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A COMPLEX FROM LINEAR ELASTICITY

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INTRODUCTION

This article will present just one example of a general construction known as the Bernstein-Gelfand-Gelfand (BGG) resolution. It was the motivating example from two lectures on the BGG resolution given at the 19^{th} Czech Winter School on Geometry and Physics held in Srní in January 1999. This article may be seen as a technical example to go with a more elementary introduction which will appear elsewhere [7]. In fact, there were many lectures on various aspects of the BGG resolution given by other participants in the Winter School. In particular, Čap, Slovák, and Souček presented their recently worked out curved analogue [5] for a general parabolic geometry.

I would like to thank Douglas Arnold for telling me about the linear elasticity complex and asking about its link to the de Rham complex.

THE DE RHAM COMPLEX

Apart from the final section, we shall work in three dimensions. On $\mathbb{R}^3$ we have the familiar differential operators of grad, curl, and div. In line with standard differential geometric conventions, let us take (x^1, x^2, x^3) as coördinates and write $\partial/\partial x^i = \nabla_i$ for $i = 1, 2, 3$. Then

$$u \stackrel{\text{grad}}{\longmapsto} \begin{bmatrix} \nabla_1 u \\ \nabla_2 u \\ \nabla_3 u \end{bmatrix} \qquad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \stackrel{\text{curl}}{\longmapsto} \begin{bmatrix} \nabla_2 v_3 - \nabla_3 v_2 \\ \nabla_3 v_1 - \nabla_1 v_3 \\ \nabla_1 v_2 - \nabla_2 v_1 \end{bmatrix} \qquad \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix} \stackrel{\text{div}}{\longmapsto} \nabla_1 w^1 + \nabla_2 w^2 + \nabla_3 w^3 .$$

It is convenient to adopt Einstein's summation convention—an implied sum over repeated indices. For example, $\operatorname{div} w^i = \nabla_i w^i$. Without being too precise, let us write $\mathcal{E}$ for the smooth functions of (x^1, x^2, x^3) and $\mathcal{E}_i$ or $\mathcal{E}^i$ for triples of such functions. Also introduce the alternating symbol ϵ^{ijk} with $\epsilon^{123} = 1$. Then, as is demonstrated in any

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reputable text on vector analysis, we have the resolution:

$$
\begin{array}{ccccccccccccc}
 & & & & u & \longmapsto & \nabla_i u & & w^i & \longmapsto & \nabla_i w^i & & \\
 & & & & \Cap & & \Cap & & \Cap & & \Cap & & \\
0 & \to & \mathbb{R} & \to & \mathcal{E} & \xrightarrow{\text{grad}} & \mathcal{E}_i & \xrightarrow{\text{curl}} & \mathcal{E}^i & \xrightarrow{\text{div}} & \mathcal{E} & \to & 0\,. \\
 & & & & & & \Cup & & \Cup & & & & \\
 & & & & & & v_i & \longmapsto & \epsilon^{ijk}\nabla_j v_k & & & &
\end{array}
\tag{1}
$$

This means that, in a suitably local sense, this is an exact sequence—each operator provides the integrability conditions for the one which precedes it. Certainly, if we consider functions defined on a contractible open subset of $\mathbb{R}^3$, then the resulting sequence is exact. One of the first tasks in any development of analysis on manifolds it to set up (1) on an arbitrary smooth manifold where it becomes the de Rham sequence.

The linear elasticity complex

As further notation, let us use round brackets to denote symmetrization. Thus, $t_{(ij)} = (t_{ij} + t_{ji})/2$ is the symmetric part of a 2-tensor t_{ij}. Further, let us write $\mathcal{E}_{(ij)}$ for the smooth symmetric 2-tensors. Here is another resolution in three dimensions.

$$
\begin{array}{ccccccccccccc}
 & & & & u_i & \longmapsto & \nabla_{(i}u_{j)} & & w^{ij} & \longmapsto & \nabla_j w^{ij} & & \\
 & & & & \Cap & & \Cap & & \Cap & & \Cap & & \\
0 & \to & \mathbb{T} & \to & \mathcal{E}_i & \longrightarrow & \mathcal{E}_{(ij)} & \longrightarrow & \mathcal{E}^{(ij)} & \longrightarrow & \mathcal{E}^i & \to & 0\,. \\
 & & & & & & \Cup & & \Cup & & & & \\
 & & & & & & v_{ij} & \longmapsto & \epsilon^{ikm}\epsilon^{jln}\nabla_k\nabla_l v_{mn} & & & &
\end{array}
\tag{2}
$$

It comes from the linearized theory of elasticity where u_i is the displacement, v_{ij} is the strain, w^{ij} is the stress, and $t^i = \nabla_j w^{ij}$ is the load obtained from a given stress. Though one of the differential operators of (2) is second order, there is clearly a strong resemblance to (1). The kernel $\mathbb{T}$ of the first operator has dimension 6:

$$\mathbb{T} = \{u_i \text{ s.t. } \nabla_{(i}u_{j)} = 0\} = \{u_i = a_i + \epsilon_{ijk}b^j x^k\} = \left\{\begin{bmatrix} a_i \\ b_j \end{bmatrix} \text{ for constants } a_i \text{ and } b_j\right\}.$$

From de Rham to linear elasticity

The form of this kernel provides a clue to the precise link between (1) and (2). We can also resolve $\mathbb{T}$ by the $\mathbb{T}$-valued de Rham sequence. It is the middle row of the following diagram.

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{E}^q & & \mathcal{E}_i{}^q & & \mathcal{E}^{iq} & & \mathcal{E}^q & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{T} & \to & \mathcal{E}(\mathbb{T}) & \to & \mathcal{E}_i(\mathbb{T}) & \to & \mathcal{E}^i(\mathbb{T}) & \to & \mathcal{E}(\mathbb{T}) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{E}_p & & \mathcal{E}_{ip} & & \mathcal{E}^i{}_p & & \mathcal{E}_p & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & & 0 & &
\end{array}
\tag{3}
$$

in which the vertical mappings are

$$\begin{array}{cccc} \phi^q & \psi_i{}^q & w^{iq} & t^q \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \begin{bmatrix} u_p \\ \phi^q \end{bmatrix} \begin{bmatrix} \epsilon_{pqr}x^q\phi^r \\ \phi^q \end{bmatrix} & \begin{bmatrix} v_{ip} \\ \psi_i{}^q \end{bmatrix} \begin{bmatrix} \epsilon_{pqr}x^q\psi_i{}^r \\ \psi_i{}^q \end{bmatrix} & \begin{bmatrix} \xi^i{}_p \\ w^{iq} \end{bmatrix} \begin{bmatrix} \epsilon_{pqr}x^q w^{ir} \\ w^{iq} \end{bmatrix} & \begin{bmatrix} \eta_p \\ t^q \end{bmatrix} \begin{bmatrix} \epsilon_{pqr}x^q t^r \\ t^q \end{bmatrix} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ u_p - \epsilon_{pqr}x^q\phi^r & v_{ip} - \epsilon_{pqr}x^q\psi_i{}^r & \xi^i{}_p - \epsilon_{pqr}x^q w^{ir} & \eta_p - \epsilon_{pqr}x^q t^r \end{array}.$$

Notice that the columns of (3) are exact. Also in (3) there are three compositions of the form

$$\begin{array}{ccc} * & & \\ \downarrow & & \\ * & \to & * \\ & & \downarrow \\ & & * \end{array}, \quad \text{for example} \quad \begin{array}{ccc} \phi^q & & \\ \downarrow & & \\ \begin{bmatrix} \epsilon_{pqr}x^q\phi^r \\ \phi^q \end{bmatrix} & \stackrel{\nabla_i}{\longmapsto} & \begin{bmatrix} \epsilon_{pir}\phi^r + \epsilon_{pqr}x^q\nabla_i\phi^r \\ \nabla_i\phi^r \end{bmatrix} \\ & & \downarrow \\ & & \epsilon_{pir}\phi^r \end{array}.$$

All three are simply algebraic (involve no differentiation):

$$\phi^q \mapsto -\epsilon_{ipr}\phi^r \qquad \psi_i{}^q \mapsto \delta^i{}_p\psi_r{}^r - \psi_p{}^i \qquad w^{iq} \mapsto \epsilon_{pir}w^{ir}$$

where $\delta^i{}_p$ is the Kronecker delta. The middle one is an isomorphism $\mathcal{E}_i{}^q \xrightarrow{\cong} \mathcal{E}^i{}_p$. For the others, if we write $t_{[ij]} = (t_{ij} - t_{ji})/2$ for the skew part of a 2-tensor and $\mathcal{E}_{[ij]}$ for the skew 2-tensors, then

$$\mathcal{E}^q \xrightarrow{\cong} \mathcal{E}_{[ip]} \to \mathcal{E}_{[ip]} \oplus \mathcal{E}_{(ip)} = \mathcal{E}_{ip} \quad \text{and} \quad \mathcal{E}^{iq} = \mathcal{E}^{(iq)} \oplus \mathcal{E}^{[iq]} \to \mathcal{E}^{[iq]} \xrightarrow{\cong} \mathcal{E}_p .$$

It is now simply a matter of diagram chasing to cancel these superfluous isomorphisms from (3) to obtain (2). For example, $u_p \in \mathcal{E}_p$ has a unique lift $U \in \mathcal{E}(\mathbb{T})$ so that $\nabla_i U \in \mathcal{E}_i(\mathbb{T})$ projects to a symmetric tensor in $\mathcal{E}_{ip}$. Specifically,

$$U = \begin{bmatrix} u_p + x^q\nabla_{[p}u_{q]} \\ \frac{1}{2}\epsilon^{jkq}\nabla_j u_k \end{bmatrix} \Rightarrow \nabla_i U = \begin{bmatrix} \nabla_i u_p + \nabla_{[p}u_{i]} + x^q\nabla_i\nabla_{[p}u_{q]} \\ \frac{1}{2}\epsilon^{jkq}\nabla_i\nabla_j u_k \end{bmatrix}$$

which projects to $\nabla_{(i}u_{p)}$ and, sure enough, $u_i \mapsto \nabla_{(i}u_{j)}$ is the first differential operator of (2). The exactness and other properties of (2) are simply inherited from the corresponding properties of the de Rham sequence (3), the only slightly cunning ingredient being the choice of mappings in the exact sequence

$$(4) \qquad \begin{array}{ccccccccc} & & \phi^q & \mapsto & \begin{bmatrix} \epsilon_{pqr}x^q\phi^r \\ \phi^q \end{bmatrix} & & & & \\ & & \cap & & \cap & & & & \\ 0 & \to & \mathcal{E}^q & \to & \mathcal{E}(\mathbb{T}) & \to & \mathcal{E}_p & \to & 0. \\ & & & & \cup & & \cup & & \\ & & & & \begin{bmatrix} u_p \\ \phi^q \end{bmatrix} & \mapsto & u_p - \epsilon_{pqr}x^q\phi^r & & \end{array}$$

For all intents and purposes then, (2) and (3) are equivalent. Roughly speaking, introducing the extra variables $\phi^q, \psi_i{}^q, \ldots$ in (3) is the standard manœuvre for writing a second order operator as a system of first order operators.

Consequences

The equivalence of (2) and (3) has consequences other than exactness of (2). Recall the argument for exactness. It is that (3) is manifestly exact as a vector-valued version of the de Rham complex. Then, after cancellation of certain extraneous terms, we obtain (2).

Roughly speaking then, anything which is true of the de Rham complex should have a counterpart for linear elasticity. There are at least two further possible instances of this. The first is Hodge theory. On a compact Riemannian manifold, the p-forms enjoy an orthogonal decomposition into three parts, the harmonic forms, the image of the exterior derivative, and the image of its dual. For vector fields on $\mathbb{R}^3$, this is often called the Helmholtz decomposition, splitting a general vector field into its potential and solenoidal parts. Corresponding decompositions in linear elasticity may be found in [8] and more generally in [9].

On a more speculative note, finite element approximation schemes for the de Rham sequence are well-understood [1]. For example, one such scheme comes down to simplicial approximation and is very much related to the isomorphism between de Rham cohomology and simplicial cohomology. It should be possible to track this through (3) to obtain finite element methods for linear elasticity.

Linear elasticity on $\mathbb{RP}_3$

Consider real projective 3-space:

$$\mathbb{RP}_3 = \{\text{1-dimensional linear subspaces } L \text{ of } \mathbb{R}^4\}\,.$$

There is a tautologically defined line bundle on $\mathbb{RP}_3$ which associates to the point $L \in \mathbb{RP}_3$ the linear space L itself. It is a sub-bundle of the trivial vector bundle whose fibre is $\mathbb{R}^4$ over every point. Let us denote the local smooth sections (more precisely the sheaf of germs of smooth sections) of the tautological line bundle by $\mathcal{E}(-1)$. Also, write $\mathcal{E}^i$ and $\mathcal{E}_i$ for the tangent and cotangent bundles respectively. Then, there is the Euler sequence

$$0 \to \mathcal{E}(-1) \to \mathcal{E}(\mathbb{R}^4) \to \mathcal{E}^i(-1) \to 0 \tag{5}$$

where $\mathcal{E}(\mathbb{R}^4)$ denotes the smooth $\mathbb{R}^4$-valued functions and $\mathcal{E}^i(-1)$ is the tensor product of $\mathcal{E}^i$ and $\mathcal{E}(-1)$. An immediate consequence is the exact sequence

$$0 \to \mathcal{E}^i(-2) \to \mathcal{E}(\Lambda^2\mathbb{R}^4) \to \mathcal{E}^{[ij]}(-2) \to 0\,. \tag{6}$$

Also, if we fix a volume form on $\mathbb{R}^4$, then $\mathcal{E}^{[ijk]}(-4)$ is trivialized by a canonical section which we shall denote by ϵ^{ijk}. There is a similar canonical section ϵ_{ijk} of $\mathcal{E}_{[ijk]}(4)$. This has two consequences. The first is that the exact sequence (6) may be rewritten

$$0 \to \mathcal{E}^q(-2) \to \mathcal{E}(\Lambda^2\mathbb{R}^4) \to \mathcal{E}_p(2) \to 0$$

and the second is that the de Rham sequence on $\mathbb{RP}_3$ may be written

$$0 \to \mathbb{R} \to \mathcal{E} \to \mathcal{E}_i \to \mathcal{E}^i(-4) \to \mathcal{E}(-4) \to 0\,.$$

Combining these two observations gives the diagram

$$\begin{array}{ccccccccccccc}
 & & & & & & & & 0 & & & & \\
 & & & & & & & & \downarrow & & & & \\
 & & & & 0 & & 0 & & \mathcal{E}_p(-2)\oplus\mathcal{E}^{(iq)}(-6) & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \| & & \downarrow & & \\
 & & & & \mathcal{E}^q(-2) & & \mathcal{E}_i{}^q(-2) & & \mathcal{E}^{iq}(-6) & & \mathcal{E}^q(-6) & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda\mathbb{R}^4 & \to & \mathcal{E}(\Lambda\mathbb{R}^4) & \to & \mathcal{E}_i(\Lambda\mathbb{R}^4) & \to & \mathcal{E}^i(\Lambda\mathbb{R}^4)(-4) & \to & \mathcal{E}(\Lambda\mathbb{R}^4)(-4) & \to & 0\,. \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{E}_p(-2) & & \mathcal{E}_{ip}{}^q(2) & & \mathcal{E}^i{}_p(-2) & & \mathcal{E}_p(-2) & & \\
 & & & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
 & & & & 0 & & \mathcal{E}^q(-2)\oplus\mathcal{E}_{(ip)}(2) & & 0 & & 0 & & \\
 & & & & & & \downarrow & & & & & & \\
 & & & & & & 0 & & & & & &
\end{array}$$

It is an elementary matter to check that this coincides with (3) when written out in standard local coördinates. Therefore, the same diagram chase as before gives the resolution

$$0 \to \Lambda^2\mathbb{R}^4 \to \mathcal{E}_i(2) \to \mathcal{E}_{(ij)}(2) \to \mathcal{E}^{(ij)}(-6) \to \mathcal{E}^i(-6) \to 0\,, \qquad .$$

extending to $\mathbb{RP}_3$ the linear elasticity sequence (2). Notice that on $\mathbb{RP}_3$, there are no choices to be made: (6) is simply derived from the Euler sequence. When restricted to $\mathbb{R}^3 \hookrightarrow \mathbb{RP}_3$, it becomes (4). What previously may have seemed a rather *ad hoc* choice of splitting becomes quite natural once we choose to compactify $\mathbb{R}^3$ as $\mathbb{RP}_3$.

Linear elasticity as BGG

To discuss BGG in any sort of generality much notation is required. The notation comes from representation theory (since BGG has a dual formulation constructed from Lie algebras). Consider $\mathbb{RP}_3$ as a homogeneous space: the group $\mathrm{SL}(4,\mathbb{R})$ of real 4×4 matrices of unit determinant acts on $\mathbb{RP}_3$ and if we take the first standard axis in $\mathbb{R}^4$ as basepoint, then the stabilizer subgroup is

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \text{ with unit determinant} \right\}.$$

The main thing we need is a notation for the irreducible homogeneous vector bundles on $\mathbb{RP}_3$. Homogeneous means that $\mathrm{SL}(4,\mathbb{R})$ acts compatible with the way it is already acting on $\mathbb{RP}_3$ and irreducible means that there is no non-trivial sub-bundle with the same property. The general such bundle is determined by an irreducible representation of P and Following [3], we may write them as $\mathcal{E}(\overset{p}{\times}\!\!-\!\!\overset{q}{\bullet}\!\!-\!\!\overset{r}{\bullet})$ where the parameter p can be any real number but q and r must be positive integers. In fact, this constitutes half of the irreducible homogeneous vector bundles on $\mathbb{RP}_3$. The other half differ by an overall twist. As is explained in [6], the notation comes from the Dynkin diagram of $\mathrm{SL}(4,\mathbb{R})$. The cross over the first node specifies the parabolic subgroup. The notation $\overset{p}{\times}\!\!-\!\!\overset{q}{\bullet}\!\!-\!\!\overset{r}{\bullet}$ determines a representation of P by means of its highest weight and hence a homogeneous bundle. Here are some examples:

$$\begin{aligned}
&\mathcal{E}(\overset{0}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}) = \mathcal{E} = \text{the trivial bundle}\\
&\mathcal{E}(\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet}) = \mathcal{E}_i = \text{the cotangent bundle (or bundle of 1-forms)}\\
&\mathcal{E}(\overset{-3}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}) = \mathcal{E}_{[ij]} = \text{the bundle of 2-forms}\\
&\mathcal{E}(\overset{1}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}) = \mathcal{E}^i = \text{the tangent bundle.}
\end{aligned}$$

Tensors with more complicated symmetries are best described by Young tableau. These tableau are the classical instrument for describing both the irreducible representations of $\mathrm{GL}(n,\mathbb{C})$ and how they may be realised inside $\otimes^k\mathbb{C}^n$ for suitable k. In some sense this is an alternative to classifying with highest weights but it is more concrete. In fact, not only do these tableau say how the representation occurs inside $\otimes^k\mathbb{C}^n$, but they actually choose a splitting. In this context they are often called Young projectors. Though misleading in their simplicity, the most elementary examples are $\square\!\square$ which takes the symmetric part of a 2-tensor and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ which takes the skew part. Generally,

$$\mathcal{E}(\overset{p}{\times}\!\!-\!\!\overset{q}{\bullet}\!\!-\!\!\overset{r}{\bullet}) = \Big[\underbrace{\begin{smallmatrix}\square\cdots\square\\\square\cdots\square\end{smallmatrix}}_{r}\underbrace{\square\cdots\square}_{q}\,\mathcal{E}_i\Big](p+2q+3r)$$

where the Young tableau, having $q+2r$ boxes, acts on $\mathcal{E}_i$ to give a bundle of covariant $(q+2r)$-tensors subject to the symmetries imposed by this tableau. The quantity $p+2q+3r$ in round brackets is the 'projective weight' [2] of the tensor.

Now we can write down BGG resolutions on $\mathbb{RP}_3$, the simplest being de Rham:

$$0 \to \mathbb{R} \to \mathcal{E}(\overset{0}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}) \to \mathcal{E}(\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet}) \to \mathcal{E}(\overset{-3}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}) \to \mathcal{E}(\overset{-4}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}) \to 0\,.$$

Of course, this sequence makes perfectly good sense on any 3-dimensional smooth manifold but on $\mathbb{RP}_3$ it has the addition feature having $\mathrm{SL}(4,\mathbb{R})$ acting on it and being invariant under this action. In particular, the space of constant functions, namely $\mathbb{R}$, should be regarded as the trivial representation of $\mathrm{SL}(4,\mathbb{R})$. The general irreducible representation of $\mathrm{SL}(4,\mathbb{R})$ is given by a highest weight which may be recorded as non-negative integers attached to the nodes of the corresponding Dynkin diagram. The general BGG resolution on $\mathbb{RP}_3$ is the sequence of bundles and differential operators

$$\mathcal{E}(\overset{a}{\times}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet}) \to \mathcal{E}(\overset{-a-2}{\times}\!\!-\!\!\overset{a+b+1}{\bullet}\!\!-\!\!\overset{c}{\bullet}) \to \mathcal{E}(\overset{-a-b-3}{\times}\!\!-\!\!\overset{a}{\bullet}\!\!-\!\!\overset{b+c+1}{\bullet}) \to \mathcal{E}(\overset{-a-b-c-4}{\times}\!\!-\!\!\overset{a}{\bullet}\!\!-\!\!\overset{b}{\bullet}) \to 0$$

which resolves the representation $\overset{a}{\bullet}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet}$ of $\mathrm{SL}(4,\mathbb{R})$ just as the de Rham resolution does the constants. The differential operators are no longer first order but have orders $a+1$, $b+1$, $c+1$ respectively. The linear elasticity sequence is the special case

$$0 \to \overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet} \to \mathcal{E}(\overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet}) \to \mathcal{E}(\overset{-2}{\times}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{0}{\bullet}) \to \mathcal{E}(\overset{-4}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{2}{\bullet}) \to \mathcal{E}(\overset{-5}{\times}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}) \to 0$$

or, as tensors,

$$0 \to \Lambda^2\mathbb{R}^4 \to \mathcal{E}_i(2) \xrightarrow{\nabla_j} \mathcal{E}_{(ij)}(2) \xrightarrow{\nabla_k\nabla_l} \boxplus\,\mathcal{E}_k(2) \xrightarrow{\epsilon^{ilm}\nabla_m} \mathcal{E}_{[jk]}(-2) \to 0\,.$$

The Calabi complex

It was also noticed by Paul Bressler that the linear elasticity complex on $\mathbb{RP}_3$ is a special case of a complex constructed by Calabi [4]. The Calabi complex is itself a special case of the BGG resolution. Extending our notation to $\mathbb{RP}_n$ in the obvious way, the Calabi complex is

$$0 \to \overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} \to \mathcal{E}(\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet})$$
$$\to \mathcal{E}(\overset{-2}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet})$$
$$\to \mathcal{E}(\overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet})$$
$$\to \mathcal{E}(\overset{-5}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet})$$
$$\to \mathcal{E}(\overset{-6}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet})$$
$$\ddots \qquad\qquad \ddots$$
$$\to \mathcal{E}(\overset{-n-1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet})$$
$$\to \mathcal{E}(\overset{-n-2}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}) \to 0$$

In fact, Calabi derives it from the de Rham sequence much as is done in this article.

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