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## COMPLEX METHODS IN REAL INTEGRAL GEOMETRY

Michael Eastwood<sup>†</sup>

Though I take responsibility for the exposition, the work described here is joint with Toby N. Bailey and C. Robin Graham and specific attributions are made for each lecture. Conversations with A. Rod Gover, Lionel J. Mason, and Michael A. Singer have been extremely useful. Indeed, the methods are a direct continuation of joint work of Bailey, Gover, Mason, and myself [4] which itself continues from earlier work of the same authors [3]. These lectures were presented at the 16<sup>th</sup> Winter School on Geometry and Physics, Srní, Czech Republic, January 1996.

The aim is to present a general machine which will analyse some real integral transforms using complex methods. The machine will be illustrated by its application to the X-ray transform. An elaboration upon these aims will be postponed until the second lecture.

### Lecture One

#### The Involutive Structure on a Totally Real Blow-Up

T.N. Bailey M.G. Eastwood C.R. Graham

Let  $\Omega$  be a connected complex  $n$ -dimensional manifold and  $M \subset \Omega$  a totally real, real-analytic submanifold. An example would be  $\mathbb{R}^n \subset \mathbb{C}^n$  and, in general, one can choose coördinates to put  $M \subset \Omega$  locally into this form. Let  $\eta : \tilde{\Omega} \rightarrow \Omega$  denote the real blow-up of  $\Omega$  along  $M$  and denote by  $\Sigma$  the exceptional variety  $\eta^{-1}(M)$ . Thus,

$$\begin{array}{ccc} \Sigma & \subset & \tilde{\Omega} \\ \downarrow & & \downarrow \eta \\ M & \subset & \Omega \end{array}$$

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with  $\Sigma$  a smooth hypersurface in  $\tilde{\Omega}$  whose fibre over  $m \in M$  is the real  $(n-1)$ -dimensional projective space of normal directions at  $m$  to  $M$  in  $\Omega$ . The mapping  $\eta: \tilde{\Omega} \setminus \Sigma \rightarrow \Omega \setminus M$  is a diffeomorphism. Therefore,  $\tilde{\Omega} \setminus \Sigma$  has a complex structure.

**Proposition 1** *The complex structure on  $\tilde{\Omega} \setminus \Sigma$  extends to  $\tilde{\Omega}$  as an involutive structure.*

*Proof.* An involutive structure (in the sense of Treves [18]) is a complex sub-bundle  $T^{0,1}$  of the complexified tangent bundle  $\mathcal{CT}$ , closed under Lie bracket. Equivalently, by setting  $\Lambda^{1,0} = (T^{0,1})^\perp$ , it is a sub-bundle of the bundle of complex-valued 1-forms which generates a differentially closed ideal. Of course, the notation is selected to generalise that for complex manifolds. The involutive structure on  $\tilde{\Omega}$  is obtained by pulling back the complex structure on  $\Omega$ , specifically  $\Lambda_{\tilde{\Omega}}^{1,0} = \eta^* \Lambda_{\Omega}^{1,0}$ . Evidently, this will be closed provided it is indeed a sub-bundle. This is ensured by  $M$  being totally real. Rather than check this abstractly, it is convenient for later use to work in local coördinates (and here we use that  $M$  is real-analytic):

$$\begin{array}{c} (z, s^1, t^1, s^2, t^2, \dots, s^{n-1}, t^{n-1}) \in \mathbb{C} \times \mathbb{R}^{2n-2} \\ \downarrow \eta \\ (z, w^1, w^2, \dots, w^{n-1}) = (z, s^1 + zt^1, s^2 + zt^2, \dots, s^{n-1} + zt^{n-1}) \in \mathbb{C}^n. \end{array}$$

Charts of this form cover  $\Sigma$ , the image of each being a double wedge in  $\Omega$  with edge in  $M$ . Writing  $z = x + iy$ , the hypersurface  $\Sigma$  is defined in these coördinates by  $y = 0$ . The bundle  $\Lambda_{\tilde{\Omega}}^{1,0}$  is spanned by  $dz, dw^1, dw^2, \dots, dw^{n-1}$  so  $\Lambda_{\tilde{\Omega}}^{1,0}$  is spanned by

$$dz, ds^1 + z dt^1 + t^1 dz, ds^2 + z dt^2 + t^2 dz, \dots, ds^{n-1} + z dt^{n-1} + t^{n-1} dz$$

which are manifestly linearly independent.  $\square$

Dually, the bundle  $T^{0,1}$  on  $\tilde{\Omega}$  is spanned by the commuting vector fields

$$\bar{Z} = \frac{\partial}{\partial \bar{z}} \quad \text{and} \quad \bar{W}_j = \frac{\partial}{\partial t^j} - z \frac{\partial}{\partial s^j} \quad \text{for } j = 1, \dots, n-1. \quad (1)$$

These vector fields extend the Cauchy-Riemann equations on  $\tilde{\Omega} \setminus \Sigma$  across  $\Sigma$ . On  $\Sigma$  the vector fields  $\bar{W}_j$  are real and span  $T^{0,1} \cap \bar{T}^{0,1}$ . They generate the fibres of  $\eta: \tilde{\Omega} \rightarrow M$ .

To any real smooth hypersurface in a smooth manifold, there is a canonically associated locally constant line bundle. Write  $\tilde{\mathcal{E}}$  for this bundle associated to  $\Sigma \subset \tilde{\Omega}$ . We recall its construction. It is defined by transition functions taking the values  $\pm 1$ . Choose a covering of  $\Sigma$  by open sets  $U_\alpha$  in  $\tilde{\Omega}$  with  $\Sigma$  defined as the jump discontinuity of a chosen Heaviside function  $H_\alpha$  (for example, sign  $y$  in local coördinates as above). Add  $\tilde{\Omega} \setminus \Sigma$  as a final set to give a covering of  $\tilde{\Omega}$ . Now specify transition functions:

- on  $U_\alpha \cap (\tilde{\Omega} \setminus \Sigma)$  use  $H_\alpha$
- on  $U_\alpha \cap U_\beta$  use  $H_\alpha / H_\beta$ .

By definition,  $\tilde{\mathcal{E}}$  is trivial on  $\tilde{\Omega} \setminus \Sigma$  and we write  $H$  as the trivialising section (i.e. the function which is identically 1 in our chosen trivialisation). When restricted to a fibre  $\eta^{-1}(m)$  for  $m \in M$ , the bundle  $\tilde{\mathcal{E}}$  becomes the standard bundle of twisted functions on  $\mathbb{R}\mathbb{P}_{n-1}$  whose sections correspond to odd functions on  $S^{n-1}$ .

Let  $V$  be a holomorphic vector bundle on  $\Omega$  and denote by  $\mathcal{O}(V)$  (respectively  $\mathcal{E}(V)$ ) its sheaf of germs of holomorphic (respectively smooth) sections. The cohomology spaces  $H^r(\Omega, \mathcal{O}(V))$  will be realised as Dolbeault cohomology. From this point of view the bundle  $V$  is equipped with a  $\bar{\partial}$ -operator, the initial operator in the usual complex:

$$\mathcal{E}(V) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1}(V) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,2}(V) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{E}^{0,n}(V) \rightarrow 0.$$

An involutive structure is all that is needed to define a complex of  $\bar{d}$ -operators

$$\mathcal{E} = \mathcal{A}^{0,0} \xrightarrow{\bar{d}} \mathcal{A}^{0,1} \xrightarrow{\bar{d}} \mathcal{A}^{0,2} \xrightarrow{\bar{d}} \dots \tag{2}$$

where  $\mathcal{A}^{0,r}$  is the sheaf of germs of smooth sections of the  $r^{\text{th}}$  exterior power of  $\Lambda^{0,1} = \Lambda^1/\Lambda^{1,0}$ . This is true on  $\tilde{\Omega}$  where the operators may also be coupled with  $\eta^*V$  (a bundle compatible with the involutive structure in the sense of [4]). Let  $\tilde{V} = \tilde{\mathcal{E}} \otimes \eta^*V$ . Since  $\tilde{\mathcal{E}}$  is given by locally constant transition functions, this bundle is also compatible with the involutive structure and we obtain a complex

$$\Gamma(\tilde{\Omega}, \mathcal{A}^{0,0}(\tilde{V})) \xrightarrow{\bar{d}} \Gamma(\tilde{\Omega}, \mathcal{A}^{0,1}(\tilde{V})) \xrightarrow{\bar{d}} \Gamma(\tilde{\Omega}, \mathcal{A}^{0,2}(\tilde{V})) \xrightarrow{\bar{d}} \dots \tag{3}$$

whose cohomology we shall denote by  $H_{\bar{d}}^r(\tilde{\Omega}, \tilde{V})$ . (Warning: there is no reason to suppose these are sheaf cohomologies.) The main purpose of this lecture is to prove the following:

**Theorem 1** *There is an exact sequence*

$$0 \rightarrow \Gamma(\Omega, \mathcal{O}(V)) \rightarrow \Gamma(M, \mathcal{E}(V)) \rightarrow H_{\bar{d}}^1(\tilde{\Omega}, \tilde{V}) \rightarrow H^1(\Omega, \mathcal{O}(V)) \rightarrow 0$$

and isomorphisms

$$H_{\bar{d}}^r(\tilde{\Omega}, \tilde{V}) \cong H^r(\Omega, \mathcal{O}(V)) \quad \text{for } r \geq 2$$

except for  $r = n - 1$  and  $r = n$  when  $n$  is odd in which case there is an exact sequence

$$0 \rightarrow H^{n-1}(\Omega, \mathcal{O}(V)) \rightarrow H_{\bar{d}}^{n-1}(\tilde{\Omega}, \tilde{V}) \rightarrow \Gamma(M, \mathcal{E}(V)) \rightarrow H^n(\Omega, \mathcal{O}(V)) \rightarrow H_{\bar{d}}^n(\tilde{\Omega}, \tilde{V}) \rightarrow 0.$$

The proof proceeds by the following Lemmata.

**Lemma 1** *The complex*

$$0 \rightarrow \Gamma(\Omega, \mathcal{E}^{0,0}(V)) \xrightarrow{\bar{\partial}} \Gamma(\Omega, \mathcal{E}^{0,1}(V)) \xrightarrow{\bar{\partial}} \Gamma(\Omega, \mathcal{E}^{0,2}(V)) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Gamma(\Omega, \mathcal{E}^{0,n}(V)) \rightarrow 0$$

is formally exact along  $M$  except in the zeroth position where the formal cohomology is  $\Gamma(M, \mathcal{E}(V))$ .

*Proof.* To say that the complex is formally exact at  $\Gamma(\Omega, \mathcal{E}^{0,r}(V))$  is to say that if  $\omega \in \Gamma(\Omega, \mathcal{E}^{0,r}(V))$  satisfies  $d\omega \sim 0$  (meaning that  $d\omega$  vanishes to infinite order along  $M$ ), then there exists  $\nu \in \Gamma(\Omega, \mathcal{E}^{0,r-1}(V))$  so that  $d\nu \sim \omega$  (meaning that  $\omega - d\nu$  vanishes to infinite order along  $M$ ). To say that the zeroth formal cohomology is  $\Gamma(M, \mathcal{E}(V))$  is to say that

- every smooth section  $f_0$  of  $V$  along  $M$  may be extended to a smooth section  $f$  of  $V$  over  $\Omega$  so that  $\bar{\partial}f \sim 0$
- if  $g \in \Gamma(\Omega, \mathcal{E}(V))$  satisfies  $\bar{\partial}g \sim 0$  and  $g|_M = 0$ , then  $g \sim 0$ .

(This is the well-known manoeuvre of taking an almost analytic extension.)

Let us check the zeroth cohomology first. Choose local coördinates  $z^j = x^j + iy^j$  with  $M$  given by  $y^1 = y^2 = \dots = y^n = 0$ . The vector fields

$$\bar{\partial}_j = \frac{1}{2} \left( \frac{\partial}{\partial x^j} + i \frac{\partial}{\partial y^j} \right)$$

are transverse to  $M$  so, in these coördinates, we may define the extension of a function off  $M$  to infinite order by specifying all derivatives  $\bar{\partial}_j \dots \bar{\partial}_k f$  along  $M$ . These may be chosen arbitrarily (provided they are symmetric in the indices  $j \dots k$ ) and, in particular, we may take  $f|_M = f_0$  and all higher derivatives zero. This is precisely a local version of what we want to prove in the case when  $V$  is trivial. A standard partition of unity argument gives the global result and also allows non-trivial  $V$ .

For higher cohomology first use the zeroth result to extend any partition of unity on  $M$  to  $\Omega$  so that it is  $\bar{\partial}$ -closed to infinite order along  $M$ . This reduces to the local case where we can use the coördinates above. Thus, we are given

$$\underbrace{\omega_{kl\dots m}}_r = \omega_{[kl\dots m]} \text{ such that } \bar{\partial}_h \dots \bar{\partial}_i \bar{\partial}_j \omega_{kl\dots m}|_M = 0$$

and we are required to specify  $\bar{\partial}_h \dots \bar{\partial}_i \bar{\partial}_j \bar{\partial}_k \nu_{l\dots m}$  on  $M$ , skew in the indices  $l \dots m$  so that

$$\bar{\partial}_h \dots \bar{\partial}_i \bar{\partial}_j \bar{\partial}_k \nu_{l\dots m}|_M = \bar{\partial}_h \dots \bar{\partial}_i \bar{\partial}_j \omega_{kl\dots m}|_M.$$

This is easily accomplished by setting  $\nu_{l\dots m}|_M = 0$  and

$$\underbrace{\bar{\partial}_h \dots \bar{\partial}_i \bar{\partial}_j \bar{\partial}_k \nu_{l\dots m}}_d|_M = \frac{rd}{r+d-1} \bar{\partial}_h \dots \bar{\partial}_i \bar{\partial}_j \omega_{kl\dots m}|_M \quad \text{for } d \geq 1. \quad \square$$

For the next lemma, recall the local coördinates introduced on  $\tilde{\Omega}$  and the vector field  $\bar{Z}$  constructed from them. Let us write  $\Sigma_0$  and  $\tilde{\Omega}_0$  for the charts of  $\Sigma$  and  $\tilde{\Omega}$  covered by this coördinate system.

**Lemma 2** *Given a smooth function  $f_0$  on  $\Sigma_0$  and a smooth function  $g$  on  $\tilde{\Omega}_0$ , there is a smooth solution  $f$  on  $\tilde{\Omega}_0$  to*

$$\bar{Z}f \sim g \quad \text{with } f|_{\Sigma_0} = f_0$$

*and  $f$  is formally unique along  $\Sigma$ . If  $f_0$  has compact support then  $f$  can be chosen to have compact support.*

*Proof.* This is a formal non-characteristic Cauchy problem for a complex vector field and is proved inductively by power series calculation. More generally, the Cauchy-Kovalevski theorem, usually stated in the real-analytic category, is true on the level of formal power series.  $\square$

**Lemma 3** *Let  $\tilde{\mathcal{C}}$  denote the sheaf of locally constant complex-valued twisted functions on  $\mathbb{R}\mathbb{P}_{n-1}$ .*

$$\text{Then } \begin{cases} H^r(\mathbb{R}\mathbb{P}_{n-1}, \tilde{\mathcal{C}}) = 0 & \text{for all } r \text{ except} \\ H^{n-1}(\mathbb{R}\mathbb{P}_{n-1}, \tilde{\mathcal{C}}) = \mathbb{C} & \text{if } n \text{ is odd.} \end{cases}$$

*Proof.* Let  $\Gamma_{\text{odd}}(S^{n-1}, \Lambda^r)$  denote the smooth  $r$ -forms on  $S^{n-1}$  satisfying  $\iota^*\omega = -\omega$  where  $\iota$  is the antipodal mapping. Then

$$H^r(\mathbb{R}\mathbb{P}_{n-1}, \tilde{\mathcal{C}}) = H^r(\Gamma_{\text{odd}}(S^{n-1}, \Lambda^*))$$

and the result follows from the de Rham theorem on  $S^{n-1}$ . Notice that the mapping  $H^{n-1}(\mathbb{R}\mathbb{P}_{n-1}, \tilde{\mathcal{C}}) \rightarrow \mathbb{C}$  may be defined by integrating a twisted volume form over  $\mathbb{R}\mathbb{P}_{n-1}$  (a well-defined procedure when  $n$  is odd though  $\mathbb{R}\mathbb{P}_{n-1}$  is not orientable).  $\square$

**Lemma 4** *The complex*

$$0 \rightarrow \Gamma(\tilde{\Omega}, \mathcal{A}^{0,0}(\tilde{V})) \xrightarrow{\bar{d}} \Gamma(\tilde{\Omega}, \mathcal{A}^{0,1}(\tilde{V})) \xrightarrow{\bar{d}} \Gamma(\tilde{\Omega}, \mathcal{A}^{0,2}(\tilde{V})) \xrightarrow{\bar{d}} \dots \xrightarrow{\bar{d}} \Gamma(\tilde{\Omega}, \mathcal{A}^{0,n}(\tilde{V})) \rightarrow 0$$

*is formally exact along  $\Sigma$  if  $n$  is even. If  $n$  is odd it is formally exact except at  $\Gamma(\tilde{\Omega}, \mathcal{A}^{0,n-1}(\tilde{V}))$  where the formal cohomology may be identified with  $\Gamma(M, \mathcal{E}(V))$ .*

*Proof.* We use the vector fields (1) to write out the  $\bar{d}$ -operator in local coördinates. An  $r$ -form  $\omega$  may be realised as a pair

$$\underbrace{\alpha_{kl\dots m}}_{r-1} = \alpha_{[kl\dots m]} \quad \underbrace{\beta_{jkl\dots m}}_r = \beta_{[jkl\dots m]}$$

(dual to  $\bar{Z} \wedge \bar{W}_k \wedge \bar{W}_l \wedge \dots \wedge \bar{W}_m$  and  $\bar{W}_j \wedge \bar{W}_k \wedge \bar{W}_l \wedge \dots \wedge \bar{W}_m$ , respectively) and  $\bar{d}\omega \sim 0$  means

$$\bar{W}_{[j}\alpha_{kl\dots m]} \sim \bar{Z}\beta_{jkl\dots m} \quad \text{and} \quad \bar{W}_{[i}\beta_{jkl\dots m]} \sim 0$$

whilst  $\omega \sim d\nu$  reads

$$\alpha_{kl\dots m} \sim \bar{Z}\delta_{kl\dots m} - \bar{W}_{[k}\gamma_{l\dots m]} \quad \text{and} \quad \beta_{jkl\dots m} \sim \bar{W}_{[j}\delta_{kl\dots m]}$$

for suitable  $\gamma_{l\dots m} = \gamma_{[l\dots m]}$  and  $\delta_{kl\dots m} = \delta_{[kl\dots m]}$ . Recall that the vector fields  $\bar{W}_j$  are tangent on  $\Sigma$  to the fibres of  $\eta$ . The forms  $\beta$  and  $\delta$  expressed above in local coördinates have a well-defined meaning on  $\Sigma$  as relative  $r$ - and  $(r-1)$ -forms respectively. On each fibre  $\eta^{-1}(m) \cong \mathbb{R}\mathbb{P}_{n-1}$  for  $m \in M$ , the bundle  $\eta^*V$  plays no rôle whilst the bundle  $\tilde{\mathcal{E}}$  is the bundle of twisted functions. From Lemma 3, we may always solve for  $\delta$  on  $\Sigma$  unless  $n$  is odd and  $r = n-1$ . Having found  $\delta|_{\Sigma}$  we may take  $\gamma$  to be identically zero, using Lemma 2 and a partition of unity to extend  $\delta$  off  $\Sigma$  so that

$$\bar{Z}\delta_{kl\dots m} \sim \alpha_{kl\dots m}.$$



- A smooth function on  $\Omega$  vanishing to infinite order along  $M$  is equivalent to a smooth function on  $\tilde{\Omega}$  vanishing to infinite order along  $\Sigma$ .
- If  $f$  is a smooth function on  $\tilde{\Omega}$  vanishing to infinite order along  $\Sigma$ , then  $Hf$  is a smooth twisted function on  $\tilde{\Omega}$  vanishing to infinite order along  $\Sigma$  and vice versa.

Here, recall that  $H \in \Gamma(\tilde{\Omega}, \tilde{\mathcal{E}})$  is the canonically defined trivialising section. These observations apply equally well to sections of a vector bundle on  $\Omega$  and its pull back to  $\tilde{\Omega}$ . They combine to give an isomorphism of complexes

$$\Gamma(M \subset \Omega, \mathcal{E}^{0,\bullet}(V)) \cong \xrightarrow{xH} \Gamma(\Sigma \subset \tilde{\Omega}, \mathcal{A}^{0,\bullet}(\tilde{V}))$$

and, hence, induced isomorphisms on cohomology. Substituting into the various exact sequences above completes the proof.  $\square$

**Corollary 1** *The spaces  $H_d^r(\tilde{\Omega}, \tilde{V})$  are finite-dimensional except for  $r = 1$  when  $n$  is arbitrary and for  $r = n - 1$  when  $n$  is odd.*

It is interesting to compare this with the Hodge theory discussed by Hanges and Jacobowitz [12]. The involutive structure on  $\tilde{\Omega}$  is elliptic except on  $\Sigma$  but the Levi form is degenerate unless  $n = 2$ . In this case  $\tilde{\Omega}$  is Example  $V_2$  on page 501 of [12] save for a change of coördinates and the Levi form has one positive and one negative eigenvalue. Then [12, Theorem on p. 501] implies that the Laplacian is hypoelliptic with one loss of derivative on 0-forms and 2-forms and standard arguments [12, pp. 509–510] imply that the cohomology is finite-dimensional in degree 0 and 2.

## Lecture Two

### Geometry on the Correspondence Space

T.N. Bailey M.G. Eastwood

The prototype for our discussions will be the X-ray transform introduced by John [13] in 1938. A compactified version of this transform starts with a smooth function  $f$  on the three-sphere  $S^3$  and for each plane  $P$  through the origin in  $\mathbb{R}^4$ , integrates  $f$  over the geodesic  $\gamma = P \cap S^3$ :

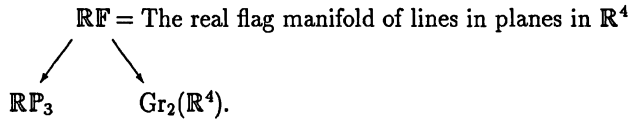
$$\phi(P) = \frac{1}{2\pi} \int_{\gamma} f.$$

The smooth function  $\phi$  is defined on the Grassmannian  $\text{Gr}_2(\mathbb{R}^4)$  of two-planes in  $\mathbb{R}^4$  and satisfies a second order linear differential equation, the ultrahyperbolic wave equation. In a closely related case, by arguing that both spaces are irreducible under an action of  $\text{SL}(4, \mathbb{R})$ , Guillemin and Sternberg [11] conclude that the X-ray transform is an isomorphism:

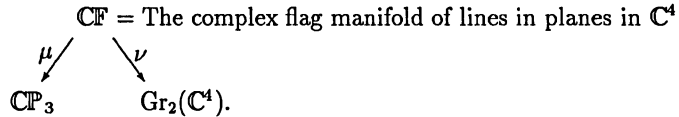
$$\left\{ \begin{array}{l} \text{Smooth even} \\ \text{functions on } S^3 \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Smooth even solutions of the} \\ \text{ultrahyperbolic wave equation on } \text{Gr}(\mathbb{R}^4) \end{array} \right\}. \quad (5)$$

Woodhouse [19] reaches the same conclusion by expanding both side in (generalised) spherical harmonics. John’s original article [13] proves a corresponding result for sufficiently smooth functions on  $\mathbb{R}^3$  subject to decay conditions at infinity. In these lectures we intend to prove (5) by complex methods.

The X-ray transform may be viewed in terms of the real correspondence



Properties of the Penrose transform may be deduced (as, for example, in [8]) by analysing the complex correspondence



To analyse the X-ray transform we shall use a halfway house



where  $F \equiv \nu^{-1}(\text{Gr}_2(\mathbb{R}^4))$ ,  $\tau \equiv \nu|_F$ , and  $\eta \equiv \mu|_F$ . The fibres of  $\nu$  are  $\mathbb{C}\mathbb{P}_1$ ’s so the same is true of  $\tau$ . However,  $\eta$  is no longer a fibration:

**Proposition 2** *The mapping  $\eta$  is the real blow-up of  $\mathbb{C}\mathbb{P}_3$  along  $\mathbb{R}\mathbb{P}_3$ .*

*Proof.* Let us view the complex correspondence in local coördinates on  $\mathbb{C}\mathbb{F}$ . A generic line inside a generic plane in  $\mathbb{C}^4$  may be written as

$$\text{span} \left\{ \begin{pmatrix} 1 \\ z \\ s^1 + zt^1 \\ s^2 + zt^2 \end{pmatrix} \right\} \subset \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ s^1 \\ s^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ t^1 \\ t^2 \end{pmatrix} \right\}$$

and  $(z, s^1, t^1, s^2, t^2) \in \mathbb{C}^5$  give standard affine coördinates on  $\mathbb{C}\mathbb{F}$ . The mapping  $\mu$  is then given by

$$(z, s^1, t^1, s^2, t^2) \longmapsto (z, s^1 + zt^1, s^2 + zt^2)$$

using standard affine coördinates on  $\mathbb{C}\mathbb{P}_3$ . The submanifold  $F$  is where  $s^1, t^1, s^2, t^2$  are real. This is the coördinate system used in Lecture One for a real blow-up.  $\square$

Since the fibres of  $\tau$  are complex, we may consider the bundle  $\Lambda_\tau^{0,1}$  of relative  $(0, 1)$ -forms. The complex structure on each fibre is induced from its image in  $\mathbb{C}\mathbb{P}_3$  under  $\eta$ . Therefore, the composition  $\eta^* \Lambda_{\mathbb{C}\mathbb{P}_3}^{1,0} \rightarrow \Lambda_F^1 \rightarrow \Lambda_\tau^{0,1}$  is zero and we obtain a well-defined homomorphism of vector bundles

$$\Lambda_F^{0,1} \longrightarrow \Lambda_\tau^{0,1}$$

where  $\Lambda_F^{0,1}$  defines the involutive structure of Lecture One. We shall verify soon that this is a surjection. In other words, the complex structure on the fibres of  $\tau$  may be obtained by restricting the involutive structure on  $F$ .

Let  $\iota$  denote the inclusion  $\iota : F \hookrightarrow \mathbb{C}\mathbb{P}$ . Then  $\eta = \mu \circ \iota$  so  $\eta^* \Lambda_{\mathbb{C}\mathbb{P}_3}^{1,0} = \iota^*(\mu^* \Lambda_{\mathbb{C}\mathbb{P}_3}^{1,0})$ . Thus, the composition

$$\iota^* \Lambda_{\mathbb{C}\mathbb{P}}^{1,0} \longrightarrow \Lambda_F^1 \longrightarrow \Lambda_F^{0,1}$$

kills  $\mu^* \Lambda_{\mathbb{C}\mathbb{P}_3}^{1,0}|_F$ . Hence, there is a well-defined homomorphism

$$\Lambda_\mu^{1,0}|_F \longrightarrow \Lambda_F^{0,1}$$

where  $\Lambda_\mu^{1,0}$  denotes the bundle  $\Lambda_{\mathbb{C}\mathbb{P}}^{1,0}/\mu^* \Lambda_{\mathbb{C}\mathbb{P}_3}^{1,0}$  of  $(1, 0)$ -forms along the fibres of  $\mu$ . Let us write  $\Lambda_\eta^{1,0}$  for  $\Lambda_\mu^{1,0}|_F$ . We have constructed the following homomorphisms

$$0 \rightarrow \Lambda_\eta^{1,0} \longrightarrow \Lambda_F^{0,1} \longrightarrow \Lambda_\tau^{0,1} \rightarrow 0. \tag{7}$$

**Proposition 3** *The sequence (7) is exact.*

*Proof.* In our usual local coördinates on  $F$ ,

$$\begin{aligned} \Lambda_\eta^{1,0} &= \frac{\text{span}\{dz, ds^1, dt^1, ds^2, dt^2\}}{\text{span}\{dz, ds^1 + z dt^1, ds^2 + z dt^2\}} \\ &\cong \text{span}\{dt^1, dt^2\} \\ \Lambda_F^{0,1} &= \frac{\text{span}\{dz, d\bar{z}, ds^1, dt^1, ds^2, dt^2\}}{\text{span}\{dz, ds^1 + z dt^1, ds^2 + z dt^2\}} \\ &\cong \text{span}\{d\bar{z}, dt^1, dt^2\} \\ \Lambda_\tau^{0,1} &= \frac{\text{span}\{d\bar{z}, ds^1, dt^1, ds^2, dt^2\}}{\text{span}\{ds^1, dt^1, ds^2, dt^2\}} \\ &\cong \text{span}\{d\bar{z}\} \end{aligned}$$

with the obvious mappings. The result follows. □

Let us see how this fits with the general real blow-up  $\eta : \tilde{\Omega} \rightarrow \Omega$  along  $M$  as in Lecture One. The involutive structure on  $\tilde{\Omega}$  is sufficient to define holomorphic curves in  $\tilde{\Omega}$ . We shall suppose that we are given a foliation of  $\tilde{\Omega}$  by such curves. Then we may simply define  $\Lambda_\eta^{1,0}$  so that (7) is exact. In order to proceed further we shall also suppose that the space of leaves of the foliation is a smooth manifold  $X$ . In other words, there is a submersion  $\tau : \tilde{\Omega} \rightarrow X$  whose fibres are the leaves of the foliation.

Finally, let us consider how (7) fits with the involutive  $\bar{d}$ -complex (2). It induces a filtering of this complex. Abusing notation somewhat, and allowing the fibres of  $\tau$  an arbitrary dimension,

$$\begin{array}{rcccccc}
 \mathcal{A}^{0,0} & = & \mathcal{E} & & & & \\
 \downarrow \bar{d} & & \downarrow \bar{\partial}_\tau & & & & \\
 \mathcal{A}^{0,1} & = & \mathcal{E}_\tau^{0,1} & + & \mathcal{E}(\Lambda_\eta^{1,0}) & & \\
 \downarrow \bar{d} & & \downarrow \bar{\partial}_\tau & & \downarrow \bar{\partial}_\tau & & \\
 \mathcal{A}^{0,2} & = & \mathcal{E}_\tau^{0,2} & + & \mathcal{E}_\tau^{0,1}(\Lambda_\eta^{1,0}) & + & \mathcal{E}(\Lambda_\eta^{2,0}) & (8) \\
 \downarrow \bar{d} & & \downarrow \bar{\partial}_\tau & & \downarrow \bar{\partial}_\tau & & \downarrow \bar{\partial}_\tau & \\
 \mathcal{A}^{0,3} & = & \mathcal{E}_\tau^{0,3} & + & \mathcal{E}_\tau^{0,2}(\Lambda_\eta^{1,0}) & + & \mathcal{E}_\tau^{0,1}(\Lambda_\eta^{2,0}) & + & \mathcal{E}(\Lambda_\eta^{3,0}) \\
 \downarrow \bar{d} & & \downarrow \bar{\partial}_\tau & & \downarrow \bar{\partial}_\tau & & \downarrow \bar{\partial}_\tau & & \downarrow \bar{\partial}_\tau \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots
 \end{array}$$

where  $\Lambda_\eta^{p,0} = \Lambda^p(\Lambda_\eta^{1,0})$ ,  $\Lambda_\tau^{0,q} = \Lambda^q(\Lambda_\tau^{0,1})$ , and  $\mathcal{E}_\tau^{0,q} = \mathcal{E}(\Lambda_\tau^{0,q})$ , the sheaf of germs of smooth sections of  $\Lambda_\tau^{0,q}$ . The right hand sides in this diagram merely indicate the composition factors defined by the filtering. There are induced differential operators

$$\bar{\partial}_\tau : \mathcal{E}_\tau^{0,q}(\Lambda_\eta^{p,0}) \longrightarrow \mathcal{E}_\tau^{0,q+1}(\Lambda_\eta^{p,0}).$$

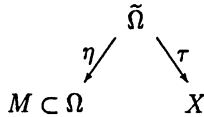
For  $p = 0$ , this is the Dolbeault resolution along the fibres of  $\tau$ . We shall write  $\mathbb{C}\mathcal{E}$  for the sheaf it resolves. It is the sheaf of germs of smooth functions which are holomorphic along the fibres of  $\tau$  and we shall refer to such functions as partially holomorphic. Similarly, we shall write  $\mathbb{C}\mathcal{E}_\eta^p$  for the sheaf of germs of partially holomorphic sections of  $\Lambda_\eta^{p,0}$ . We may view the partially holomorphic structure on  $\Lambda_\eta^{p,0}$  as arising from the diagram (8). Alternatively, in the X-ray case,  $\Lambda_\eta^{p,0}$  is obtained by restricting the holomorphic bundle  $\Lambda_\mu^{1,0}$  on  $\mathbb{C}\mathbb{F}$  to  $F$ . Of course, any holomorphic vector bundle on  $\mathbb{C}\mathbb{F}$  will be partially holomorphic when restricted to  $F$ .

### Lecture Three

#### Pushing Down

T.N. Bailey M.G. Eastwood

Lecture Two ended with



where

- $\Omega$  is a complex manifold.
- $M$  is a totally real, real-analytic submanifold.
- $\tilde{\Omega}$  is the real blow-up of  $\Omega$  along  $M$  endowed with the involutive structure of Lecture One.
- $X$  is a smooth manifold.
- $\tau : \tilde{\Omega} \rightarrow X$  is a submersion with complex fibres of complex dimension one.

From now on we shall assume that  $\Omega$ , and hence  $\tilde{\Omega}$ ,  $X$ , and the fibres of  $\tau$ , are compact. The prototype is (6).

Suppose  $V$  is a holomorphic vector bundle on  $\Omega$  and recall the complex (3). Combined with the short exact sequence (7), we obtain a spectral sequence

$$E_0^{p,q} = \Gamma(\tilde{\Omega}, \mathcal{E}_\tau^{0,q}(\Lambda_\eta^{p,0} \otimes \tilde{V})) \implies H_d^{p+q}(\tilde{\Omega}, \tilde{V}).$$

This is the  $E_0$ -level of the spectral sequence associated with the filtered complex (8). It will enable us to interpret the involutive cohomology spaces  $H_d^p(\tilde{\Omega}, \tilde{V})$  down on  $X$ . This interpretation follows exactly the general method explained by Bailey [1] in relation to the Penrose transform. The same methods were used by Schmid [16] in 1967 in relating the discrete series representations realised as Dolbeault cohomology on  $G/T$  to a realisation as solutions of his  $\mathcal{D}$ -operator on  $G/K$ . More recently, similar techniques appear in the theses of Singer [17] and Wong [20].

It is easy to pass to the  $E_1$ -level. The differentials at the  $E_0$ -level are the  $\bar{\partial}_\tau$ -operators. Thus,

**Theorem 2** *There is a spectral sequence*

$$E_1^{p,q} = \Gamma(X, \tau_* \mathbb{E}_\eta^p(\tilde{V})) \implies H_d^{p+q}(\tilde{\Omega}, \tilde{V}).$$

(Those familiar with the Penrose transform will recognise this spectral sequence from its counterpart there (for example, [5, p. 308], [6, Theorem 7.3.1], or [8, Theorem 4.1]). Notice that since the fibres of  $\tau$  are compact, Dolbeault cohomology spaces along these







Consequently, there is an X-ray transform

$$\mathcal{X} : \Gamma(\mathbb{R}\mathbb{P}_3, \mathcal{E}(\Lambda^1)) \longrightarrow \Gamma(\text{Gr}_2(\mathbb{R}^4), \tilde{\mathcal{E}})$$

whose range is

$$\{f \in \Gamma(\text{Gr}_2(\mathbb{R}^4), \tilde{\mathcal{E}}) \text{ s.t. } \mathcal{D}df = d\rho \text{ for some } \rho \in \Gamma(\text{Gr}_2(\mathbb{R}^4), \tilde{\mathcal{E}}^2)\}$$

where  $\mathcal{D} : \tilde{\mathcal{E}}^1 \rightarrow \tilde{\mathcal{E}}^3$  is the operator of [9, Proposition 3.3]. (To establish this carefully requires a good deal of further argument, cf. [14].) The kernel of  $\mathcal{X}$  is identified with the twisted anti-self-dual Maxwell fields of zero charge. Alternatively, in view of Example 2, the kernel consists of the exact 1-forms. The transform  $\mathcal{X}$  is geometrically extremely natural. A point of  $\text{Gr}_2^+(\mathbb{R}^4)$  gives an oriented geodesic on  $\mathbb{R}\mathbb{P}_3$  over which a smooth 1-form may be integrated. Changing the orientation changes the sign of this integral so there results a twisted smooth function on  $\text{Gr}_2(\mathbb{R}^4)$ . From this interpretation it is clear that  $\mathcal{X}$  annihilates the exact forms. The converse is a result of Michel [15]. It generalises considerably [2]. That one can identify the range of  $\mathcal{X}$  in terms of differential equations on  $\text{Gr}_2(\mathbb{R}^4)$  is, perhaps, more subtle.

**Example 5** An X-ray transform can be constructed and analysed starting with any irreducible homogeneous vector bundle on  $\mathbb{C}\mathbb{P}_3$ . The corresponding exercise for the Penrose transform is carried out in [7] and a complete analysis in the X-ray case would follow similar lines. Certainly, the necessary direct images are computed in [7]. The other ingredient, needed to compute the analytic cohomology in  $\mathbb{C}\mathbb{P}_3$  which appears in Theorem 1, is the Bott-Borel-Weil Theorem—see, for example, [6].

### Concluding Remarks

The X-ray transform considered in this section could, in principle, be analysed by decomposing the spaces in question into harmonics under the action of  $SO(4)$ . On the other hand, the main analytical ingredient involved in applying Theorems 1 and 2 is the Bott-Borel-Weil Theorem, either on  $\mathbb{C}\mathbb{P}_3$  or on the fibres of  $\tau$ , each isomorphic to  $\mathbb{C}\mathbb{P}_1$ . The Bott-Borel-Weil Theorem can itself be proved using (generalised) spherical harmonics. Thus, one possible point of view on this approach to the X-ray transform is that the geometry of (6) is automatically organising the spherical harmonics to effect the analysis. Presumably, a similar point of view can be adopted in any situation with a sufficient degree of symmetry. Of course, the Bott-Borel-Weil Theorem is an extremely efficient method of organising these harmonics. It would be interesting to find some less symmetrical examples where the analytic cohomology of Theorem 1 is beyond the Bott-Borel-Weil Theorem. Certainly, the Penrose transform goes well beyond this.

There are two limitations inherent in this development. The first comes about by using a real blow-up. Here one replaces a totally real submanifold  $M$  of a complex manifold with its space of normal directions. This is the right thing to do for a real integral transform defined by integrals over one-dimensional real cycles. For cycles of

dimension  $k$  one would have to modify the blow-up procedure so as to replace  $M$  with its space of normal  $k$ -planes. Presumably, this is not a serious impediment. A tighter restriction seems to be the requirement of  $M$  having a compact complexification with the real cycles in  $M$  inducing a foliation of the blow-up by complex submanifolds. At the moment it is far from clear how stringent a requirement this is. Many more examples are needed to illuminate this issue.

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