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ON THE SECOND ORDER ABSOLUTE DIFFERENTIATION

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ABSTRACT. First we compare two different approaches to the second order absolute differentiation on an arbitrary fibered manifold. Then we extend the second approach to connections on the functional bundle of all smooth maps between the fibers over the same base point of two fibered manifolds over the same base. (For the first approach, this problem was solved in [4].)

There are two different approaches to the second order absolute differentiation in the case of a principal or linear connection Γ. The first one constructs $\nabla^2_{\Gamma,\Lambda}$ by means of an auxiliarly linear connection Λ on the base manifold, [17], which is related to the ideas of tensor calculus. The second one applies another geometric idea by C. Ehresmann, [6], and constructs ∇^2_Γ by means of Γ only. In Section 1 we recall the first construction in the case of a connection Γ on an arbitrary fibered manifold $\pi : Y \to M$, which has been developed recently in [1]. In Section 2 we generalize Ehresmann's approach to connections on a finite-dimensional groupoid to the groupoid $\mathcal{G}Y$ of all diffeomorphisms between the individual fibers of Y. We use systematically the structure of a smooth space in the sense of Frölicher on $\mathcal{G}Y$. The groupoid approach clarifies directly that the values of ∇^2_Γ are semiholonomic 2-jets. But it is remarkable that it also interprets some prolongation procedures for connections on Y from a new point of view. In Section 3 we present a construction of ∇^2_Γ by using second tangent bundles, which we need for a generalization in Section 6. Then we comment on some differences between $\nabla^2_{\Gamma,\Lambda}$ and ∇^2_Γ.

The second part of the present paper is devoted to a functional version of the second order absolute differentiation. Consider two locally trivial fibered manifolds $p_1 : Y_1 \to M$, $p_2 : Y_2 \to M$ over the same base and the bundle of all fiber maps

$$\mathcal{F}(Y_1, Y_2) = \bigcup_{x\in M} C^\infty(Y_{1x}, Y_{2x}), \tag{1}$$

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which is a smooth space in the sense of Frölicher, [2]. The first approach to the second order absolute differentiation on $\mathcal{F}(Y_1, Y_2)$ was studied in [4], so that we go directly to the second one. In Section 5 we define the absolute differential $\nabla_\Gamma f$ of any smooth map f of a manifold N into $\mathcal{F}(Y_1, Y_2)$ with respect to a connection Γ on $\mathcal{F}(Y_1, Y_2)$. Then we construct $\nabla^2_\Gamma f$ by using the machinery of second tangent bundles. In Section 7 we deduce for a finite order connection Γ on $\mathcal{F}(Y_1, Y_2)$ and a section $s : M \to \mathcal{F}(Y_1, Y_2)$ that the deviation of semiholonomic 2-jet $\nabla^2_\Gamma s(x)$ coincides up to the sign with the curvature of Γ at $s(x)$.

If we deal with finite dimensional manifolds and maps between them, we always assume they are of class C^∞, i.e. smooth in the classical sense. On the other hand, the concept of smoothness in the infinite dimension is due to Frölicher, [7], see also [3].

1. An auxiliary linear connection on the base. On an arbitrary fibered manifold $\pi : Y \to M$, a connection can be defined as a section $\Gamma : Y \to J^1 Y$, see e.g. [13]. We denote by $v_\Gamma : TY \to VY$ its vertical projection. If $s : M \to Y$ is a section, we define its absolute differential by

$$\nabla_\Gamma s = v_\Gamma \circ Ts\,, \tag{2}$$

i.e. we construct the vertical projection of the tangent map of s. Hence $\nabla_\Gamma s$ is a section $M \to VY \otimes T^*M$. Let x^i, y^p be some local fiber coordinates on Y and let Γ be expressed by

$$dy^p = F^p_i(x, y)\, dx^i \tag{3}$$

and s by $y^p = s^p(x)$. Then the coordinate form of (2) is

$$\frac{\partial s^p}{\partial x^i} - F^p_i(x, s(x))\,. \tag{4}$$

There is a canonical isomorphism $i_Y : V(J^1 Y \to M) \to J^1(VY \to M)$, [13], p. 255. If we compose the vertical tangent map $V\Gamma : VY \to VJ^1 Y$ with i_Y, we obtain a connection $\mathcal{V}\Gamma := i_Y \circ V\Gamma$ on $VY \to M$, which is called the vertical prolongation of Γ. If Y^p are the additional coordinates on VY, then the equations of $\mathcal{V}\Gamma$ are (3) and

$$dY^p = \frac{\partial F^p_i}{\partial y^q} Y^q\, dx^i\,. \tag{5}$$

Let Λ be a linear connection on TM and Λ^* be the dual connection on T^*M. Since $\mathcal{V}\Gamma$ is semilinear, we can construct the tensor product $\mathcal{V}\Gamma \otimes \Lambda^*$, which is a connection on $VY \otimes T^*M$, [1], [13]. If Y^p_i are the tensor coordinates on $VY \otimes T^*M$ and $\Lambda^i_{jk}(x)$ are the Christoffel symbols of Λ, then the coordinate expression of $\mathcal{V}\Gamma \otimes \Lambda^*$ is (3) and

$$dY^p_i = \left(\frac{\partial F^p_j}{\partial y^q} Y^q_i - \Lambda^k_{ij} Y^p_k \right) dx^j\,. \tag{6}$$

Hence we can construct the second order absolute differential

$$\nabla^2_{\Gamma,\Lambda} s = \nabla_{\mathcal{V}\Gamma\otimes\Lambda^*}(\nabla_\Gamma s)\,. \tag{7}$$

If we have a vector bundle $\pi : E \to M$, then $VE \approx E \times_M E$. If Γ is a linear connection on E, then $\mathcal{V}\Gamma$ coincides with the product $\Gamma \times \Gamma$. For every section $s : M \to E$, we have $pr_2 \circ \nabla_\Gamma s : M \to E \otimes T^*M$ and $\nabla^2_{\Gamma,\Lambda} s$ is identified with $(\nabla_\Gamma s, \nabla_{\Gamma\otimes\Lambda^*}(\nabla_\Gamma s))$. This is the classical tensor approach, [16]. In particular, if E is a tensor power of TM and T^*M and Γ is the corresponding tensor power of a linear connection Λ on TM and of its dual Λ^*, we take Λ again for the auxiliary linear connection on TM. Then we obtain the classical procedures of tensor calculus.

2. The groupoid approach. Let $\pi : Y \to M$ be a locally trivial fibered manifold. We write $IsoC^\infty(Y_x, Y_y)$ for the set of all diffeomorphisms of Y_x into Y_y.

Definition 1. The set

$$\mathcal{G}Y = \bigcup_{(x,y)\in M\times M} IsoC^\infty(Y_x, Y_y) \tag{8}$$

is called the groupoid of all diffeomorphisms of the fibers of Y or the groupoid of Y.

We are going to show that $\mathcal{G}Y$ is a smooth space in the sense of Frölicher.

In general, let $p_1 : Y_1 \to M_1$ and $p_2 : Y_2 \to M_2$ be two locally trivial fibered manifolds. Then we define

$$\mathcal{F}ib(Y_1, Y_2) = \bigcup_{(x,y)\in M_1\times M_2} C^\infty(Y_{1x}, Y_{2y}) \tag{9}$$

We denote by $p : \mathcal{F}ib(Y_1, Y_2) \to M_1 \times M_2$ the canonical projection. Consider the product projections $pr_1 : M_1 \times M_2 \to M_1$, $pr_2 : M_1 \times M_2 \to M_2$ and construct the pullbacks $\bar{Y}_1 = pr_1^* Y_1$, $\bar{Y}_2 = pr_2^* Y_2$, which are fibered manifolds over $M_1 \times M_2$. Then we have defined $\mathcal{F}(\bar{Y}_1, \bar{Y}_2)$ in the sense of (1). By the definition of pullback,

$$\mathcal{F}ib(Y_1, Y_2) = \mathcal{F}(\bar{Y}_1, \bar{Y}_2)\,. \tag{10}$$

This introduces the structure of a smooth space on $\mathcal{F}ib(Y_1, Y_2)$, [2]. In other words, for every manifold N a map $f : N \to \mathcal{F}ib(Y_1, Y_2)$ is smooth, if the base map $p \circ f : N \to M_1 \times M_2$ is of class C^∞ and the induced map

$$\widetilde{f} : f_1^* Y_1 \to Y_2\,, \tag{11}$$

$\widetilde{f}(u, y_1) = f(u)(y_1)$, $f_1 = pr_1 \circ p \circ f$, $u \in N$, $y_1 \in Y_1$, $f_1(u) = p_1(y_1)$, is of class C^∞. Moreover, r-jets of N into $\mathcal{F}ib(Y_1, Y_2)$ are introduced by

$$J^r(N, \mathcal{F}ib(Y_1, Y_2)) := J^r(N, \mathcal{F}(\bar{Y}_1, \bar{Y}_2))\,,$$

where the right-hand side was defined in [4]. In the case of product bundles $Y_1 = M_1 \times Q_1$, $Y_2 = M_2 \times Q_2$, we have

$$J^r(N, \mathcal{F}ib(Y_1, Y_2)) = J^r(N, M_1 \times M_2) \times_N C^\infty(Q_1, J^r(N, Q_2))\,, \tag{12}$$

where the subscript α indicates that we consider the maps into the fibers of the jet projection $\alpha : J^r(N, Q_2) \to N$.

The inclusion $\mathcal{G}Y \subset \mathcal{F}ib(Y, Y)$ defines the structure of smooth space on $\mathcal{G}Y$. We shall write $a = pr_1 \circ p : \mathcal{G}Y \to M$, $b = pr_2 \circ p : \mathcal{G}Y \to M$. The following definition extends an idea by Ehresmann, [6], to the infinite dimensional space $\mathcal{G}Y$.

Definition 2. An element of connection on $\mathcal{G}Y$ at $x \in M$ is a 1-jet at x of a smooth map $\sigma : U \to \mathcal{G}Y$ of a neighbourhood U of x satisfying

$$a\sigma(u) = x, \quad b\sigma(u) = u \quad \text{for all} \quad u \in U \quad \text{and} \quad \sigma(x) = \mathrm{id}_{Y_x}. \tag{13}$$

The set of all elements of connection on $\mathcal{G}Y$ will be denoted by $Q\mathcal{G}Y$. Since $Q\mathcal{G}Y \subset J^1(M, \mathcal{G}Y)$, it is a smooth space as well. The source jet map is a projection $Q\mathcal{G}Y \to M$.

Every $A \in Q_x\mathcal{G}Y$, $A = j^1_x\sigma(u)$, defines a section $\widetilde{A} : Y_x \to J^1_xY$, $\widetilde{A}(y) = j^1_x\sigma(u)(y)$. Conversely, Proposition 5 of [19] implies that for every section $B : Y_x \to J^1_xY$ there exists a neighbourhood U of $x \in M$ and a map $\sigma : U \to \mathcal{F}ib(Y, Y)$ satisfying (13) such that $B = \widetilde{j^1_x\sigma}$. Since $\sigma(x) = \mathrm{id}_{Y_x}$, the map $\sigma : U \times Y_x \to Y$ is local diffeomorphism in a neighbourhood of $\{x\} \times Y_x$. In this local sense, connections on Y correspond to smooth sections $\Gamma : M \to Q\mathcal{G}Y$.

Given a section $s : M \to Y$ and an element of connection $A = j^1_x\sigma(u) \in Q_x\mathcal{G}Y$, we define the absolute differential $\nabla_A s$ by

$$\nabla_A s = j^1_x \left(\sigma^{-1}(u)(s(u))\right) \in J^1_x(M, Y_x), \tag{14}$$

where $\sigma^{-1}(u)$ denotes the inverse diffeomorphism, so that $\sigma^{-1}(u)(s(u))$ is a local map $M \to Y_x$. We are going to show that (14) coincides with (2) for $\widetilde{A} = \Gamma|Y_x$. In some local coordinates x^i, y^p, let $\bar{y}^p = f^p(u, y)$ be the coordinate expression of $\sigma(u)$. Then $\widetilde{A} : Y_x \to J^1_xY$ is given by

$$F^p_i(x, y) = \frac{\partial f^p(x, y)}{\partial x^i}. \tag{15}$$

If $y^p = s^p(u)$ is the coordinate form of s, $\sigma^{-1}(u)(s(u))$ is expressed by $\widetilde{f}^p(u, s(u))$, where $\widetilde{f}^p(u, y)$ is the inverse diffeomorphism of $\sigma(u)$. Then the coordinate form of $j^1_x\sigma^{-1}(u)(s(u))$ is

$$\frac{\partial \widetilde{f}^p(x, y)}{\partial x^i} + \frac{\partial \widetilde{f}^p(x, y)}{\partial y^q} \frac{\partial s^q}{\partial x^i}. \tag{16}$$

But $\sigma(x) = \mathrm{id}_{Y_x}$ implies $\partial\widetilde{f}^p(x, y)/\partial y^q = \delta^p_q$. Differentiating $\sigma^{-1}(u) \circ \sigma(u) = \mathrm{id}_{Y_x}$, we obtain $\partial\widetilde{f}^p(x, y)/\partial x^i = -\partial f^p(x, y)/\partial x^i = -F^p_i(x, y)$. Hence (16) concides with (4).

Using this point of view, we interpret $\nabla_\Gamma s$ as a section of the union

$$J^1(M, Y, \pi) := \bigcup_{x \in M} J^1(M, Y_x), \tag{17}$$

which is a fibered manifold over M. Every diffeomorphism $\varphi : Y_x \to Y_y$ is extended into a map $J^1(\mathrm{id}_M, \varphi) : J^1(M, Y_x) \to J^1(M, Y_y)$. This defines an injection $\mathcal{G}Y \hookrightarrow \mathcal{G}(J^1(M, Y, \pi))$. Hence every element of connection $A = j^1_x\sigma(u)$ on $\mathcal{G}Y$ is extended into an element of connection A_1 on $\mathcal{G}(J^1(M, Y, \pi))$ defined by

$$A_1 = j^1_x J^1(\mathrm{id}_M, \sigma(u)). \tag{18}$$

The correctness of this definition follows from the coordinate expressions. The local coordinates x^i, y^p on Y induce jet coordinates v^i, y^p, y^p_i on each $J^1(M, Y_x)$. If $\sigma(u)$ is expressed by

$$\bar{x}^i = u^i, \quad \bar{y}^p = f^p(u, y)\,, \tag{19}$$

then the additional coordinate expression of $J^1(\mathrm{id}_M, \sigma(u))$ is

$$\bar{v}^i = v^i, \quad \bar{y}^p_i = \frac{\partial f^p(u,y)}{\partial y^q} y^q_i\,. \tag{20}$$

Hence A_1 is of the form

$$dv^i = 0, \quad dy^p = F^p_i(x,y)\,dx^i, \quad dy^p_i = \frac{\partial F^p_j}{\partial y^q} y^q_i\,dx^j\,. \tag{21}$$

Thus, every connection Γ on Y is canonically extended into a connection Γ_1 on $J^1(M, Y, \pi)$. Since $\nabla_\Gamma s$ is a section of $J^1(M, Y, \pi)$, every

$$\nabla_{\Gamma_1}(\nabla_\Gamma s)(x) = j^1_x J^1(\mathrm{id}_M, \sigma(u))^{-1}(\nabla_\Gamma s(u))$$

is a semiholonomic 2-jet of M into Y_x.

Definition 3. The map

$$\nabla^2_\Gamma s := \nabla_{\Gamma_1}(\nabla_\Gamma s) : M \to \bigcup_{x \in M} \bar{J}^2_x(M, Y_x) \tag{22}$$

is called the second absolute differential of s with respect to Γ.

Proposition 1. *The coordinate form of* $\nabla^2_\Gamma s$ *is* (4) *and*

$$\frac{\partial^2 s}{\partial x^i \partial x^j} - \frac{\partial F^p_i}{\partial x^j} - \frac{\partial F^p_i}{\partial y^q}\frac{\partial s^q}{\partial x^i} - \frac{\partial F^p_j}{\partial y^q}\frac{\partial s^q}{\partial x^i} + \frac{\partial F^p_j}{\partial y^q} F^q_i\,. \tag{23}$$

Proof. This follows directly from (4) and (21).

We remark that the idea of extending the groupoid $\mathcal{G}Y$ can be applied for prolongating connections in many similar cases. For example, every diffeomorphism $\varphi : Y_x \to Y_y$ induces the tangent map $T\varphi : V_x Y \to V_y Y$. Hence every element of connection $A = j^1_x \sigma(u)$ on $\mathcal{G}Y$ defines an element of connection $\mathcal{V}A = j^1_x T(\sigma(u))$ on the groupoid $\mathcal{G}VY$ of the vertical tangent bundle. For a connection Γ on Y, $\mathcal{V}\Gamma$ coincides with the vertical prolongation from Section 1.

3. The use of second tangent bundles. For every vector bundle $p : E \to M$, there are two vector bundle structures $\pi_E : TE \to E$ and $Tp : TE \to TM$ on TE. Moreover, we have an injection $i : E \to TE$ which identifies E_x with the tangent space $T_{0_x}(E_x)$ of the fiber E_x at its zero vector 0_x. In other words, $i(E)$ is the common kernel of both projection π_E and Tp. Using the terminology of J. Pradines, [18], [15], we say that $i(E) =: HE \subset TE$ is the *heart* of E. Clearly, if $q : D \to N$ is another vector bundle and $f : E \to D$ is a linear morphism, then Tf is a linear morphism of both vector bundle structures $\pi_E \to \pi_D$ and $Tp \to Tq$. We shall also say that Tf is linear in both directions. Moreover, $Tf(HE) \subset HD$ and the restriction $Hf : HE \to HD$ of Tf coincides with f.

Every non-holonomic 2-jet $X \in \widetilde{J}^2_x(M,N)_y$ is of the form $j^1_x\sigma$, where $\sigma : M \to J^1(M,N)$ is a section of the source projection $\alpha : J^1(M,N) \to M$, [5]. Every $\sigma(u) \in J^1_u(M,N)$ is identified with a linear map $\mu(\sigma(u)) : T_uM \to TN$, so that X defines a map

$$\mu X : TT_xM \to TT_yN, \quad \mu X = T_x\mu(\sigma(u)). \tag{24}$$

Consider the projections $\pi_{TM} : TTM \to TM$ and $T\pi_M : TTM \to TM$.

Lemma 1. (J. Pradines, [18]) *A map $A : TT_xM \to TT_yN$ represents a non-holonomic 2-jet $X \in \widetilde{J}^2_x(M,N)_y$, i.e. $A = \mu X$, iff all following conditions are fulfilled:*

(i) *A is π_T-projectable over a linear map $A_1 : T_xM \to T_yN$ and $T\pi$-projectable over a linear map $A_2 : T_xM \to T_yN$,*

(ii) *A is a linear morphism with respect to both vector bundle structures π_T and $T\pi$,*

(iii) *the heart restriction $A_0 : H_xTM \to H_yTN$ coincides with A_1.*

Moreover, X is semiholonomic, iff $A_1 = A_2$.

Proof. If $f^p(u)$, $f^p_i(u)$ is the coordinate expression of σ, then $\mu\sigma(u)$ is of the form $y^p = f^p(u)$, $Y^p = f^p_i(u)X^i$. For $T_x\mu\sigma(u)$ we find

$$Y^p = f^p_i(x)X^i, \quad dy^p = \frac{\partial f^p(x)}{\partial x^i}dx^i, \quad dY^p = \frac{\partial f^p_i(x)}{\partial x^j}X^i dx^j + f^p_i(x)\,dX^i. \tag{25}$$

This is the coordinate form of our claim. □

The absolute differentiation of sections of a fibered manifold $\pi : Y \to M$ can be extended to any map $f : N \to Y$. We define

$$\nabla_\Gamma f = v_\Gamma \circ Tf, \tag{26}$$

so that $\nabla_\Gamma f$ is a $\mathcal{VB}$-morphism $TN \to VY$ over $f : N \to Y$. If u^s are some local coordinates on N, U^s are the additional coordinates on TN,

$$x^i = f^i(u), \quad y^p = f^p(u) \tag{27}$$

is the coordinate expression of f and Γ is given by (3), then the coordinate form of $\nabla_\Gamma f$ is (27) and

$$Y^p = \left(\frac{\partial f^p}{\partial u^s} - F^p_i(f^j(u), f^q(u))\frac{\partial f^i}{\partial u^s}\right) U^s . \tag{28}$$

$\nabla_\Gamma f$ is a map with values in VY, so that we can construct its absolute differential with respect to any connection Δ on $VY \to M$. Since $\varrho : VY \to Y$ is a vector bundle, J^1VY is a vector bundle over J^1Y. A connection $\Delta : VY \to J^1VY$ is called semilinear, if it is projectable, i.e. there exists a connection $\Delta_0 : Y \to J^1Y$ satisfying $\Delta_0 \circ \varrho = (J^1\varrho) \circ \Delta$, and Δ is a $\mathcal{VB}$-morphism $VY \to J^1VY$ over Δ_0. Clearly, both $T\varrho : TVY \to TY$ and $V\varrho : VVY \to VY$ are vector bundles. If Δ is a semilinear connection, its vertical projection $v_\Delta : TVY \to VVY$ is a $\mathcal{VB}$-morphism $T\varrho \to V\varrho$ over $v_{\Delta_0} : TY \to VY$.

Proposition 2. *Let Γ be a connection on $\pi : Y \to M$, Δ be a semilinear connection on $VY \to M$ and $f : N \to Y$ be a map. Then*

$$\nabla_\Delta(\nabla_\Gamma f)(u) : TT_uN \to VV_{f(u)}Y \tag{29}$$

corresponds to a non-holonomic 2-jet of $\widetilde{J}^2_u(N, Y_{\pi(f(u))})$. If $\Delta_0 = \Gamma$, then each jet (29) is semiholonomic.

Proof. Since $\nabla_\Gamma f : TN \to VY$ is a linear morphism, $T\nabla_\Gamma f : TTN \to TVY$ is linear in both directions. Since Δ is semilinear, its vertical projection v_Δ is linear in both directions. Hence $v_\Delta \circ T\nabla_\Gamma f$ is linear in both directions over $v_{\Delta_0} \circ Tf$ and $v_\Gamma \circ Tf$. The heart map is $v_\Gamma \circ Tf$. Then our claim follows from Lemma 1. □

Proposition 3. *If we take $\Delta = \mathcal{V}\Gamma$, then for every section $s : M \to Y$ we have*

$$\nabla_{\mathcal{V}\Gamma}\nabla_\Gamma s(x) = \mu(\nabla^2_\Gamma s(x)) . \tag{30}$$

Proof. By (28), the coordinate form of $\nabla_\Gamma s$ is $y^p = s^p(x)$ and

$$Y^p = \left(\frac{\partial f^p}{\partial x^i} - F^p_i(x, s(x))\right) X^i . \tag{31}$$

Using (5), we find $\nabla_{\mathcal{V}\Gamma}\nabla_\Gamma s$ in the form corresponding to (23). □

4. Remarks. The groupoid approach to connections was invented by C. Ehresmann for Lie groupoids, which correspond to the classical principal fiber bundles, [6]. Every principal fiber bundle $\pi : P \to M$ with structure group G determines the associated groupoid PP^{-1} which can be defined as the factor space $P \times P/\sim$ with respect to the equivalence relation $(u, v) \sim (ug, vg)$, $u, v \in P$, $g \in G$. Writing uv^{-1} for such an equivalence class, we have two projections $a, b : PP^{-1} \to M$, $a(uv)^{-1} = \pi v$, $b(uv^{-1}) = \pi u$. The formula

$$(uv^{-1})\,(vw^{-1}) = uw^{-1}$$

defines a partial composition law in PP^{-1} and $e_x = uu^{-1}$ is its unit for every $x = \pi u \in M$. By definition, a Lie groupoid Φ over M is isomorphic to PP^{-1} for a principal bundle $P \to M$. If E is a fiber bundle associated with P with standard fiber S, every $v \in P_x$ determines the "frame map" $q_v : S \to E_x$, [13]. Then $q_u \circ q_v^{-1} : E_x \to E_y$, $u \in P_y$, depends on uv^{-1} only. This defines a map $PP^{-1} \to \mathcal{G}E$, which is called the action of PP^{-1} on E.

An element of connection on a Lie groupoid Φ at $x \in M$ is 1-jet of a local map $\sigma : U \to \Phi$ of a neighbourhood U of $x \in M$ satisfying $a\sigma(u) = x$, $b\sigma(u) = u$, $s(x) = e_x$, [6]. The space of all elements of connection on Φ is a fibered manifold $Q\Phi \to M$. A connection on Φ is a section $\Gamma : M \to Q\Phi$. If Φ acts on a fibered manifold $E \to M$, Γ induces a connection $\Gamma_E : E \to J^1E$, $\Gamma_E(y) = j^1_x\sigma(u)(y)$, provided $\Gamma(x) = j^1_x\sigma(u)$. In particular, $\Phi = PP^{-1}$ acts canonically on P and the connection Γ_P is principal. One verifies easily that the rule $\Gamma \to \Gamma_P$ establishes a bijection between connections on PP^{-1} and principal connections on P. For Φ acting on E, the absolute differentiation of sections of E with respect to a connection on Φ was introduced by Ehresmann, [6]. The principal bundle form of this operation was studied in [10]. Section 2 of the present paper represents a generalization of these ideas to the infinite dimensional groupoid $\mathcal{G}Y$.

We have already remarked in Section 1 that the first approach to the iterated absolute differentiation is related with the classical ideas of tensor calculus. On the other hand, the second approach is of different geometric character and its interesting applications can be found, e.g. in the theory of submanifolds of a space with Cartan connection, [9]. The connection in question determines the geometry of every submanifold N and the use of the contact elements generated by N, [13] (which are called jets of the submanifold N by some authors), eliminates any role of a linear connection on N. For example, the higher order torsions of N can be introduced in the framework of the second approach, [9].

5. Maps to the functional bundle. Consider two locally trivial fibered manifolds $p_1 : Y_1 \to M$, $p_2 : Y_2 \to M$ and the functional bundle (1). Write $p : \mathcal{F}(Y_1, Y_2) \to M$ for the canonical projection. The set $\mathcal{F}(Y_1, Y_2)$ is a smooth space in the sense of Frölicher, [2]. A connection Γ on $\mathcal{F}(Y_1, Y_2)$ is a smooth section $\Gamma : \mathcal{F}(Y_1, Y_1) \to J^1\mathcal{F}(Y_1, Y_2)$, [2]. For every smooth map $f : N \to \mathcal{F}(Y_1, Y_2)$, we can construct the tangent map $Tf : TN \to T\mathcal{F}(Y_1, Y_2)$. Using the vertical projection $v_\Gamma : T\mathcal{F}(Y_1, Y_2) \to V\mathcal{F}(Y_1, Y_2)$ of Γ, we define the absolute differential

$$\nabla_\Gamma f = v_\Gamma \circ Tf : TN \to V\mathcal{F}(Y_1, Y_2)\,. \tag{32}$$

By linearity, (32) can be considered as a map $N \to V\mathcal{F}(Y_1, Y_2) \otimes T^*N$.

We remark that the construction of $\nabla_\Gamma f$ can be reduced to the absolute differentiation of a section of an induced bundle with respect to the induced connection analogously to the classical case of fibered manifolds. In general, if $g : N \to M$ is a map, we construct the induced bundles g^*Y_i, $i = 1, 2$,

$$g^*Y_i = \{(u, y_i) \in N \times Y_i,\ g(u) = p_i(y_i)\}$$

and define $g^*\mathcal{F}(Y_1, Y_2) = \mathcal{F}(g^*Y_1, g^*Y_2)$, which is a smooth space over N. If $s : M \to \mathcal{F}(Y_1, Y_2)$ is a smooth section, the formula $(g^*s)(u) = s(g(u))$ defines the induced section $g^*s : N \to g^*\mathcal{F}(Y_1, Y_2)$. The rule $j^1_x s \mapsto j^1_u g^* s$ defines a map $J^1_x\mathcal{F}(Y_1, Y_2) \to J^1_u(g^*\mathcal{F}(Y_1, Y_2))$, $x = g(u)$. In this way, every connection $\Gamma : \mathcal{F}(Y_1, Y_2) \to J^1\mathcal{F}(Y_1, Y_2)$ induces a connection $g^*\Gamma : g^*\mathcal{F}(Y_1, Y_2) \to J^1 g^*\mathcal{F}(Y_1, Y_2)$. Every smooth map $f : N \to \mathcal{F}(Y_1, Y_2)$ with $g = p \circ f$ defines a section $g^* f : N \to g^*\mathcal{F}(Y_1, Y_2)$. Then we have an identification

$$\nabla_\Gamma f \approx \nabla_{g^*\Gamma} g^* f . \tag{33}$$

Since $\Gamma : \mathcal{F}(Y_1, Y_2) \to J^1\mathcal{F}(Y_1, Y_2)$ is a kind of differential operator, one can characterize an r-th order connection, $r \geq 1$, [2]. We recall that every $X \in J^1_x\mathcal{F}(Y_1, Y_2)_\varphi$ is identified with an affine bundle morphism $\widetilde{X} : J^1_x Y_1 \to J^1_x Y_2$ over $\varphi : Y_{1x} \to Y_{2x}$, whose derived linear morphism is $T\psi \otimes \mathrm{id}_{T^*M}$. We say that Γ is of order r, if the condition $j^r_y\varphi = j^r_y\psi$, $\varphi, \psi \in C^\infty(Y_{1x}, Y_{2x})$, $y \in Y_{1x}$ implies

$$\widetilde{\Gamma}(\varphi)|(J^1Y_1)_y = \widetilde{\Gamma}(\psi)|(J^1Y_1)_y , \tag{34}$$

i.e. the restriction of the associated maps $\widetilde{\Gamma}(\varphi)$, $\widetilde{\Gamma}(\psi) : J^1_x Y_1 \to J^1_x Y_2$ to the fiber $(J^1Y_1)_y$ over y coincide.

Write $\mathcal{F}\mathcal{J}^r(Y_1, Y_2) = \bigcup_{x\in M} J^r(Y_{1x}, Y_{2x})$, which is a finite dimensional manifold. If x^i, y^p or x^i, z^a are some local fiber coordinates on Y_1 or Y_2, respectively, then the induced coordinates on $\mathcal{F}\mathcal{J}^r(Y_1, Y_2)$ are x^i, y^p, z^a_α, where α is a multiindex of the range equal to the range of y^p with $0 \leq |\alpha| \leq r$. Let $S(J^1Y_1, J^1Y_2)$ be the space of all affine maps $(J^1Y_1)_y \to (J^1Y_2)_z$ with the derived linear map of the form $B \otimes \mathrm{id}_{T^*_x M}$, $B \in V_z Y_2 \otimes V^*_y Y_1$. An r-th order connection Γ determines the associated map $\mathcal{G} : \mathcal{F}\mathcal{J}^r(Y_1, Y_2) \to S(J^1Y_1, J^1Y_2)$ by (34). Its coordinate form is

$$z^a_i = z^a_p y^p_i + \Phi^a_i(x^i, y^p, z^a_\alpha) , \quad 0 \leq |\alpha| \leq r . \tag{35}$$

We say that Φ^a_i is the coordinate expression of Γ. Analogously to [2], if $x^i = f^i(u)$, $z^a = f^a(u, y)$ is the coordinate form of $f : N \to \mathcal{F}(Y_1, Y_2)$, then the coordinate expression of $\nabla_\Gamma f$ is

$$\left(\frac{\partial f^a(u,y)}{\partial u^s} - \Phi^a_i(x^i(u), y^p, \partial_\alpha f^a(u,y))\frac{\partial f^i}{\partial u^s}\right) U^s . \tag{36}$$

6. The second order procedure. In the remaining two sections we assume Γ is a finite order connection. Its vertical prolongation $\mathcal{V}\Gamma : V\mathcal{F}(Y_1, Y_2) \to J^1 V\mathcal{F}(Y_1, Y_2)$ is a semilinear connection, [4]. Thus, for every map $F : N \to \mathcal{F}(Y_1, Y_2)$ we construct the iterated absolute differential

$$\nabla_{\mathcal{V}\Gamma}(\nabla_\Gamma f) : TTN \to VV\mathcal{F}(Y_1, Y_2) . \tag{37}$$

We are going to deduce that the value of (37) at each $u \in N$ corresponds to a semiholonomic 2-jet of N into $C^\infty(Y_{1x}, Y_{2x})$, $x = p(f(u))$.

The non-holonomic and semiholonomic 2-jets of N into any functional bundle $\mathcal{F}(Y_1, Y_2)$ ($C^\infty(Y_{1x}, Y_{2x})$ is the case of one-point base) can be introduced as a special case of the iterated 2-jets studied in [4]. In particular, for the product bundles $Y_1 = M \times Q_1$, $Y_2 = M \times Q_2$, we have $\mathcal{F}(Y_1, Y_2) = M \times C^\infty(Q_1, Q_2)$ and Section 6 of [4] gives the following identifications

$$\widetilde{J}^2(N, M \times C^\infty(Q_1, Q_2)) = \widetilde{J}^2(N, M) \times_N C^\infty_\alpha(Q_1, \widetilde{J}^2(N, Q_2)), \tag{38}$$

$$\bar{J}^2(N, M \times C^\infty(Q_1, Q_2)) = \bar{J}^2(N, M) \times_N C^\infty_\alpha(Q_1, \bar{J}^2(N, Q_2)), \tag{39}$$

where the subscript α indicates that we consider the maps into the fibers of the jet prolongation $\alpha : \widetilde{J}^2(N, Q) \to N$ or $\alpha : \bar{J}^2(N, Q) \to N$. On the other hand, as a special case of Proposition 1 of [12], we obtain another trivialization formula

$$TT(M \times C^\infty(Q_1, Q_2)) = TTM \times C^\infty(Q_1, TTQ_2). \tag{40}$$

Every element $X \in \widetilde{J}^2_u(N, \mathcal{F}(Y_1, Y_2))_\psi$ is of the form $X = j^1_u\sigma(v)$. Each $\sigma(v) \in J^1_v(N, \mathcal{F}(Y_1, Y_2))$ is identified with a linear map $\mu\sigma(v) : T_vN \to T\mathcal{F}(Y_1, Y_2)$, so that X defines a map

$$\mu X : TT_uN \to TT_\psi\mathcal{F}(Y_1, Y_2), \quad \mu X = T_u\mu(\sigma(v)). \tag{41}$$

Proposition 4. *For every $u \in N$, there exists a unique element*

$$\nabla^2_\Gamma f(u) \in \bar{J}^2_u(N, C^\infty(Y_{1x}, Y_{2x}))_{f(u)},$$

$x = p(f(u))$, *satisfying*

$$\nabla_{\mathcal{V}\Gamma}\nabla_\Gamma f(u) = \mu(\nabla^2_\Gamma f(u)). \tag{42}$$

Proof. In the same way as in the proof of Proposition 2 we deduce that $\nabla_{\mathcal{V}\Gamma}\nabla_\Gamma f$ satisfies the functional modification of Lemma 1 with the semiholonomicity condition. Using the trivializations (39) and (40), we can apply Lemma 1 pointwise. □

7. Relations to the curvature. For a finite order connection Γ with the coordinate expression Φ^a_i from (35), the additional coordinate expression of $\mathcal{V}\Gamma$ is

$$\frac{\partial\Psi^a_i}{\partial z^b}Z^b + \cdots + \frac{\partial\Phi^a_i}{\partial z^a_\alpha}Z^b_\alpha \tag{43}$$

with $Z^b_\alpha = dz^b_\alpha$, [4]. If $z^a = f^a(x, y)$ is the coordinate expression of a section $s : M \to \mathcal{F}(Y_1, Y_2)$, then we obtain the coordinate form $\nabla_\Gamma s$ as a special case of (36)

$$\left(\frac{\partial f^a(x, y)}{\partial x^i} - \Phi^a_i(x^i, y^p, \partial_\alpha f^a(x, y))\right)X^i =: f^a_i X^i. \tag{44}$$

Hence the coordinate form of the "second order term" in $\nabla_{\mathcal{V}\Gamma}\nabla_\Gamma f$ is

$$\left(\frac{\partial}{\partial x^j}(f^a_i) - \frac{\partial\Phi^a_j}{\partial z^b}f^p_i - \cdots - \frac{\partial\Phi^a_j}{\partial z^b_\alpha}\partial_\alpha(f^p_i)\right)X^i \otimes dx^j \tag{45}$$

Analogously to the formula (36) of [4], we find that the alternation in i and j of (45) is $-(C\Gamma)(s(x))$, where $C\Gamma$ is the curvature of Γ, [2].

We recall that every semiholonomic 2-jet $X \in \bar{J}^2_u(N, \mathcal{F}(Y_1, Y_2))_\psi$ determines the deviation $\Delta X \in T_\psi\mathcal{F}(Y_1, Y_2) \otimes \Lambda^2 T^*_u N$, whose coordinate expression is just the alternation of the "second order" component of X, [2]. Hence we have proved

Proposition 5. *For every finite order connection Γ on $\mathcal{F}(Y_1, Y_2)$ and every section $s : M \to \mathcal{F}(Y_1, Y_2)$, we have*

$$\Delta\left(\nabla^2_\Gamma s(x)\right) = -C\Gamma(s(x)), \qquad x \in M. \tag{46}$$

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