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IMPOSSIBLE EINSTEIN-WEYL GEOMETRIES

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There are several ways of viewing the Einstein-Weyl equations. The one adopted in [1] is to view them as a system of partial differential equations for a smooth one-form α defined on a Riemannian manifold. The Rho tensor is a trace adjusted multiple of the Ricci tensor. In three dimensions it is equivalent to the Riemannian curvature and may be characterised in terms of the metric connection ∇ by

$$\begin{aligned}&\langle W,(\nabla_X\nabla_Y-\nabla_Y\nabla_X)Z\rangle=\\&\qquad \mathrm{P}(W,X)\langle Y,Z\rangle-\mathrm{P}(W,Y)\langle X,Z\rangle+\mathrm{P}(Y,Z)\langle W,X\rangle-\mathrm{P}(X,Z)\langle W,Y\rangle\end{aligned}$$

for arbitrary smooth vector fields W,X,Y,Z. The Einstein-Weyl equations say that the trace-free symmetric part of

$$\nabla\alpha\;+\;\alpha\otimes\alpha\;+\;\mathrm{P}$$

vanishes. Given a Riemannian metric, we may ask whether there are any solutions of these equations. As explained in [1], this question is most difficult in three dimensions. Joint work with Paul Tod [2] finds all local solutions of the Einstein-Weyl equations in three dimensions where the background metric is homogeneous with unimodular isometry group. In particular, we show that there are no solutions with Nil or Sol as background. The aim of this article is to present these two special cases. The complications of [2] necessitated by a comprehensive analysis are quite severe. These complications are much less severe for Nil and Sol. In particular, the computer algebra used extensively in [2] is no longer essential. Of course, this is joint work with Paul Tod. In fact, these two cases were completed as a hitherto unpublished prelude to [2]. For the sake of brevity, the motivation behind various convenient choices will be suppressed. They can be found in [2].

1. Nil

This metric may be written in local coördinates:

$$dx^2+dy^2+(dz-2xdy)^2.$$

The vector fields

$$e_1=\partial/\partial x\qquad e_2=\partial/\partial y+2x\partial/\partial z\qquad e_3=\partial/\partial z$$

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are orthonormal and commute according to

$$[e_1, e_2] = 2e_3 \qquad [e_1, e_3] = 0 \qquad [e_2, e_3] = 0\,.$$

With respect to this orthonormal frame, let the smooth one-form α have components X, Y, Z. The Einstein-Weyl equations become

$$\begin{array}{l} e_1X + X^2 - 2 \;=\; e_2Y + Y^2 - 2 \;=\; e_3Z + Z^2 + 2 \\ e_2X + e_1Y + 2XY = 0 \qquad e_3X + e_1Z + 2Y + 2XZ = 0 \\ e_3Y + e_2Z - 2X + 2YZ = 0\,. \end{array} \tag{1}$$

We can introduce auxiliary variables P, Q, R, Λ to rewrite this system as the following set of equations

$$\begin{array}{lll} e_1X = -X^2 + 2 + \Lambda & e_2X = -Z - R - XY & e_3X = -Y + Q - XZ \\ e_1Y = Z + R - XY & e_2Y = -Y^2 + 2 + \Lambda & e_3Y = X - P - YZ \\ e_1Z = -Y - Q - XZ & e_2Z = X + P - YZ & e_3Z = -Z^2 - 2 + \Lambda\,. \end{array}$$

Also, introduce smooth functions S, T, U by

$$e_1\Lambda = S - \Lambda X + X \qquad e_2\Lambda = T - \Lambda Y + Y \qquad e_3\Lambda = U - \Lambda Z - 3Z\,.$$

Then it turns out that all partial derivatives of all ten variables may be determined by differentiating these twelve equations. For example,

$$\begin{array}{rcl} e_1e_2X - e_2e_1X &=& e_1(-Z - R - XY) - e_2(-X^2 + 2 + \Lambda) \\ &=& -e_1Z - e_1R - Xe_1Y - Ye_1X + 2Xe_2X - e_2\Lambda \end{array}$$

and substituting for e_1Z, e_1Y, e_1X, e_2X, $e_2\Lambda$ and simplifying gives

$$e_1e_2X - e_2e_1X = -2Y + Q - 2XZ - e_1R - 3XR - T\,.$$

Alternatively,

$$e_1e_2X - e_2e_1X = [e_1, e_2]X = 2e_3X = -2Y + 2Q - 2XZ$$

so we conclude that

$$e_1R = -Q - 3RX - T\,.$$

The complete system has thirty equations of which the following are typical

$$\begin{array}{rcl} e_1X &=& -X^2 + 2 + \Lambda \\ &\vdots& \\ e_3\Lambda &=& U - \Lambda Z - 3Z \\ e_1P &=& -4 - 2PX + QY + RZ \\ &\vdots& \\ e_1R &=& -Q - 3RX - T \\ &\vdots& \\ e_1S &=& 12 - 3P^2 + 3Q^2 + 3R^2 - 3XS \\ e_2S &=& -U - 6PQ + R - \Lambda R + 4Z - 2YS - XT \\ &\vdots& \\ e_3U &=& -24 + 3P^2 + 3Q^2 - 3R^2 - 3ZU\,. \end{array}$$

Further differentiation now yields polynomial constraints on these variables which eventually turn out to be inconsistent. To see this, it is convenient, as a bookkeeping device, to consider the following complex combinations

$$\xi \equiv X + iY \qquad \pi \equiv P + iQ \qquad \sigma \equiv S + iT \tag{2}$$

and the differential operator

$$\delta \equiv e_1 + ie_2\,. \tag{3}$$

Then, for example, the equations

$$e_1X - e_2Y = -X^2 + Y^2 \qquad \text{and} \qquad e_1Y + e_2X = -2XY$$

may be combined as the real and imaginary parts of the single complex equation $\delta\xi = -\xi^2$. Together with Z and R, we obtain from the complete Einstein-Weyl system, the following closed sub-system:

$$\begin{aligned}
\delta\xi &= -\xi^2 \\
\delta\pi &= -3\xi\pi \\
\delta\sigma &= -3\xi\sigma - 6\pi^2 \\
\delta Z &= -\xi Z + i\xi + i\pi \\
\delta R &= -3\xi R + i\pi + i\sigma\,.
\end{aligned}$$

Since e_1 and e_2 both commute with e_3, the following expression

$$\Omega = \frac{e_2e_3S - e_3e_2S + e_1e_3T - e_3e_1T + 2ie_2e_3T - 2ie_3e_2T}{24}$$

should vanish. The right hand side may be computed from the Einstein-Weyl system. Its vanishing is a polynomial constraint on the ten variables. In fact, it may be written as a complex polynomial in ξ, Z, π, R, σ. A computation gives

$$\Omega = -\xi^2 - \pi\sigma + i\pi^2 Z - i\xi\pi R\,.$$

This expression may be further differentiated by using the sub-system. This easily leads to further constraints. In particular, it is not hard to check that

$$\delta^2\Omega + 16\xi\delta\Omega + 56\xi^2\Omega = -30\xi^4$$

and so for any local solution of the Einstein-Weyl equations, ξ must vanish. In other words X and Y must vanish separately. (Alternatively, by running through the same argument with starting $\bar{\xi} \equiv X - iY$ and $\bar{\delta} \equiv e_1 - ie_2$ etcetera, we may conclude that X and Y must vanish separately even if we allow complex-valued solutions.) Substituting back into (1) gives

$$e_1Z = 0 \qquad e_2Z = 0 \qquad e_3Z + Z^2 + 4 = 0\,.$$

Since $[e_1, e_2] = 2e_3$, the first two equations imply that $e_3Z = 0$. Thus, Z is constant and the final equation reads $Z^2 + 4 = 0$ which has has no real solutions. The complex solutions $Z = \pm 2i$ correspond to the real solutions noted in [3] on the Lorentzian version of NIL.

2. Sol

This metric may be written in local coördinates:

$$dx^2 + e^{2x}dy^2 + e^{-2x}dz^2 .$$

The vector fields

$$e_1 = \partial/\partial x \qquad e_2 = e^{-x}\partial/\partial y \qquad e_3 = e^{x}\partial/\partial z$$

are orthonormal and commute according to

$$[e_1, e_2] = -e_2 \qquad [e_1, e_3] = e_3 \qquad [e_2, e_3] = 0 .$$

With respect to this orthonormal frame, let the smooth one-form α have components X, Y, Z. The Einstein-Weyl equations become

$$\begin{array}{l} e_1X + X^2 - 2 \;=\; e_2Y + X + Y^2 \;=\; e_3Z - X + Z^2 \\ e_2X + e_1Y - Y + 2XY = 0 \qquad e_3X + e_1Z + Z + 2XZ = 0 \\ e_3Y + e_2Z + 2YZ = 0 . \end{array} \tag{4}$$

We can introduce auxiliary variables P, Q, R, Λ to rewrite this system as the following set of equations

$$\begin{array}{lll} e_1X = -X^2 + 2 + \Lambda & e_2X = Y - R - XY & e_3X = -Z + Q - XZ \\ e_1Y = R - XY & e_2Y = -X - Y^2 + \Lambda & e_3Y = -P - YZ \\ e_1Z = -Q - XZ & e_2Z = P - YZ & e_3Z = X - Z^2 + \Lambda . \end{array} \tag{5}$$

Also, introduce smooth functions S, T, U by

$$e_1\Lambda = S - \Lambda X + X \qquad e_2\Lambda = T - \Lambda Y - Y \qquad e_3\Lambda = U - \Lambda Z - Z .$$

As for Nil, it turns out that all partial derivatives of all ten variables may be determined by differentiating these twelve equations. The same complex combinations (2) and differential operator (3) together with Z and R gives rise to a slightly more complicated closed sub-system:

$$\begin{array}{rcl} \delta\xi &=& -\xi^2 + \xi + 2 \\ \delta\pi &=& -3\xi\pi + \pi \\ \delta\sigma &=& -3\xi\sigma - 6\pi^2 + 6 - 6\xi + \sigma - 2iR \\ \delta Z &=& -\xi Z + i\pi \\ \delta R &=& -3\xi R + i\sigma - 2i . \end{array}$$

As for Nil there is a constraint expressible in the same variables:

$$\begin{array}{rcl} \Omega &=& \dfrac{e_2e_3S - e_3e_2S + e_1e_3T - e_3e_1T - e_3T + 2ie_2e_3T - 2ie_3e_2T}{24} \\ &=& -\pi\sigma + i\pi^2 Z - i\xi\pi R - \pi/2 - i\xi Z/2 + 7iZ/6 . \end{array}$$

It turns out that

$$\Psi \equiv \delta\Omega + 7\xi\Omega - 2\Omega$$

is independent of σ and R. It is linear in Z and cubic in π. Another differentiation reduces it to being linear in both variables:

$$\delta\Psi + 9\xi\Psi - 3 = p(\xi)Z + q(\xi)\pi$$

for certain polynomials $p(\xi)$ and $q(\xi)$. It is clear from the sub-system that this form is preserved by further differentiation:

$$\delta(\delta\Psi + 9\xi\Psi - 3) = r(\xi)Z + s(\xi)\pi .$$

It follows that either Z and π are identically zero or the determinant

$$p(\xi)s(\xi) - q(\xi)r(\xi) = 100(63\xi^4 - 372\xi^3 + 708\xi^2 - 384\xi - 31)i/9 \tag{6}$$

is zero. Suppose that Z vanishes. Then (5) reduces to

$$\begin{array}{ll} e_1X = -X - X^2 + 2 & e_2X = Y - R - XY \\ e_1Y = R - XY & e_2Y = -2X - Y^2 + \Lambda \end{array} \tag{7}$$

Recall that $[e_1, e_2] = -e_2$. We can solve

$$0 = e_1e_2X - e_2e_1X + e_2X \qquad \text{and} \qquad 0 = e_1e_2Y - e_2e_1Y + e_2Y$$

to complete the system with

$$e_1R = -R - 3RX \qquad \text{and} \qquad e_2R = -4 - 3RY .$$

Now we obtain an algebraic constraint

$$0 = e_1e_2R - e_2e_1R + e_2R = -8 - 12X - 6R^2$$

from which R may be eliminated by a further differentiation:

$$0 = e_1(8 + 12X + 6R^2) + 2(3X + 1)(8 + 12X + 6R^2) = 60X^2 + 60X + 40 . \tag{8}$$

It follows that X is constant. However, the first equation of (7) forces this constant to be -2 or 1 both of which are incompatible with (8). It follows that (6) vanishes and, in particular, that ξ is constant. Even if we allow complex solutions, by running through the same argument with starting $\bar{\xi} \equiv X - iY$ and $\bar{\delta} \equiv e_1 - ie_2$ etcetera, we may conclude that both X and Y are constant. Substituting back into (4) gives

$$\begin{array}{lll} 0 = (X + 1)(X - 2) - Y^2 & 0 = (2X - 1)Y & \\ e_1Z = -(2X + 1)Z & e_2Z = -2YZ & e_3Z = (X + 2)(X - 1) - Z^2 . \end{array} \tag{9}$$

Recall that $[e_1, e_3] = e_3$. Therefore,

$$0 = e_1e_3Z - e_3e_1Z - e_3Z = 2((X + 1)Z^2 + X(X + 2)(X - 1))$$

and Z is constant. Now (9) reduces to a set of polynomial equations

$$\begin{array}{lll} Y^2 = (X + 1)(X - 2) & (2X - 1)Y = 0 & \\ (2X + 1)Z = 0 & YZ = 0 & Z^2 = (X + 2)(X - 1) \end{array}$$

which are easily seen to be inconsistent.

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