E}(\Omega^{-(n-2)/2} \varphi) \doteq \Omega^{-(n-2)/2} (E\varphi);$$

that is, E is conformally covariant of biweight $((n-2)/2, (n-2)/2)$. Since the terminal weight of E matches the initial weight of $\mathcal{R}$, the composition T is conformally covariant of biweight $((n-2)/2, n/2)$.

The Dirichlet-to-Robin operator is a most interesting and quite concrete example of a pseudo-differential conformal covariant. Note that

$$T = \Delta_{\partial M}^{1/2} + \text{lower order},$$

by virtue of the relation

$$\Delta_M = -N^2 + \Delta_{\partial M} + \text{lower order}.$$

On the round sphere S^n, we can obtain T as an intertwining operator for $\mathrm{SO}_0(n+1,1)$ ([8], equation (2.16)); here it turns out that

$$T = \sqrt{\Delta_{S^n} + \left(\frac{n-1}{2}\right)^2}. \tag{3}$$

(This can also be computed from the characterization of the Δ_{S^n} eigenfunctions as restrictions to the unit sphere of homogeneous harmonic polynomials in $\mathbb{R}^{n+1}$.) Various Dirichlet-to-Neumann operators B have recently become important in the theory of *electrical impedance tomography* [34, 50], which may be characterized as the attempt to reconstruct information about the conformal factor Ω (given some background metric) from measurements of B performed on ∂M (i.e. from *nondestructive testing*).

Examples 1.4 Back in the realm of differential operators, the first-order *Stein-Weiss operators* [49], or *gradients*, are conformally covariant, by a result of Fegan [26]. Let g be Riemannian, and let $\mathbb{V}$ be an irreducible vector bundle with structure group $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$. Since the defining and spin representations of these groups are faithful, these $\mathbb{V}$ are exactly the irreducible components of the tensor (or tensor-spinor) bundles on an oriented Riemannian (or Riemannian spin) manifold. The covariant derivative ∇ carries (sections of) $\mathbb{V}$ to (sections of) $T^*M \otimes \mathbb{V}$, which is generally reducible under the structure group:

$$T^*M \otimes \mathbb{V} \cong_{\mathfrak{so}(n)} \mathbb{W}_1 \oplus \ldots \oplus \mathbb{W}_N\,, \tag{4}$$

where the $\mathbb{W}_i$ are irreducible, and of course N depends on $\mathbb{V}$. As it happens, (4) is a *multiplicity free* decomposition; that is, $\mathbb{W}_i \cong_{\mathfrak{so}(n)} \mathbb{W}_j \Rightarrow i = j$. Thus the composition

$$G_i : \mathbb{V} \xrightarrow{\quad\nabla\quad} T^*M \otimes \mathbb{V} \xrightarrow{\quad\mathrm{Proj}_i\quad} \mathbb{W}_i$$

is a well-defined first-order differential operator between irreducible bundles. Fegan showed that each G_i is conformally covariant, with biweight depending on the highest weights of the $\mathfrak{so}(n)$ modules to which $\mathbb{V}$ and $\mathbb{W}_i$ are associated. All the natural first order differential operators from Riemannian geometry, for example, d, δ, the Dirac operator $\nabla\!\!\!/$, the twistor operator on spinors, and the S operator of elasticity theory are gradients, or direct sums of gradients.

Examples 1.5 A new class of conformal covariants on differential forms was discovered in 1981 [6]; these were the earliest known conformal covariants which are not versions of operators known in Physics, and the first to involve the Ricci tensor r. Let g be pseudo-Riemannian, and let

$$\begin{aligned} J &= \frac{K}{2(n-1)}, \\ V &= \frac{1}{n-2}(r - Jg). \end{aligned}$$

Consider the following operator on the k-form bundle $\Lambda^k M$:

$$D_{2,k} := (s+1)\delta d + (s-1)d\delta + (s+1)(s-1)(J - 2V\cdot), \quad s = s_k = \frac{n-2k}{2},$$

where $V\cdot$ is the natural (derivation) action of a two-tensor on the Grassmann algebra. Then $D_{2,k}$ is conformally covariant of biweight $(s-1, s+1)$. Special cases are the conformal Laplacian Y when $k = 0$, and the *Maxwell operator* δd on *vector potentials* ($(n-2)/2$-forms when n is even).

Example 1.6 Inspired by the example of $D_{2,k}$ and its use of the Ricci tensor, Stephen Paneitz [43] in 1983 introduced a fourth-order conformal covariant on functions, motivated by problems of gauge fixing for the Maxwell equations. This operator was independently discovered by Eastwood and Singer [25], who were also motivated by the gauge fixing problem, and by Riegert [46] in dimension 4, in connection with the *trace anomaly.* For pseudo-Riemannian metrics g, let

$$\begin{aligned} Q &= \tfrac{1}{2}nJ^2 - 2|V|^2 + \Delta J, \\ T &= (m-2)J - 4V\cdot \quad \text{on one-forms.} \end{aligned}$$

Then

$$P := \Delta^2 + \delta T d + \frac{n-4}{2} Q$$

is conformally covariant of biweight $((n-4)/2, (n+4)/2)$.

Now (the vacuum) *Maxwell's equations* on an even-dimensional pseudo-Riemannian manifold are

$$dF = \delta F = 0,$$

on an $n/2$-form F. Modulo cohomological obstructions, we may take a *vector potential* A with $dA = F$, since F is closed. As a condition on A, Maxwell's equations just become $\delta dA = 0$; thus the term *Maxwell operator* for the conformal covariant δd on $(n-2)/2$-forms. A can be altered by a closed summand without affecting the physical field F; in particular, if u is an $(n-4)/2$-form, addition of du to A has no effect; this is a change of *gauge.* By imposing the extra restriction $\delta A = 0$ (so that the form Laplacian $\delta d + d\delta$ annihilates A), we cut down somewhat on the amount of gauge freedom.

Both Paneitz and Eastwood-Singer were concerned about the lack of conformal covariance of the gauge condition $\delta A = 0$: though δ is conformally covariant on $\Lambda^{(n-2)/2}$, its initial weight does not agree with that of δd (namely 0). On doubly-covered compactified Minkowski space $\mathcal{M} = S^1 \times S^3$ in four dimensions, the *Coulomb* gauge condition

$$\delta A = 0, \qquad \iota(d\tau)A = 0, \tag{5}$$

with τ the *time* parameter on the S^1 factor and ι interior multiplication, completely fixes the gauge: two gauge equivalent Maxwell vector potentials satisfying (5) must be equal. When A is pulled back by a conformal transformation of $\mathcal{M}$, however, the resulting potential A' is generally not in the space C cut out by the conditions (5). As Paneitz pointed out, however, A' is in the somewhat larger, conformally invariant space $C + d\mathcal{N}(P)$. In other words, though conformal transformations take us out of C, we do not need access to all gauge changes du in order to return; the relatively small class of du with $u \in \mathcal{N}(P)$ suffice. (See [8], Section 4.b for a more complete discussion.) Paneitz also notes that a similar result holds in a more general class of conformally flat Lorentz 4-manifolds, by virtue of the possibility of embedding them conformally in $\mathcal{M}$.

Eastwood and Singer made a different observation. First note that $P = (\Delta\delta + \delta T)d$ in dimension 4, and consider the gauge condition

$$(\Delta\delta + \delta T)A = 0. \tag{6}$$

This gauge can be imposed, given an arbitrary A, by solving $Pf = (\Delta\delta + \delta T)A$ and replacing A by $A - df$. The only gauge changes which preserve (6) are now those from $d\mathcal{N}(P)$. Though the null space of $\Delta\delta + \delta T$ is not conformally invariant, its intersection $\mathcal{N}(\Delta\delta+\delta T)\cap\mathcal{N}(\delta d)$ with Maxwell potentials is; this makes (6) a *conformally invariant gauge* for the Maxwell equations.

Examples 1.7 Also inspired by the construction of $D_{2,k}$ and its use of the Ricci tensor, Wünsch [53] constructed an analogous operator $F_{2,p}$ on the bundle TFS^p of trace-free symmetric p-tensors. All these second-order operators were later subsumed in a classification of all second-order conformal covariants; this was first distributed as [9], and has now appeared in published form in [12]. The Wünsch operator $F_{2,2}$ was used in an essential way in [11], in the analysis of the extremal problem for the functional determinant on S^6 (see Lecture 3 below).

Examples 1.8 In [30], Graham, Jenne, Mason, and Sparling showed the existence of a conformal covariant P_m of even order m on scalar functions whenever the dimension n is odd, or $m \leq n$. P_m has leading term $\Delta^{m/2}$, and generalizes the conformal Laplacian $Y = P_2$ and the Paneitz operator $P = P_4$. The operator P_6 had also been constructed earlier, in [53]. It is shown in [11] that there is a P_m of the form

$$S_{m-1}d + \frac{n-m}{2}Q_m\,, \tag{7}$$

where S_{m-1} is a natural differential operator and Q_m is a local scalar invariant with coefficients that are rational in n. It is also shown in [11] that in the conformally flat category, P_m is unique and formally self-adjoint. P_m potentially figures quite prominently in spectral theoretic questions in higher dimensions. An important open question is that of whether P_m and Q_m can be chosen to be formally self-adjoint for general (not necessarily conformally flat) metrics, and rational in the dimension n, without disturbing the validity of (7).

The above is really a discussion of individual conformal covariants, or classes of such, that are known in some concrete way, with a bias toward those that have become important in the spectral theory of differential operators. Not touched upon are important classification results for conformally covariant differential operators. Working on *model spaces* like compactified Minkowski space (for Lorentz metrics) or the sphere (for Riemannian metrics), Baston, Eastwood, Jakobsen, and Rice [33, 24, 3, 4] have exploited the theory of Verma module embeddings and the Bernstein-Gelfand-Gelfand resolution to obtain far-reaching classification results for differential operators covariant under the conformal transformation groups of these spaces, or in short, *differential conformal transformation covariants.* Baston, Eastwood, and Rice also prove results on generalizations of these operators to conformal covariants on arbitrarily curved manifolds. A survey of known results is given in [48]. It is known, however, that not all conformal transformation covariants generalize to conformal covariants in the arbitrarily curved setting: Graham [29] shows that a general P_6 (in the sense of Example 1.8) cannot exist in dimension 4; yet this operator exists as a conformal transformation covariant on compactified Minkowski space, and on S^4 (see [8], Sec. 2.c and Remark 2.23).

Correction 1.9 At an earlier meeting of the Winter School [10], the present author claimed that a fourth-order conformal transformation covariant on one-forms ([8], Sec. 3.d) has no arbitrarily curved generalization in dimension 4. The proof given in [10] was in error, and Graham [29] has shown that there is such a generalization.

Open problems:

1.a Formulate and attack the classification problem for pseudo-differential conformal covariants, and conformally covariant boundary problems. Can an asymptotic expansion of the symbol of a putative conformal covariant or covariants with leading symbol $|\xi|$ be used to get information on the basic problems of electrical impedance tomography?

1.b Exactly which differential conformal transformation covariants have arbitrarily curved generalizations?

1.c What is the connection between the Paneitz operator and the Maxwell equations, or Yang-Mills equations, on general (not necessarily conformally flat) 4-manifolds?

1.d Does the construction of the GJMS operator P_m automatically yield a formally self-adjoint operator, with P_m and $Q_m = P_m 1$ rational in the dimension n, satisfying (7)?

1.e Assuming a positive answer to 1.d, consider the curvature prescription problems

$$P_m u = \frac{n-m}{2}\bar{Q}_m u^{(n+m)/(n-m)}, \qquad u = \Omega^{(n-m)/2} \tag{8}$$

analogous to the Yamabe problem, for $m < n$. In analogy with the passage from the Yamabe problem for $n > 2$ to the Gauss curvature prescription problem when $n = 2$, consider also the "analytic continuation" of (8) to dimension m:

$$P_n\omega + Q_n = \bar{Q}_n e^{n\omega}. \tag{9}$$

(8) and (9) govern the conformal change of the quantities Q_m under $\bar{g} = \Omega^2 g$, where $\Omega = e^\omega$. Can we conformally deform to constant $\bar{Q}_m$? Which functions can be realized as the $\bar{Q}_m$ of some some $\bar{g}$?

1.f Do the differential form operators (Examples 1.5) carry information useful in the problem of prescribing the Ricci tensor?

Lecture 2: The analytic content of conformal covariants

The classification of differential conformal covariants centers has centered on algebraic techniques involving Verma modules. For more general pseudo-differential conformal covariants, the theory of Harish-Chandra modules, and especially intertwining operators for principal series representations, will be necessary. The differential conformal transformation covariants then turn up for a discrete set of values of a certain continuous parameter (essentially the conformal weight). Going the other way, a complete understanding of the spectra of the differential intertwinors can sometimes be used to get an understanding of the nonlocal intertwinors, through analytic continuation, or through a functional equation. (See [16], esp. Sec. 5.)

Recently, intriguing new applications of conformal covariants in harmonic analysis and the spectral theory of differential operators have come into view. Since the (differentiable) dependence of the covariant on the conformal weight enters explicitly, the use of nonlocal conformal covariants is necessary. A part of this picture has long been visible. For example, we can add a nonlinear term to the Yamabe operator,

$$N(u) = Yu + c|u|^{(n+2)/(n-2)}$$

for c a constant, without disturbing conformal covariance. This nonlinear operator turns up in the Yamabe equation

$$Yu = \frac{n-2}{4(n-1)}\bar{K}u^{(n+2)/(n-2)}. \tag{10}$$

A positive function u solves (10) iff $\bar{g} = \Omega^2 g$ has constant scalar curvature $\bar{K}$, where $u = \Omega^{(n-2)/2}$. In fact, (10) is just the conformal covariance relation for Y, applied to the constant function 1. The central issue in the solution of the Yamabe problem [54, 51, 2, 47] is the sharp form of inequality describing the Sobolev embedding

$$L_1^2 \hookrightarrow L^{2n/(n-2)}, \tag{11}$$

or, equivalently, minimization of the Yamabe quotient $(Yu, u)_{L^2}/\|u\|^2_{2n/(n-2)}$. The problem is difficult because the embedding is *borderline*: for $q < 2n/(n-2)$, the embedding $L^2_1 \hookrightarrow L^q$ is compact; for $q > 2n/(n-2)$, there is no embedding. At the borderline, the embedding is bounded but not compact.

The behavior of the Yamabe problem as $n \downarrow 2$ is revealing: when $n = 2$, the constant scalar curvature prescription problem is

$$\Delta\omega + \tfrac{1}{2}K = \tfrac{1}{2}\bar{K}e^{2\omega}, \tag{12}$$

where $\bar{g} = e^{2\omega}g$. (The quantity $\frac{1}{2}K$ is the *Gauss curvature* in dimension 2.) The corresponding embedding of the Sobolev class L^2_1 lands in the Orlicz class e^L. On the sphere, the inequality describing this embedding is known as the *Moser-Trudinger inequality*; its sharp form is due to Onofri [39]. Such exponential class inequalities were studied by Adams [1] for domains in $\mathbb{R}^n$, and the transplantation of these to manifolds was carried out in detail by Fontana [27], generalizing many special cases in the literature. A productive way to think of the Moser-Trudinger inequality and its generalization to higher dimensional spheres [5, 21] is as an *endpoint derivative* of the embeddings $L^2_r \hookrightarrow L^{2n/(n-2r)}$ as $r \uparrow n/2$. More precisely, Beckner proves the sharp exponential class inequality in [5] by first deriving the the dual inequalities describing the embeddings

$$L^{2n/(n+2r)} \hookrightarrow L^2_{-r}, \tag{13}$$

differentiating (13) at $r = 0$ to obtain an inequality describing the embedding $L\log L \hookrightarrow L^2_{-n/2}$, and then taking the dual of this inequality.

On the sphere S^n the Sobolev embedding and exponential class inequalities have an important representation theoretic interpretation: each inequality compares two invariant Banach norms on a (scalar) *complementary series* representation of $\mathrm{SO}_0(n+1,1)$. These can be given the following elementary description.

Recall that a *conformal transformation* of a pseudo-Riemannian manifold (M, g) is a diffeomorphism $h : M \to M$ with $h \cdot g = \Omega_h^2 g$, for some $0 < \Omega_h \in C^\infty(M)$. Here $h\cdot$ is the natural pushout of tensors under a diffeomorphism ([32], p. 90, Problem 2); on covariant tensors like g, $h\cdot = (h^{-1})^*$. On the infinitesimal level, we have the notion of *conformal vector fields* X, for which $\mathcal{L}_X g = 2\omega_X g$, some $\omega_X \in C^\infty(M)$. An elementary calculation shows that the set $\mathbf{ctran}(M, g)$ of conformal transformations is a group, the set $\mathbf{cvf}(M, g)$ of conformal vector fields is a Lie algebra, and we have the *cocycle conditions*

$$\Omega_{h\circ k} = \Omega_h(h \cdot \Omega_k), \qquad \omega_{[X,Y]} = X\omega_Y - Y\omega_X, \tag{14}$$

for all $h, k \in \mathbf{ctran}(M, g)$, $X, Y \in \mathbf{cvf}(M, g)$. The connection between the finite and infinitesimal notions is made by integrating conformal vector fields to local one-parameter groups of local conformal transformations.

Because any conformal transformation can be thought of as a composition

$$(\text{conformal change})\circ(\text{isometry}) : (M, g) \xrightarrow{\quad h \quad} (M, \Omega_h^2 g) \xrightarrow{\quad \mathrm{id} \quad} (M, g),$$

a conformal covariant D of biweight (a, b) will satisfy

$$\begin{aligned} D(\Omega_h^a h \cdot \varphi) &= \Omega_h^b h \cdot (D\varphi), \\ D(\mathcal{L}_X + a\omega_X)\varphi &= (\mathcal{L}_X + b\omega_X)D\varphi \end{aligned}$$

for any $h \in \mathbf{ctran}(M,g)$ and $X \in \mathbf{cvf}(M,g)$. (If spinors are involved, we need the natural extension of $h\cdot$ and $\mathcal{L}_X$ to tensor spinors; see [40] and [35].)

By the cocycle conditions (14), the maps

$$\Omega_h^a h\cdot \qquad \text{and} \qquad \mathcal{L}_X + a\omega_X$$

are homomorphisms $\mathbf{ctran}(M,g) \to \operatorname{Aut} C^\infty(\mathbf{V})$ and $\mathbf{cvf}(M,g) \to \operatorname{End} C^\infty(\mathbf{V})$, for each irreducible tensor-spinor bundle $\mathbf{V}$. To make contact with the normalizations of Lie theory, we label these maps

$$u^{\mathbf{V}}_{a-\frac{n}{2}+j(\mathbf{V})} \qquad \text{and} \qquad U^{\mathbf{V}}_{a-\frac{n}{2}+j(\mathbf{V})}$$

respectively, where $j(\mathbf{V})$ is the *internal conformal weight* of $\mathbf{V}$: if $\mathbf{V}$ is a subbundle of $\binom{p}{q}$-tensors-$\binom{r}{s}$-spinors, then $j(\mathbf{V}) = q - p$.

If $M = S^n$ and g is the round metric, the group $G = \mathrm{SO}_0(n+1,1)$ acts effectively and conformally as follows: view S^n as the unit sphere S_1^n in $\mathbb{R}^{n+1}$; let $x = (x_0, \ldots, x_n)$ be homogeneous coordinates. Identify S_1^n with $S_1^n \times \{1\} \subset \mathbb{R}^{n+2}$. If $A \in G$, we can take the linear action of A on $(x,1)$ and then divide by $(A(x,1))_{n+1} > 0$ to land back in $S_1^n \times \{1\}$. The homomorphisms $u_r^{\mathbf{V}}$ (on the group level) and $U_r^{\mathbf{V}}$ (on the Lie algebra level) are the *principal series* representations of G. For details on translation to the conventional Lie theoretic notation and terminology, see [11], Sec. 2, and [16], Sec. 3.a.

A standard practice is to study the corresponding $(\mathfrak{g}, K)$ module. Here $\mathfrak{g} = \mathfrak{so}(n+1,1)$ and $K = \mathrm{SO}(n)$. Note that the restriction $u_r^{\mathbf{V}}|_K$ is independent of r, since $\mathrm{SO}(n)$ acts by isometries. We denote the common value of these restrictions by $u^{\mathbf{V}}$. This entails the use of a smaller section space than $C^\infty(\mathbf{V})$, namely the K-finite section space $\mathcal{E}(\mathbf{V})$, consisting of those φ for which $\operatorname{span}\{u^{\mathbf{V}}(k)\varphi \mid k \in K\}$ is finite dimensional. Alternatively, $\mathcal{E}(\mathbf{V})$ may be described as the space of finite linear combinations of spherical harmonic sections.

When no superscript $\mathbf{V}$ appears explicitly in the notation, we shall take $\mathbf{V}$ to be the trivial (scalar) bundle. In particular, $\mathcal{E}$ will be the space and (U_r, u) the homomorphisms of the *spherical principal series.* Note that r may be taken to be complex. The (U_r, u) for $r \in i\mathbb{R}$ are the *unitary spherical principal series*; these are the representations involved in the Fourier and Radon transforms on G/K. (The same can be said of the $(U_r^{\mathbf{V}}, u^{\mathbf{V}})$ for suitable bundle-valued Fourier and Radon transforms; see [17].) There is a unique (up to a constant factor) *intertwining operator* A_{2r} carrying (U_{-r}, u) to (U_r, u) for each r. By [16], Sec. 3.a, or [12], Theorem 3.20, A_{2r} has the eigenvalue

$$\mu_j^{(r)} = \frac{\Gamma(\frac{n}{2}+j+r)}{\Gamma(\frac{n}{2}+r)\Gamma(\frac{n}{2}+j-r)} \tag{15}$$

on E_j, the space of j^{th} order spherical harmonics. If $\frac{n}{2} + r \in -\mathbb{N}$, the formula is to be interpreted in the sense of analytic continuation (cancelling poles and zeros). Since the operator

$$B = \sqrt{\Delta + \left(\frac{n-1}{2}\right)^2}$$

has eigenvalue $j + (n-1)/2$ on E_j, we have

$$A_{2r} = \frac{\Gamma(B + r + \frac{1}{2})}{\Gamma(\frac{n}{2} + r)\Gamma(B - r + \frac{1}{2})}. \tag{16}$$

Note that A_1 is just B, up to a constant factor, and that B is none other than the Dirichlet-to-Robin operator for the sphere, viewed as the boundary of the $(n+1)$-ball.

Similar statements can be often be made for vector bundles; for example, see [16], Theorem 3.1 and [12], Theorem 3.12.

The pushout of the volume form $d$vol by an orientation-preserving conformal transformation h is $\Omega_h^n d\mathrm{vol}$, so the inner product $(\varphi, A_{2r}\psi)_{L^2}$ on $\mathcal{E}$ is (U_{-r}, u)-invariant for real r. Indeed, for general r, if $X \in \mathbf{cvf}(S^n, g)$,

$$\begin{aligned}
(\varphi, A_{2r}U_{-r}(X)\psi)_{L^2} &= \int_{S^n} \varphi\overline{A_{2r}U_{-r}(X)\psi}\,d\mathrm{vol} \\
&= \int_{S^n} \varphi\overline{U_r(X)A_{2r}\psi}\,d\mathrm{vol} \\
&= -\int_{S^n} \overline{A_{2r}\psi}\mathcal{L}_X(\varphi d\mathrm{vol}) + \left(\tfrac{n}{2} + \bar{r}\right)\int_{S^n} \omega_X\varphi\bar{\psi}\,d\mathrm{vol} \\
&= -\int_{S^n} (U_{-\bar{r}}(X)\varphi)\overline{A_{2r}\psi}\,d\mathrm{vol},
\end{aligned}$$

since $U_r(X) = \mathcal{L}_X + \left(\frac{n}{2} + r\right)\omega_X$. ($u$-invariance is immediate, since A_{2r} has a consistent eigenvalue (15) on each K-type E_j.) For r real and $|r| < n/2$, the eigenvalue $\mu^{(r)}$ is positive, so (U_r, u) is unitarizable. These are the spherical (i.e. scalar) *complementary series* representations.

A key point is that the (U_r, u)-invariant pre-unitary structure on $\mathcal{E}$ is an L^2_{-r} Sobolev norm for $r \in (-n/2, n/2)$. Indeed, by Stirling's formula applied to (16),

$$A_{2r} = \Gamma\left(\tfrac{n}{2} + r\right)^{-1}\Delta^r + (\text{lower order}) \tag{17}$$

as a pseudo-differential operator; the positivity of $\mu^{(r)}$ shows that the resulting norm is equivalent to $(\cdot, (\Delta + 1)^r \cdot)_{L^2}$.

The sharp Sobolev embedding inequalities, as proved by Beckner, read as follows. If $d\xi$ is normalized measure on S^n, then

$$\|f\|^2_{L^{2n/(n-2r)}(S^n, d\xi)} \leq \Gamma\left(\tfrac{n}{2} - r\right)(f, A_{2r}f)_{L^2(S^n, d\xi)}, \tag{18}$$

with equality iff f has the form $c\Omega_h^{(n-2r)/2}$ for some h in the conformal transformation group $\mathrm{SO}_0(n+1, 1)$, and some constant c.

Note that the conformal transformations are generated by the orientation-preserving conformal transformations, together with the isometries. It is also worth noting the explicit form of the conformal factors Ω_h of the sphere. In homogeneous coordinates

$(x_0, \ldots, x_n)$, the transformations which "pull in the x_i direction" have conformal factors $\cosh t + (\sinh t)x_i$ as t runs over $\mathbb{R}$. To get a general conformal factor, we just apply the rotation group $\mathrm{SO}(n+1)$ to any of these.

The $L^{2n/(n-2r)}$ norm is *another* (U_{-r}, u)-invariant pre-Banach norm on the space $\mathcal{E}$, again by the behavior of the volume form under conformal pushout:

$$\int \left|\Omega_h^{\frac{n}{2}-r} h \cdot f\right|^{2n/(n-2r)} d\xi = \int (h \cdot |f|^{2n/(n-2r)})\Omega_h^n d\xi = \int h \cdot (|f|^{2n/(n-2r)} d\xi).$$

The proof of (18) and the statement about extremals is ultimately based on Lieb's sharp L^p-to-L^q estimates for the Riesz kernel operators $r^\lambda *$ on $\mathbb{R}^n$ [36]; the connection is made clear by the diagram

$$\begin{array}{ccc} L^2_r(S^n, d\xi) & \xrightarrow{\ \iota\ } & L^q(S^n, d\xi) \\ \uparrow A_{-2r} & & \uparrow A_{-2r} \\ L^2_{-r}(S^n, d\xi) & \xleftarrow[\ \iota\ ]{} & L^p(S^n, d\xi) \end{array}$$

in which $p = 2n/(n+2r)$, $q = 2n/(n-2r)$, and the maps ι are Sobolev embeddings.

When $r = n/2$, we reach the endpoint of the complementary series, and simultaneously, the end of the series of borderline Sobolev embeddings. By uniqueness, A_n is a constant multiple of the GJMS operator (recall Examples 1.8) P_n in even dimensions n; (17) fixes the normalization at $A_n = P_n/\Gamma(n)$. Beckner's exponential class inequality (see also [21]) says that

$$\frac{n}{2}(\omega, A_n\omega)_{L^2(S^n,d\xi)} - \log \int_{S^n} e^{n(\omega-\bar{\omega})} d\xi \geq 0 \tag{19}$$

(where $\bar{\omega} = \int \omega d\xi$), with equality iff e^ω has the form $c\Omega_h$ for some $h \in \mathrm{SO}_0(n+1, 1)$ and some constant c.

The nature of the extremals in both (18) and (19) strongly suggests the interpretation of the *functions* f and ω being estimated as *metrics*; namely, the conformal metrics $g_\omega = e^{2\omega}g$, where $f = e^{(n-2r)\omega/2}$. With these identifications, the second term in (18), resp. (19) is

$$\left(\frac{\operatorname{vol} g_\omega}{\operatorname{vol} g_0}\right)^{\frac{n-2r}{n}}, \quad \text{resp.} \quad -n\bar{\omega} + \log \frac{\operatorname{vol} g_\omega}{\operatorname{vol} g_0}.$$

The value of this interpretation is borne out by the analysis of the extremal problem for the functional determinant in the conformal class of the round metric g (Lecture 3).

2.a What are the sharp forms of the Sobolev embeddings $L^2_r \hookrightarrow L^{2n/(n-2r)}$ on arbitrarily curved manifolds? Do they involve arbitrarily curved generalizations of the pseudo-differential operators A_{2r}? (See [23].)

2.b Widom [52] shows that for the exponential class inequality on S^1, the sharp constant $n/2 = 1/2$ can be divided by $k+1$ if we assume that e^ω is orthogonal to all trigonometric polynomials of degree $\leq k$. Can this be generalized to S^n? I.e., can we improve the best constant under the assumption that e^ω, or $e^{n\omega}$, is orthogonal to all spherical harmonics of degree $\leq k$?

2.c Do bundle-valued complementary series intertwinors give sharp forms of bundle-valued borderline Sobolev embeddings?

2.d What are the analogues of (18,19) for CR (tangential Cauchy-Riemann) geometry on the odd spheres?

2.e A principle explored in more detail in [16], Sec. 5 states that "smaller" bundles generally have wider complementary series. Assuming a positive answer for 2.c, what happens to the Sobolev embeddings that are "beyond the fringe" of the narrower complementary series?

Lecture 3: The functional determinant

It is perhaps surprising that elliptic differential operators like the Laplacian Δ on an n-dimensional Riemannian manifold have an associated quantity that deserves to be called a determinant. Indeed, the eigenvalues of the Laplacian grow at the asymptotic rate $\lambda_j \sim \text{const} \cdot j^{2/n}$ [28], so the product of the λ_j is badly divergent.

However, if A is a finite square positive symmetric matrix with eigenvalues λ_j, and $\zeta_A(s)$ is the complex function $\sum \lambda_j^{-s}$, then $\zeta_A'(0) = -\log\det A$. This can be used to generalize the determinant to the situation in which A is an operator like the Laplacian. Suppose A is a formally self-adjoint differential operator with positive definite leading symbol on a Riemannian vector bundle V over a compact, n-dimensional Riemannian manifold (M,g). Then necessarily A is elliptic, and the order of A is an even number 2ℓ. (To avoid trivialities, we always assume $\ell \neq 0$.) The eigenvalues λ_j of A are bounded below (though a finite number may be negative), and $\lambda_j \sim \text{const} \cdot j^{2\ell/n}$. We define the zeta function of A by

$$\zeta_A(s) = \sum_{\lambda_j \neq 0} |\lambda_j|^{-s}.$$

The range of summation and the absolute value signs are just artificial devices to handle the possibility of zero and negative eigenvalues. We could also use, for example, $\sum_{\lambda_j>0} \lambda_j^{-s}$, which differs from our $\zeta_A(s)$ by an entire function of s. Our conventions in this regard affect later formulas, but have no conceptual effect on the problems we consider.

The eigenvalue growth rate shows immediately that $\zeta_A(s)$ is holomorphic in the half-plane $\{\mathrm{Re}(s) > n/2\ell\}$. There is also an analytic continuation to a meromorphic function on the complex plane, with isolated simple poles. The best way to see this is to use the small-time asymptotic expansion of the heat operator trace. Let $f \in C^\infty(M)$; then

$$\mathrm{Tr}_{L^2} f e^{-tA} \sim a_i(A,f) t^{(i-n)/2\ell}, \qquad t \downarrow 0,$$

where

$$a_i(A,f) = \int_M f U_i[A] d\mathrm{vol}_g . \tag{20}$$

Here $U_i[A]$ is a local invariant of the symbol of A, and the $U_{\mathrm{odd}} = 0$. More specifically, $U_i[A]$ is a polynomial in the total symbol of A, with coefficients that depend smoothly on the leading symbol.

This is actually more information than we need just to describe the operator trace $\mathrm{Tr}_{L^2} e^{-tA}$. The formula involving the auxiliary function f, however, is useful in describing the *variation* of the heat operator trace and other spectral quantities. In particular, it will be necessary in the study of the conformal variation of the functional determinant.

The heat operator trace asymptotics and the zeta function are related by the Mellin transform,

$$(\mathcal{M}F)(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} F(t) dt.$$

Indeed, this provides the required analytic continuation of $\zeta_A(s)$, and shows that $\Gamma(s)\zeta_A(s)$ has, at worst, simple poles at the $(n-i)/2\ell$, for $i \in 2\mathbb{N}$. (See, e.g., [19].) In particular, $\zeta_A(s)$ is regular at $s = 0$, and we may define the determinant by

$$|\det A| = e^{-\zeta_A'(0)}, \qquad \mathrm{sgn}\det A = (-1)^{\#\{\lambda_j < 0\}}.$$

Now let D be a conformal covariant of biweight (a,b), and suppose that $A = D^h$ is formally self-adjoint with positive definite leading symbol for some $h \in \mathbb{Z}^+$. For example, we could take $D = A = Y$, the conformal Laplacian; or we could take $D = \nabla\!\!\!\!/$, the Dirac operator, and $A = \nabla\!\!\!\!/^2$. We shall be interested in how $\det A$ varies within a conformal class $\{g_\omega = e^{2\omega} g_0 \mid \omega \in C^\infty(M)\}$ of metrics. Adopting the convention that all objects computed in the metric g_ω will bear the subscript ω, what we wish to compute is

$$\frac{\det A_\omega}{\det A_0}. \tag{21}$$

It will be convenient to have some mechanism for ignoring uniform scale changes; i.e., conformal factors ω that are just constants. One way to do this is to demand that $\mathrm{vol}(g_\omega)$ is always 1. A better way, in terms of making contact with sharp inequalities like (18,19), is to renormalize the zeta function. Since the operator root D of A has conformal biweight (a,b), it has in particular *level* $b-a$, in the sense that $D_\alpha = e^{-(b-a)\alpha} D_0$ for α constant. A thus has level $h(b-a)$, and the level agrees with the order 2ℓ, by considerations of naturality and positive definiteness of the leading symbol (see [11]). As a result, the function

$$Z_{A,g}(s) = \mathrm{vol}(g)^{-2\ell s/n}\zeta_{A_g}(s) \tag{22}$$

is insensitive to uniform scaling. What we shall calculate is the functional

$$\mathcal{D}(A, g_0, \omega) := -Z'_{A,g_\omega}(0) + Z'_{A,g_0}(0). \tag{23}$$

The information carried by (23) is equivalent to that carried by (21). In fact,

$$Z'_{A,g}(0) = \zeta'_{A_g}(0) - \frac{2\ell}{n}\zeta_{A_g}(0)\log\mathrm{vol}(g).$$

Thus (23) is the log of (21) if $\mathrm{vol}(g_\omega) = \mathrm{vol}(g_0)$. Note that although the choice of a background metric g_0 is prominent in our formulation, this is just a convenience: one could instead speak of determinant quotients within a conformal class, without reference to g_0, by forming the appropriate two-metric functional. Our use of the background metric is analogous to the use of an origin o to represent a two-point kernel function $K(x,y)$ as a one-point function $K(x,\mathrm{o})$, in the presence of a transitive symmetry group.

A key issue in dealing with the functional $\mathcal{D}(A, g_0, \omega)$ is the

Conformal Index Theorem 3.1 [18] *Under the assumptions above, the quantities $\zeta_{A_g}(0)$, $q_{A,g} := \dim\mathcal{N}(A_g)$, $a_n(A_g)$, and $\#\{\lambda_j < 0\}$ are invariant under conformal change of metric. Here $a_n(A_g) := a_n(A_g, 1)$ is the integrated heat invariant from (20), and the λ_j are the eigenvalues of A, counted with multiplicity. The first three of these invariants are related by $a_n(A_g) = \zeta_{A_g}(0) + q_{A,g}$. If $\omega \in C^\infty(M)$, then*

$$(d/d\varepsilon)a_i(A_{\varepsilon\omega}, 1) = (n-i)a_i(A_{\varepsilon\omega}, \omega). \tag{24}$$

Note that the conformal invariance of $a_n(A_g)$ is really the special case $i = n$ of (24). This equation may be viewed as a formula for the conformal variation of the integrated heat invariant $a_i(A)$, or, from the opposite point of view, as a formula for the conformal *primitive*, or integral, of the local heat invariant $a_i(A, f)$, provided $i \neq n$. When $i = n$, the "missing primitive" for $a_n(A, f)$ is supplied by the functional determinant:

Polyakov Principle 3.2 [19] *Under the assumptions above,*

$$(d/d\varepsilon)\zeta'_{A_{\varepsilon\omega}}(0) = 2\ell\left(a_n(A_{\varepsilon\omega}, \omega) - \sum_{\lambda_j=0}\int_M \omega(|\varphi_j|^2 d\mathrm{vol})_{\varepsilon\omega}\right), \tag{25}$$

where $\{\varphi_j\}$ is an orthonormal basis of the null space of A.

It is easy to see that the last term in (25) has its own conformal primitive: take the variation in the direction of another conformal factor η, and verify that the result is symmetric in ω and η. (One first uses the conformal covariance of D to show that if $\{\psi_j\}$ is a basis for $\mathcal{N}(D_g) = \mathcal{N}(A_g)$, then $\{\Omega^{-a}\psi_j\}$ is a basis for $\mathcal{N}(D_{\Omega^2 g}) = \mathcal{N}(A_{\Omega^2 g})$.) The formal calculation giving the Polyakov Principle (ignoring analytic issues and the effect of a possible null space) goes as follows: let $D = D_{\varepsilon\omega}$. Then

$$\frac{d}{d\varepsilon}\zeta'_{A_{\varepsilon\omega}}(0) = \frac{d}{ds}\Big|_{s=0}\left(\frac{d}{d\varepsilon}\mathrm{Tr}\, D^{-hs}\right) = \frac{d}{ds}\Big|_{s=0}\left(-hs\mathrm{Tr}\left\{\frac{d}{d\varepsilon}D\right\}D^{-hs-1}\right). \qquad (26)$$

But the conformal covariance relation for D gives

$$\frac{d}{d\varepsilon}D = -(b-a)\omega D + a[D,[\omega]] = -\frac{2\ell}{h}\omega D + a[D,[\omega]].$$

(We apologize for the notational collision resulting from using square brackets both for the commutator and the multiplication operator.) Thus the quantity in (26) is

$$\frac{d}{ds}\Big|_{s=0}\left(2\ell s\mathrm{Tr}\,\omega D^{-hs}\right) = 2\ell\cdot\mathrm{Tr}\,\omega D^{-hs} = 2\ell\cdot\mathrm{Tr}\,\omega A_{\varepsilon\omega}^{-s}.$$

Here we have used various trace identities, and the regularity of the local zeta function $\mathrm{Tr}\, fA^{-s}$ at $s=0$. For the full justification, see [18, 19].

Knowing the conformal variation of $\zeta'_A(0)$, we can integrate along the conformal curve of metrics $g_{\varepsilon\omega}$, $\varepsilon \in [0,1]$, to get $\zeta'_{A_\omega}(0) - \zeta'_{A_0}(0)$. To do this explicitly, one needs explicit knowledge of the quantity $a_n(A,f)$ appearing in the variational formula. The combinatorial complexity of the heat invariants a_i increases quickly with i; since here we are interested in $i = n$, only low dimensions n will be immediately accessible. When n is odd, $a_n(A,f)$ vanishes, so the only contribution to the variation of the determinant is the relatively trivial one due to the null space. Thus the determinant as a functional on a conformal class is uninteresting in odd dimensions. (For odd-dimensional manifolds *with boundary*, however, the conformal behavior of the determinant is interesting: the a_{odd}, instead of being 0, are integrals of local invariants of the boundary.)

In fact, the process of integrating in ε is especially simple, given explicit knowledge of the a_n: by the scale homogeneity of the heat invariants together with qualitative results on the conformal variation of local invariants (see, e.g., [7], Proposition 1.8), we know that $U_n(A_{\varepsilon\omega})(d\mathrm{vol})_{\varepsilon\omega}$ is polynomial in ε. (The exponential-in-ε weight factors for U_n and $d$vol cancel.) The computation can be streamlined further by computing conformal variations of quantities that might appear; this "table of variations" may then be used in reverse as a "table of primitives". Slightly more sophisticated is analytic continuation of (24) in the dimension n; this involves setting up the calculation in such a way that all coefficients one meets are guaranteed to be rational in n. Both of these streamlined approaches are discussed in detail in [11].

The lowest even dimensions are, of course, 2, 4, and 6. Explicit information is available in each of these dimensions; for $n = 2$, this is the classic work of Polyakov, Onofri, and others [44, 45, 39]. The formula for $n = 4$ was given in [20], and a formula for conformally flat 6-manifolds was given in [11]. On the spheres in these dimensions, these formulas, together with the sharp inequalities in [5], immediately give extremal results for the determinant. We shall concentrate on the case of the sphere for the remainder of this section; see [41, 42, 14, 23] and the references mentioned just above for results on other manifolds.

Theorem 3.3 [39] *If g_0 is the round metric on S^2,*

$$-\mathcal{D}(\Delta, g_0, \omega) = \mathcal{D}(\nabla\!\!\!/^2, g_0, \omega) = \frac{1}{3}\left(-\log \int_{S^2} e^{2(\omega-\bar{\omega})}d\xi + \int_{S^2} \omega(\Delta_0\omega)d\xi\right).$$

Thus as a result of (19), $\mathcal{D}(\Delta, g_0, \omega)$ is maximized, and $\mathcal{D}(\nabla\!\!\!/^2, g_0, \omega)$ is minimized, exactly when e^ω is a multiple of some conformal transformation factor Ω_h. Another way of saying this is: Any extremal metric conformal to and having the same volume as g_0 is diffeomorphic to g_0.

Note that (19) is Onofri's sharp form of the Moser-Trudinger inequality in the case $n = 2$, and that the operator A_2 that appears there is just Δ_0. In fact, Onofri proved more. Dimension 2 is special in that the space of metrics, modulo conformal changes and diffeomorphisms, is a finite-dimensional object. The just-mentioned quotient of the space of metrics on S^2 is a one-point space: one can get from any Riemannian conformal class on S^2 to any other via a diffeomorphism. Since spectral invariants are diffeomorphic invariants, one can assert that *the scale-normalized functional determinant of Δ_g is maximized, and that of $\nabla\!\!\!/^2_g$ minimized, exactly at the metrics $c\varphi^* g_0$, where φ is a diffeomorphism of S^2 and c is a positive constant.* As a result, *any* extremal metric with the same volume as g_0 is diffeomorphic to g_0.

In dimension 4, new phenomena turn up. There is no hope of cutting across conformal classes, at least in such an elementary way as in dimension 2, so results on S^4 are confined to the standard conformal class (at least so far). As described above, the computation of the determinant formula begins with a computation of the heat invariant U_n, here U_4. If A is a geometric, orientation-insensitive operator (with no special conformal behavior assumed), $U_4[A]$ must be a linear combination of K^2, $|r|^2$, $|C|^2$, and ΔK, where C is the Weyl conformal curvature tensor. If $A = D^h$ is a power of a conformal covariant as above, then the Conformal Index Theorem imposes one linear relation on the coefficients. A way of stating this relation that has turned out to be extremely productive is the following:

$$U_4[A] = \beta_1[A]|C|^2 + \beta_2[A]Q + \beta_3[A]\Delta J. \tag{27}$$

(Recall the definition of the Paneitz quantity Q from Example 1.6.) Using Q in the basis for the level 4 local $O(n)$-invariants was one of the main ideas in [20]; in the process of integrating along conformal paths, it inevitably brings the Paneitz operator P explicitly into the determinant formula.

Theorem 3.4 [20] *If g_0 is the round metric on S^4 and A is as above, then*

$$-\mathcal{D}(A, g_0, \omega) = \frac{3}{2}\beta_2\left(-\log \int_{S^4} e^{4(\omega-\bar{\omega})}d\xi + \frac{1}{3}\int_{S^4} \omega(P_0\omega)d\xi\right)$$

$$+\frac{1}{2}\beta_3 \int_{S^4} \left\{ (J^2 d\mathrm{vol})_\omega - (J^2 d\mathrm{vol})_0 \right\}.$$

Note that the β_2 term above is exactly the quantity asserted positive by the $n = 4$ case of the exponential class inequality (19), and that the operator A_4 appearing there is $\frac{1}{6}P_0 = \frac{1}{6}\Delta_0(\Delta_0 + 2)$. A less obvious point is:

Remark 3.5 The β_3 term in Theorem 3.4 is estimated by the sharp Sobolev embedding inequality (18), with $n = 4$ and $r = 1$. Indeed, with these parameters, $A_{2r} = A_2 = Y_0/2 = (\Delta_0 + J_0)/2$. By (18) and the Schwartz inequality, if f is nowhere zero, then

$$2\|f\|_4^2 \le \int_{S^4} f(\Delta_0 + J_0) f \, d\xi \le \|f^2\|_2 \left\| \frac{\Delta_0 f}{f} + J_0 \right\|_2.$$

Taking $f = e^\omega$ and using the Yamabe equation (10), we have

$$2 \le \left\| \frac{Y_0 e^\omega}{e^\omega} \right\|_2 = \left\| \frac{e^{3\omega} J_\omega}{e^\omega} \right\|_2 = \left(\int_{S^4} (J^2 d\xi)_\omega \right)^{1/2}.$$

Here $(d\xi)_\omega = d\mathrm{vol}_\omega / \mathrm{vol}(g_0)$. Thus the L^2 norm of J, with everything computed in the metric $e^{2\omega} g_0$, is minimized exactly when e^ω is (up to a positive constant factor) a conformal transformation factor Ω_h.

Remark 3.6 A more general form of the argument in Remark 3.5 goes as follows. Suppose n is even, and let r be an integer $< n/2$. Put $S = (n - 2r)Q_{2r}/2$, so $P_{2r} = P + S$, where P annihilates constants. Then if f is nowhere zero, (18) and Hölder's inequality give

$$\|f\|_{2n/(n-2r)}^2 \le \int_{S^n} f(P_0 + S_0) f \, d\xi \le \|f^2\|_{n/(n-2r)} \left\| \frac{P_0 f}{f} + S_0 \right\|_{n/2r}.$$

Taking $f = e^{(n-2r)\omega/2}$ and using the prescription equation (8) (as we are entitled to do in the conformally flat case), we get

$$S_0 \le \left\| \frac{e^{(n+2r)\omega/2} S_\omega}{e^{(n-2r)\omega/2}} \right\|_{n/2r} = \left(\int_{S^n} (|S|^{n/2r} d\xi)_\omega \right)^{2r/n}.$$

Note that $S_0 = (P_{2r})_0 1 = \Gamma(\frac{n}{2} + r)/\Gamma(\frac{n}{2} - r) > 0$, so the $L^{n/2r}$ norm of S, with everything evaluated in the metric $e^{2\omega} g_0$, is minimized exactly when e^ω is a positive constant times a conformal transformation factor Ω_h. This more general inequality can already be put to use in dimension 6; see (28) below.

Theorem 3.4 and Remark 3.5 combine to show that the functional $\mathcal{D}(A, g_0, \omega)$ is extremized exactly when e^ω has the form $c\Omega_h$, *provided* the universal constants $\beta_2[A]$ and $\beta_3[A]$ have the same sign. When A is the conformal Laplacian Y or the square $\nabla\!\!\!/^2$ of the Dirac operator, the signs actually *do* agree: $-$ for Y, and $+$ for $\nabla\!\!\!/^2$. (See [20] for exact values. Note that the β_2 of the current discussion is half of the β_2 in that paper.) We have:

Theorem 3.7 [14] *If g_0 is the round metric on S^4, then $\mathcal{D}(Y, g_0, \omega)$ is minimized, and $\mathcal{D}(\nabla^2, g_0, \omega)$ is maximized, exactly when e^ω is a multiple of some conformal transformation factor Ω_h. As a result, any metric which is extremal in the conformal class of g_0, and has the same volume as g_0, is diffeomorphic to g_0.*

It seems rather lucky that the signs of β_2 and β_3 should agree for these very basic choices of A. This luck persists in dimension 6 [11], and for boundary problems in dimension 4 [15, 22]; in both cases several constants (not just 2) must have the same sign in order to couple the inequalities involved. There are also indications that in the problem of cutting across conformal classes, in which the sign of β_1 begins to play a role, that this sign "is what it needs to be" to give a chance at some results.

However, an "unlucky" example is also known. The Paneitz operator P in dimension $n = 4$ has $\beta_2\beta_3 < 0$. (See [13] for exact values.) This, of course, does not rule out an analogue of Theorem 3.7 for the functional $\mathcal{D}(P, g_0, \omega)$. It shows, however, that this functional is more delicate than the ones corresponding to Y and ∇^2, and that analysis of this functional will involve comparison of the *gaps* in the exponential class and borderline Sobolev embedding inequality. Given the formula for $\mathcal{D}(P, g_0, \omega)$, one can compute rather easily that at the round metric $\omega = 0$ (which is of course a critical metric for both functionals, that with coefficient β_2 and that with coefficient β_3), the second variation of $\mathcal{D}(P, g_0, \omega)$ is positive semidefinite, and positive definite in directions not tangent to curves of conformal transforms of the round metric. Thus a reasonable conjecture is that $\mathcal{D}(P, g_0, \omega)$ is minimized at the round metric and its conformal transforms.

A calculation on S^6 [11] gives positive results for the extremal problems involving $\mathcal{D}(Y, g_0, \omega)$ and $\mathcal{D}(\nabla^2, g_0, \omega)$. Some invariant theory, together with the Conformal Index Theorem 3.1, shows that for operators A as above, our problem is to extremize

$$\begin{aligned} a_1[A]&\left\{-\log\int_{S^6} e^{6(\omega-\bar{\omega})}d\xi + \frac{1}{40}\int_{S^6}\omega((P_6)_0\omega)d\xi\right\} \\ &+a_2[A]\int_{S^6}(|dJ|^2 d\xi)_\omega + a_3[A]\int_{S^6}((|dJ|^2+2J^3)d\xi)_\omega \\ &+a_4[A]\int_{S^6}\left(\left(|dJ|^2+\frac{28}{5}J^3-\frac{48}{5}J|V|^2\right)d\xi\right)_\omega, \end{aligned}$$

where the $a_i[A]$ are constants. The constancy of the conformal index allows us to remove the invariant $\operatorname{tr} V^3 = V^i{}_j V^j{}_k V^k{}_i$, at the cost of shifting the whole expression by a constant. Thus for suitable $a_i[A]$, the above is an expression for $\mathcal{D}(A, g_0, \omega)+c[A]$, for some constant $c[A]$.

The expression with coefficient $a_1[A]$ is nonnegative, and vanishes exactly at conformal transforms of g_0, by Beckner's Theorem. The expression with coefficient $a_2[A]$ is easy to analyze: it is nonnegative, and vanishes iff J_ω has constant scalar curvature. By Obata's Theorem [38], J_ω has constant scalar curvature iff g_ω is a conformal transform of g_0.

The expression with coefficient $a_3[A]$ brings us in contact with the Yamabe number

$$6 = q(S^6) = \inf_{u \in C^\infty(S^6)} \frac{(Yu, u)_{L^2(d\xi)}}{\|u\|^2_{L^3(d\xi)}}.$$

This is a conformal invariant, meaning that we can evaluate Y and $d\xi$ in any of the metrics g_ω and get the same answer. In fact, if

$$\mathcal{Y}(\omega, u) = \frac{(Y_\omega u, u)_{L^2((d\xi)_\omega)}}{\|u\|^2_{L^3((d\xi)_\omega)}},$$

then $\mathcal{Y}(\omega + \eta, u) = \mathcal{Y}(\omega, e^{2\eta}u)$, by the conformal covariance relation for Y (Example 1.1 above). Working in the metric g_ω and choosing $u = J_\omega$, we get

$$\int_{S^6} \left(\left(|dJ|^2 + 2J^3 \right) d\xi \right)_\omega \geq 6 \left(\int_{S^6} (|J|^3 d\xi)_\omega \right)^{2/3} \geq 6J_0^2 = 54. \tag{28}$$

The second $\geq$ in (28) comes from Remark 3.6, with $n = 6$ and $r = 1$. If g_ω has the form $h \cdot g_0$ for h in the conformal group, equality holds in each $\geq$. But by Remark 3.6, equality can *only* hold in the second $\geq$ if $g_\omega = h \cdot g_0$. Thus the expression with coefficient $a_3[A]$ is at least 54, and has the same extremals as the a_1 and a_2 terms.

The estimate on the a_4 term features the Wünsch operator $F = F_{2,2}$ described in Example 1.7. It is somewhat analogous to the last estimate, in that F is applied to the *Einstein tensor* $b = V - Jg/n$. The idea, which yields an inequality for $n \geq 5$ is as follows. F has positive definite leading symbol, so it is reasonable to ask whether it is positive definite, in the sense that $(F\varphi, \varphi)_{L^2} \geq 0$ for all smooth trace-free symmetric tensors φ, with equality iff $\varphi = 0$. (Note that the trace-free condition is conformally invariant.) Since F is conformally covariant, the positive definiteness of F is a conformally invariant condition: it holds at $g_\omega = e^{2\omega} g_0$ if and only if it holds at g_0. By analyzing bundle-valued principal series representations of $SO_0(n+1, 1)$, one finds that F is indeed positive definite on (S^n, g_0) for $n \geq 5$. In particular, $((Fb)_\omega, b_\omega)_{L^2((d\xi)_\omega)} \geq 0$, with equality iff $b = 0$. By Obata's Theorem again, $b = 0$ iff g_ω is a positive constant multiple of a conformal transform of g_0. This gives a sharp inequality in dimension $n \geq 5$, essentially describing the Sobolev embedding of $L^2_1(\mathrm{TFS}^2)$ into $L^{2n/(n-2)}(\mathrm{TFS}^2)$ on the sphere. (See [11] for details.)

In dimension $n = 6$, the expression with coefficient $a_4[A]$ is just $((Fb)_\omega, b_\omega)_{L^2((d\xi)_\omega)}$ shifted by a multiple of the conformal index. The upshot is that the a_4 term too attains its minimum exactly at conformal transforms of the round metric; the minimum value is 108.

To sum up:

Theorem 3.8 [11] *If the $a_i[A]$, $i = 1, 2, 3, 4$, are all nonnegative (resp. nonpositive) and at least one is nonzero, then $\mathcal{D}(A, g_0, \omega)$ is minimized (resp. maximized) exactly* *at positive constants times the metrics $g_\omega = h \cdot g_0$, for h a conformal transformation of (S^6, g_0).*

All $a_i[Y]$ are negative, and all $a_i[\nabla\!\!\!/^2]$ are positive; see [11] for exact values. Thus we have:

Theorem 3.9 *On S^6, $\mathcal{D}(Y, g_0, \omega)$ is maximized and $\mathcal{D}(\nabla\!\!\!/^2, g_0, \omega)$ minimized exactly at positive constants times the metrics $g_\omega = h \cdot g_0$. As a result, any metric which is extremal in the conformal class of g_0, and has the same volume as g_0, is diffeomorphic to g_0.*

Open problems:

3.a Let P be the Paneitz operator. Are there critical metrics for $\mathcal{D}(P, g_0, \omega)$ other than constant multiples of the round metric g_0 and its conformal transforms $h \cdot g_0$, h in the conformal group of (S^4, g_0)?

3.b For the conformal Laplacian Y and the square of the Dirac operator, results on the extremal problem for the functional determinant obtained so far give a "checkerboard" pattern:

	Y	$\nabla\!\!\!/^2$
dim. 2 :	max	min
dim. 4 :	min	max
dim. 6 :	max	min

Should this be expected to continue in higher dimensions? (See [11], Sec. 8 for an inconclusive discussion.)

3.c The sphere S^n, equipped with its standard conformal class, is a "model space" for conformal geometry, in the sense that it has a conformal diffeomorphism group, namely $\mathrm{SO}_0(n+1,1)$, of the largest possible dimension. Similar things can be said of three other series of geometries, spaces, and groups:

	space	**geometry**	**transformation group**
(real case)	S^n	conformal	$\mathrm{SO}_0(n+1,1)$
(complex case)	S^{2n+1}	CR	$\mathrm{SU}(n+1,1)$
(quaternionic case)	S^{4n+3}		$\mathrm{Sp}(n+1,1)$
(octonionic case)	S^{15}		$F_{4(-20)}$

The last "series" consists of just one space/group. (See [16] for details.) In particular, CR, or *tangential Cauchy-Riemann* geometry is the subject of intense current interest among analysts. How does the extremal problem for determinants of operators natural to these geometries, for example the CR sub-Laplacian, interact with harmonic analysis on these spaces? In particular, what are the CR inequalities analogous to (18) and (19)? Do they estimate the determinant of the CR sub-Laplacian?

Epilogue

In dimension 4, the Paneitz operator P and Paneitz curvature quantity Q may be important in Quantum Gravity. Starting with Riegert [46] in 1984, attempts have been made by physicists to include contributions from P in an action functional for gravity. Because of the interplay between P and the Maxwell equations, and thus the Yang-Mills equations, a theory based on P would seem to be apt for natural coupling to matter field equations.

For various reasons, it has been difficult to construct a quantum theory starting from the classical Einstein equations for the metric. Thus, despite the beauty and simplicity of the Einstein functional, the search for an action which incorporates the right classical features, and in addition produces produces a quantum theory, goes on. One of the themes of the work described in these lectures is *uniformization* – driving the geometry to a uniform state by requiring that some natural functional take an extreme value. This is typically associated with a constant curvature condition – the Polyakov formula (25) essentially says (among other things) that the curvature quantity $a_n(A, \omega)$ should be constant at a conformal critical metric. Gravitational theories have the same general character: the Einstein equations, like the Yamabe and conformal relativity problems mentioned under Example 1.1 above, are all uniformity conditions on the geometry (at least before coupling to possible matter fields). Different uniformity conditions typically have solutions in common, so one need not lose the familiar solutions to Einstein's equations by using an alternative action functional.

The Einstein equations assert a uniform state for the Einstein tensor in empty space, or assert that the Einstein tensor compensates the nonuniform effect of matter fields. One way to retain some of the content of the Einstein equations in choosing alternative, higher order equations, is to require that the new action functional be improved by the *Einstein flow*, a variant of the Ricci flow studied by Hamilton, De Turck, and others. Some work in this direction is currently being done by Matt Gursky; the paper [31] is good background for this topic. Specifically, one could ask whether any conformal metric $\Omega^2 g_0$ on S^n having constant Paneitz quantity Q, for $n \geq 4$, is diffeomorphic to g_0.

The analogous statement for K (and $n \geq 3$) is Obata's Theorem (recall Lecture 3), and its proof goes as follows. Let $b = V - Jg/n$ be the (trace-free) Einstein tensor. Then with all covariant derivatives and curvatures computed in the metric $g = \Omega^2 g_0$,

$$b_{ij} = -\Omega^{-1}\left\{\Omega_{|ij} + \frac{1}{n}(\Delta\Omega)g_{ij}\right\},$$

so

$$\begin{aligned} 0 \leq \int \Omega|b|^2 &= -\int b^{ij}\left\{\Omega_{|ij} + \frac{1}{n}(\Delta\Omega)g_{ij}\right\} \\ &= -\int b^{ij}\Omega_{|ij} \end{aligned}$$

$$= \int b^{ij}{}_{|j}\Omega_{|i}\,.$$

Since the "other" Einstein tensor $E = V - Jg$ is divergence free and J is constant, b is divergence free, so the last line above equals 0. We conclude that $b = 0$, whence g has constant sectional curvature (since the Weyl tensor C also vanishes by virtue of conformal flatness), and thus g is diffeomorphic to g_0.

A rationale for the appearance of the Einstein tensor(s) in this proof is as follows. The *total metric variation* of a curvature quantity H is the contravariant 2-tensor $T = T[H]$ for which

$$\overset{\bullet}{g} = \eta \quad \Longrightarrow \quad \left(\int H\, d\mathrm{vol}_g\right)^{\bullet} = \int T^{ij}\eta_{ij}\, d\mathrm{vol}_g\,.$$

Here the bullet • denotes the variation operator $(d/d\varepsilon)|_{\varepsilon=0}$ for a smooth (not necessarily conformal) curve of metrics g_ε. T is automatically divergence free. For the scalar curvature K, the total metric variation $T[K]$ is, up to normalization, the divergence free Einstein tensor E; this is, in fact, the derivation of the Einstein equations. The Obata argument may be seen as a way to take advantage of the fact that the Einstein flow tends to "evenly distribute" the total scalar curvature, and to "improve" (lessen) the functional $\int K$. (Note that $|b|^2 = E^{ij}b_{ij}$, since b is traceless.)

For $n \geq 5$, to work with the Paneitz quantity instead of the scalar curvature, we may take the total metric variation $T = T[Q]$ and compute $\int \Omega T^{ij} b_{ij}$. Here the integrand is not manifestly nonnegative, as it is in the Obata argument; nevertheless, careful integration by parts, together with some hard analysis (work of Gursky) shows that this integral may be written as a sum of nonnegative terms; when all these vanish, we must be at a metric diffeomorphic to the round one. When $n = 4$, $T[Q]$ vanishes, but rational continuation in the dimension still makes an argument possible, using

$$\left.\frac{T[Q]}{n-4}\right|_{n=4}$$

in place of $T[Q]$. The positivity result may be interpreted as saying that, roughly speaking, the Einstein flow improves the functional $\int Q$, and evenly distributes the total Paneitz curvature.

It is also possible to investigate the same Obata type question with other local invariants in place of Q; specifically, quantities like $U_4[P]$. (See [13] for an exact formula for this invariant.) The relevance of the constant U_4 condition in dimension 4 comes from the Polyakov formula: $U_4 \equiv$ const is the Euler-Lagrange equation arising from varying the determinant (and either fixing volume, or compensating for volume dilation as in (22)). As a result, a successful Obata argument in the case of $U_4[P]$ would say that $\mathcal{D}(P, g_0, \omega)$ has no critical points ω, other than the "obvious" ones, for which $e^{2\omega}g_0$ is diffeomorphic to g_0.

In another direction, Carlo Morpurgo [37] has discovered that the dual to Beckner's exponential class inequality (19), the *logarithmic Hardy-Littlewood-Sobolev inequality* (see [21]), also appears in spectral extremal problems. This inequality describes the embedding of the Orlicz class $L \log L$ in $L^2_{-n/2}$. Specifically, the rightmost pole of the function $Z_{A,g}(s)$ from (22), for suitable A, is at $s = n/2\ell$. "Subtracting the pole", i.e. taking

$$\left[Z_{A,g}(s) - \frac{\mathrm{Res}_{s=n/2\ell} Z_{A,g}(s)}{s - n/2\ell}\right]_{s=n/2\ell},$$

yields a spectral invariant whose dominant term is the difference between the two sides in the log HLS inequality. Morpurgo has since related other quantities encoded in the zeta function to other inequalities in the harmonic analysis of the sphere (work in progress). The basic idea is to attempt to estimate $Z_{A,g}(s)$ as a function of g within a conformal class, for generic real s, and make conclusions for the case in which $n/2 - s\ell$ is a nonnegative integer.

An open problem:

For which divergence free symmetric 2-tensor local invariants T at a given level h can $\int \Omega T^{ij} b_{ij}$ be bounded (on either side) by 0? Here we work, as above, in a metric $\Omega^2 g_0$ conformal to the round metric on the sphere. For which local scalar invariants H does $T[H]$ have this property? What happens when we ask the same question for manifolds other than the sphere (with, say, locally symmetric background metrics), especially in the non-conformally flat case?

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