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ABSTRACT. We describe explicitly the kernels of higher spin twistor operators on standard even dimensional Euclidean space $\mathbb{R}^{2l}$, standard even dimensional sphere S^{2l}, and standard even dimensional hyperbolic space $\mathbb{H}^{2l}$, using realizations of invariant differential operators inside spinor valued differential forms. The kernels are finite dimensional vector spaces (of the same cardinality) generated by spinor valued polynomials on $\mathbb{R}^{2l}$, S^{2l}, $\mathbb{H}^{2l}$.

1. TWISTOR SPINORS ON CONFORMALLY FLAT SPACES - ANALYTIC APPROACH

In this article, we describe explicitly the kernels of higher spin twistor operators on standard even dimensional Euclidean space $\mathbb{R}^{2l}$, standard even dimensional sphere S^{2l}, and standard even dimensional hyperbolic space $\mathbb{H}^{2l}$, using realizations of invariant differential operators inside spinor valued differential forms. Along the way, we rederive the results of [2], concerning the analytical derivation of twistor spinors on standard Euclidean space $\mathbb{R}^n$, sphere S^n with standard metric of positive constant curvature and hyperbolic space H^n with standard metric of negative constant curvature. Our aim is to give alternative representational proves to analytical computations presented for basic twistor operator in [2], and to extend them in a natural way to the case of higher spin twistor operators. The basic idea is to use the results [22], and further decompose the kernels of higher spin twistor operators on S^n from $K = Spin(n+1)$-types on $M = Spin(n)$-types, and to identify these $M = Spin(n)$-modules inside the space of spinor valued polynomials on $\mathbb{R}^n$ using standard embedding $\mathbb{R}^n \hookrightarrow S^n$. Invariant differential operators on spinor valued polynomials on $\mathbb{R}^n$ then allow us to obtain analytic description of higher spin twistor spinors, and to recover, for the basic twistor operator, the results [2].

An important ingredient is the concept of conformal invariance of higher spin twistor operators. As a benefit of conformal invariance, similar results hold true in the case of hyperbolic space, which we realize as an open domain in $\mathbb{R}^n$ with metric conformally related to Euclidean one.

Despite the property of Dirac operators (and generally elliptic operators), that no element of the infinite dimensional space of solutions on $\mathbb{R}^n$ can be lifted to the solution

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on sphere (in other words, the solution can not be extended to the 'point at infinity'), the remarkable fact is that every element of the lifted finite dimensional kernel of higher spin twistor operators on $\mathbb{R}^n$ can be extended on the whole sphere S^n as an element of the kernel of higher spin twistor operators on sphere S^n. The fact that all solutions of higher spin twistor equations on $\mathbb{R}^n$ extend to the whole S^n can be seen from representational-theoretic behavior without writing down explicit analytical formulas for solutions. Moreover, a complete and explicit description of spaces of solutions on $\mathbb{R}^n, S^n$ and H^n can be given.

Many of the statements, mainly in the introduction, are discussed separately for the even and odd cases. However, the main Theorem6.3 holds true only in the even case $n = 2l$. The tools we use do not suffice to determine analogous statements in odd dimensions.

Let us recall the results presented in [2], describing the kernel of basic twistor operator on n-dimensional conformally flat spaces - Euclidean space $\mathbb{R}^n$, sphere S^n and hyperbolic space H^n, via analytical methods. All definitions and results of this section can be found in [2], section 1.4.

Let (M^n, g) be n-dimensional Riemannian spin manifold with spinor bundle $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ in even dimensions and $\mathbb{S}$ in odd dimensions, and let c denotes the Clifford multiplication, $c : TM \otimes \mathbb{S} \to \mathbb{S}$. Then $\mathtt{Ker}(c)$ is a subbundle of $TM \otimes \mathbb{S}$, and the projection $P_{\mathtt{Ker}(c)}$ on $\mathtt{Ker}(c)$ is given by

$$P_{\mathtt{Ker}(c)}(X \otimes s) = X \otimes s + \frac{1}{n}\sum_{i=1}^{n} e_i \otimes e_i X s\,, \tag{1}$$

where $(e_1, \ldots, e_n)$ is a local ON-basis of TM.

Definition 1.1. *Let D denotes the Dirac operator associated to a Spin-structure on (M, g). The twistor operator T on (M, g) is the composition of spinor covariant derivative $\nabla^{\mathbb{S}}$ and the projection $P_{\mathtt{Ker}c}$,*

$$T \equiv P_{\mathtt{Ker}c} \circ \nabla^{\mathbb{S}} \;:\; \Gamma(\mathbb{S}) \xrightarrow{\nabla^{\mathbb{S}}} \Gamma(TM \otimes \mathbb{S}) \xrightarrow{P_{\mathtt{Ker}c}} \Gamma(\mathtt{Ker}c)\,,$$

and its local formula is

$$Ts = \sum_{i=1}^{n} e_i \otimes (\nabla_{e_i} + \frac{1}{n} e_i D)s\,. \tag{2}$$

Definition 1.2. *A section $s \in \Gamma(\mathbb{S})$ is called a (basic)* **twistor spinor** *if s lies in the kernel of the (basic) twistor operator T.*

Twistor spinors on $\mathbb{R}^n$ are polynomials with values in the spinor bundle:

$$s(x) = s_0 + x s_1\,, \tag{3}$$

where $s_0 \in \mathbb{S}$ and $s_1 \in \mathbb{S}$ are constant (global) sections of $\mathbb{S}$. In particular, $dim\mathtt{Ker}T = 2^{[\frac{n}{2}]+1}$.

The pullback of standard Euclidean metric $g_{\mathbb{R}^n}$ on $\mathbb{R}^n$ induced by stereographic projection from the north pole of S^n is conformally equivalent to the standard round metric on S^n, and it yields twistor spinors on $\pi^{-1}(\mathbb{R}^n) \hookrightarrow S^n$ uniquely extending to twistor spinors on S^n:

$$s(x) = \frac{s_0 + x s_1}{(1+ \| x \|^2)^{\frac{1}{2}}}, \tag{4}$$

where $s_0 \in \Gamma(\mathbb{S})$ and $s_1 \in \Gamma(\mathbb{S})$ are constant sections of unique spinor bundle on S^n.
The last example we shall also deal with is the case of n-dimensional hyperbolic space H^n, realized as an open unit ball in $\mathbb{R}^n$, with the metric g_{H^n} conformally equivalent to the flat Euclidean metric on $\mathbb{R}^n$:

$$g_{H^n} = \frac{4}{(1- \| x \|^2)^2} g_{\mathbb{R}^n}. \tag{5}$$

The general form of twistor spinor on H^n is

$$s(x) = \frac{s_0 + x s_1}{(1- \| x \|^2)^{\frac{1}{2}}}, \tag{6}$$

where $s_0 \in \Gamma(\mathbb{S})$ and $s_1 \in \Gamma(\mathbb{S})$ are constant sections of unique spinor bundle on H^n (note that similarly to the cases of $\mathbb{R}^n$ and S^n, the hyperbolic space H^n is also simply connected).
Note the unified structure of the general twistor spinors (3), (4) resp. (6) on $\mathbb{R}^n$, S^n resp. H^n. In fact, the only difference comes from the 'conformal factor'.

2. Conformal invariance of higher spin twistor operators

Let (M^n, g) be a Riemannian spin manifold, and let $\mathbb{S}$ denotes the $Spin(n)$-bundle with respect to the $Spin$-structure. Then the following theorem characterizes the transformation property of the basic twistor operator under conformal transformation

$$g \to \tilde{g} = \sigma g$$

of the underlying metric g. Let $\tilde{\mathbb{S}}$ be the spinor bundle of $(M^n, \tilde{g})$.

Theorem 2.1. *Let $s \in \Gamma(\mathbb{S})$ and $\tilde{s} \in \Gamma(\tilde{\mathbb{S}})$. Then*

$$\tilde{T}\tilde{s} = (\sigma)^{-\frac{1}{4}} \{\widetilde{T((\sigma)^{-\frac{1}{4}} s)}\}. \tag{7}$$

In particular, $s \in \Gamma(\mathbb{S})$ is a twistor spinor on $(M, g) \iff (\sigma)^{\frac{1}{4}} \tilde{s} \in \Gamma(\tilde{\mathbb{S}})$ is a twistor spinor on $(M, \tilde{g})$.

Proof. See [2], section 1.4., Theorem 7. □

Recall the behavior of the spinor calculus under the conformal change of metric $\tilde{g} = \sigma g$ on spin manifold (M, g), see [2]. There is an identification $\mathbb{S} \to \tilde{\mathbb{S}}$ of the spinor bundle $\mathbb{S}$ on (M, g) and the spinor bundle $\tilde{\mathbb{S}}$ on $(M, \tilde{g})$, such that for all $X, \tilde{X} \in TM, \tilde{X} = \sigma^{-\frac{1}{2}} X$, it holds true:

$$\begin{aligned} \tilde{X}\tilde{s} &= \widetilde{\{Xs\}}, \\ \nabla^{\tilde{\mathbb{S}}}_{\tilde{X}} &= \sigma^{\frac{1}{2}} \widetilde{\{\nabla_X s\}} + \frac{1}{2} \widetilde{\{X grad(\sigma^{-\frac{1}{2}}) s\}} + \frac{1}{2} X \sigma^{-\frac{1}{2}} \{\tilde{s}\}, \forall s \in \mathbb{S}, \tilde{s} \in \tilde{\mathbb{S}}. \end{aligned} \tag{8}$$

Note, that the identification $\mathbb{S} \to \tilde{\mathbb{S}}$ in previous equations is always applied on particular expression in parenthesis.

By construction, [16],[20],[26], the higher spin twistor operators T_k are conformally invariant differential operators. Due to the normalization of conformal weight of the metric to be 2,

$$g \to \tilde{g} = \sigma^2 g, \tag{9}$$

we get:

Lemma 2.2. *The transformation law of higher spin twistor operators T_k under conformal transformation (9) is*

$$\tilde{T}_k \tilde{s} = (\sigma)^{-\frac{1}{2}} \{T_k(\widetilde{(\sigma)^{-(k-\frac{1}{2})} s})\}. \tag{10}$$

In particular, s is a k-th higher spin twistor spinor on (M, g) $\Longleftrightarrow$ $(\sigma)^{k-\frac{1}{2}} \tilde{s}$ is a k-th higher spin twistor spinor on $(M, \tilde{g})$. For $k = 1$, we reproduce the formula (7) (note that the conformal weight of the metric in [2] *is different from our).*

Proof. The explicit form of conformal weights was computed for example in [22]. □

3. Kernels of higher spin twistor operators on S^n in terms of M-types

The spinor bundle $\mathbb{S}$ on Riemannian manifold (S^n, g_0) for $n \geq 3$ exists and is unique. It is $M = Spin(n)$-bundle associated to the fundamental spinor representation of a simply connected twofold covering of $SO(n)$-bundle of ON-frames.

In [22], we computed the sets of $K = Spin(n+1)$-types lying in the kernels of invariant k-th order (spin) twistor operators on the sphere S^n, using suitable Bernstein-Gelfand-Gelfand sequence (see [8],[20],[22]) of $P = CO(n, \mathbb{R}) \rtimes \mathbb{R}^n$-modules resolving $G = Spin(n+1, 1, \mathbb{R})$-module $(\frac{2k-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_{l+1})_G$. For later purposes, we would like to carry on the decomposition further from $K = Spin(n+1)$-types on $M = Spin(n)$-types.

Lemma 3.1. *In terms of $M = Spin(n)$-types, the kernels of higher spin twistor operators T_k on the sphere S^n are*

- *in the even case $n = 2l$:*

$$\mathtt{Ker}T_{(\frac{1}{2}{}_1,\ldots,\frac{1}{2}{}_l)_M, e_1, k} \simeq \oplus_{j\in\{-1,1\}} \oplus_{p=0}^{k-1} (p+1) \, (\frac{2k-2p-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_{l-1}, \frac{j}{2}{}_l)_M. \tag{11}$$

- *in the odd case $n = 2l+1$:*

$$\mathtt{Ker}T_{(\frac{1}{2}{}_1,\ldots,\frac{1}{2}{}_l)_M, e_1, k} \simeq \oplus_{p=0}^{k-1} 2(p+1) \, (\frac{2k-2p-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_l)_M. \tag{12}$$

Proof. In [22], we determined the kernels of higher spin twistor operators T_k on S^n. In the even case, the result is

$$\mathtt{Ker}T_{(\frac{1}{2}{}_1,\ldots,\frac{1}{2}{}_l)_M, e_1, k} \simeq \oplus_{m=1}^{k} (\frac{2m-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_l)_K, \tag{13}$$

and in the odd case $(n = 2l+1)$

$$\mathtt{Ker}T_{(\frac{1}{2}{}_1,\ldots,\frac{1}{2}{}_l)_M, e_1, k} \simeq \oplus_{m=1}^{k} \{(\frac{2m-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_{l+1})_K \oplus (\frac{2m-1}{2}{}_1, \frac{1}{2}{}_2, \ldots, \frac{1}{2}{}_l, -\frac{1}{2}{}_{l+1})_K\}. \tag{14}$$

We decompose these spaces of K-types further on M-types, using appropriate type of branching rules, see [5],[18],[22], for the couple of Lie groups ($K = Spin(n+1), M = Spin(n)$).

Concerning the even case $n = 2l$, the decomposition of the sum of K-types in the kernel of T_k on M-modules results in

(15)

$$\begin{aligned}
&\oplus_{m=1}^{k} (\frac{2m-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_l)_K \simeq \oplus_{j\in\{-1,1\}} \{k(\frac{1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_{l-1}, \frac{j}{2}_l)_M \oplus \\
&\oplus (k-1)(\frac{3}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_{l-1}, \frac{j}{2}_l)_M \oplus \dots \oplus 2(\frac{2k-3}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_{l-1}, \frac{j}{2}_l)_M \oplus \\
&\oplus (\frac{2k-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_{l-1}, \frac{j}{2}_l)_M\} \\
&\simeq \oplus_{j\in\{-1,1\}} \oplus_{p=0}^{k-1} (p+1) (\frac{2k-2p-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_{l-1}, \frac{j}{2}_l)_M .
\end{aligned}$$

Especially, in the case $k = 1$ we get

$$(16) \qquad (\frac{1}{2}_1, \dots, \frac{1}{2}_l)_K \simeq (\frac{1}{2}_1, \dots, \frac{1}{2}_l)_M \oplus (\frac{1}{2}_1, \dots, \frac{1}{2}_{l-1}, -\frac{1}{2}_l)_M .$$

In the odd case $n = 2l+1$,

(17)

$$\begin{aligned}
&\oplus_{m=1}^{k} \{(\frac{2m-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_{l+1})_K \oplus (\frac{2m-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_l, -\frac{1}{2}_{l+1})_K\} \\
&\simeq 2\{k(\frac{1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_l)_M \oplus (k-1)(\frac{3}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_l)_M \\
&\oplus \dots \oplus 2(\frac{2k-3}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_l)_M \oplus \\
&\oplus (\frac{2k-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_l)_M\} \simeq \oplus_{p=0}^{k-1} 2(p+1) (\frac{2k-2p-1}{2}_1, \frac{1}{2}_2, \dots, \frac{1}{2}_l)_M ,
\end{aligned}$$

and as a special case $k = 1$ one gets

$$(18) \qquad \begin{aligned}
&(\frac{1}{2}_1, \dots, \frac{1}{2}_{l+1})_K \oplus (\frac{1}{2}_1, \dots, \frac{1}{2}_l, -\frac{1}{2}_{l+1})_K \\
&\simeq (\frac{1}{2}_1, \dots, \frac{1}{2}_l)_M \oplus (\frac{1}{2}_1, \dots, \frac{1}{2}_l)_M \simeq 2(\frac{1}{2}_1, \dots, \frac{1}{2}_l)_M ,
\end{aligned}$$

i.e. in the kernel of basic twistor operator on S^{2l+1} there is the fundamental spinor representation of M with double multiplicity. □

4. Various realizations of representations of $M = Spin(n)$

The BGG sequences in the cases of conformal spheres (see [22]) rely on identification $S^n \simeq G/P$, where $G = Spin(n+1, 1, \mathbb{R})$ is the group of conformal transformations and P is its maximal parabolic subgroup. The representation content of P is $M \times \mathbb{R}_+ = Spin(n) \times \mathbb{R}_+$. Let us denote irreducible representation of P by (λ, w), where λ is an irreducible representation of $M = Spin(n)$ and $w \in \mathbb{C}$ is a conformal weight. The following identification is easy to prove:

Theorem 4.1. *Let G be a Lie group and H its Lie subgroup. Let G/H be the corresponding homogeneous space, and let $\boldsymbol{V}(\lambda)$ be an irreducible representation of H, such that its associated vector bundle is $\mathcal{V}(\lambda) \equiv (G, \pi, H, \boldsymbol{V}(\lambda))$. Then there is bijection between the space of sections $\Gamma(M, \mathcal{V}(\lambda))$ of $\mathcal{V}(\lambda)$ and the space of H-equivariant maps from G to the representation space $\boldsymbol{V}(\lambda)$*

$$C^\infty(G, \boldsymbol{V}(\lambda))^H := \{f \in C^\infty(G, \boldsymbol{V}(\lambda)) \mid f(gh) = \lambda(h^{-1})f(g), \forall g \in G\ \forall h \in H\} \tag{19}$$

given by

$$\begin{aligned} \phi &: f \in C^\infty(G, \boldsymbol{V}(\lambda))^H \xrightarrow{\sim} s \in \Gamma(M, \mathcal{V}(\lambda)), \\ \phi &: f \longrightarrow s(x) = [p, f(p)], p \in \pi^{-1}(x), [p, f(p)] \in G \times_H \boldsymbol{V}(\lambda). \end{aligned} \tag{20}$$

The left regular action of G on $C^\infty(G, \boldsymbol{V}(\lambda))^H$

$$(\pi_G(g_1)f)(g_2) = f(g_1^{-1}g_2), f \in C^\infty(G, \boldsymbol{V}(\lambda))^H, g_1, g_2 \in G, \tag{21}$$

preserves the subspace $C^\infty(G, \boldsymbol{V}(\lambda))^H$, and hence defines the action on the space $\Gamma(M, \mathcal{V}(\lambda))$.

This theorem allows us to decompose the space of sections of associated vector bundle $\mathcal{V}_w(\lambda)$ over the homogeneous space $S^n \simeq G/P$ on G-modules. Similarly, the restrictions of the previous left regular action π_G on (left regular) representation π_K of maximal compact subgroup $K = Spin(n+1)$ and further on (left regular) representation π_M of its subgroup $M \subset K \subset G$ leads to the decomposition of $\Gamma(G/P, \mathcal{V}_w(\lambda))$ on K-modules resp. M-modules, because both π_K and π_M also preserve the space $C^\infty(G, \boldsymbol{V}_w(\lambda))^P$.

On the other hand, the support of all sections of associated vector bundles was up to now the homogeneous space S^n. We would like to extend at least some parts of the previous procedure to the base space $\mathbb{R}^n$, suitably embedded in S^n. The sphere S^n can be also regarded as a homogeneous space $S^n \simeq K/M$, where $K = Spin(n+1)$ and $M = Spin(n)$, so that K acts transitively on S^n with isotropy group $M = Spin(n)$. The round metric on S^n corresponds to translations by K of the Killing form on $\mathcal{K} = spin(n+1)$. The isometry group of this metric on S^n is just the group $M = Spin(n)$.

Let us choose a distinguished point on S^n in such a way that, without loss of generality, it corresponds in a convenient coordinate system to the north pole $N \equiv (1, 0, \ldots, 0) \in S^n \subset \mathbb{R}^{n+1}$ of standard embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$. Because this point N is fixed by the action of $M = Spin(n)$ (M is stabilizer of a distinguished point on homogeneous space $K/M \simeq S^n$), it follows that $S^n \setminus \{N\}$ is also fixed by the action of M. It is not the case that the subspace $S^n \setminus \{N\}$ of S^n is also fixed by the action of K, because K acts transitively on $K/M \simeq S^n$, and so there is always a group element in K connecting $N \in S^n$ with any other point on S^n. There is analytical diffeomorphism (stereographic projection) π, identifying $\mathbb{R}^n \simeq \mathbb{R}^{n+1}|_{x_{n+1}=0} \subset \mathbb{R}^{n+1}$ with $S^n \setminus \{N\}$. By definition of π, there is an isometry of Riemannian manifolds $(\pi^{-1}(\mathbb{R}^n), \pi^\star(g_{\mathbb{R}^n})) \equiv (\mathbb{R}^n, g_{\mathbb{R}^n})$. The pull-back of the standard metric on $\mathbb{R}^n$ to $\pi^{-1}(\mathbb{R}^n) \subset S^n$ is the conformal multiple of standard round metric on $S^n|_{\pi^{-1}(\mathbb{R}^n)}$. Moreover, this metric on $\pi^{-1}(\mathbb{R}^n) \subset S^n$ uniquely extends on the whole sphere S^n with the same conformal coefficient in the second trivialization of the sphere.

The restriction $S^n \simeq K/M \searrow K/M \setminus \{N\} \simeq S^n \setminus \{N\}$ of support of any section of $\Gamma(K/M, \mathcal{V}(\lambda))$ to $\Gamma(K/M \setminus \{N\}, \mathcal{V}(\lambda))$ implies inclusion

$$\Gamma(K/M \setminus \{N\}, \mathcal{V}(\lambda)) \subset \Gamma(K/M, \mathcal{V}(\lambda)). \tag{22}$$

By Theorem4.1 applied to the homogeneous space $S^n \simeq K/M$ and the groups K and M, we get

$$\Gamma(K/M, \mathcal{V}(\lambda)) \simeq C^\infty(K, \boldsymbol{V}(\lambda))^M, \tag{23}$$

where

$$C^\infty(K, \boldsymbol{V}(\lambda))^M := \{f \in C^\infty(K, \boldsymbol{V}(\lambda)) \mid f(km) = \lambda(m^{-1})f(k), \forall k \in K \ \forall m \in M\} \tag{24}$$

for representation λ of M and associated vector bundle $\mathcal{V}(\lambda) \equiv K \times_M \boldsymbol{V}(\lambda)$. The left regular representation π_K of K,

$$(\pi_K(k_1)f)(k_2) = f(k_1^{-1}k_2)\,,\ f \in C^\infty(K, \boldsymbol{V}(\lambda))\,,\ k_1, k_2 \in K\,, \tag{25}$$

preserves the subspace $C^\infty(K, \boldsymbol{V}(\lambda))^M$ and hence defines the action on the space $\Gamma(K/M, \mathcal{V}(\lambda))$. However, this action does not respects $\pi^{-1}(\mathbb{R}^n) \hookrightarrow S^n \simeq K/M$, but its restriction to M (as was explained in the last paragraph) $\pi_K\,|_M$ does. We summarize this in the following observation:

Observation 4.2. *Let λ be a representation of M. Then there is (infinite dimensional) left regular representation of M on the space $C^\infty(K, \boldsymbol{V}(\lambda))^M$, decomposing it on M-modules:*

$$(\pi_K\,|_M\,(m)f)(k) = f(m^{-1}k)\,,\ f \in C^\infty(K, \boldsymbol{V}(\lambda))^M\,,\ k \in K\,,\ m \in M\,. \tag{26}$$

The restrictions of all spaces of sections on the subspace $K \setminus \{eM\}$ are invariant with respect to the left regular representation $\pi_K\,|_M$ of $M = Spin(n)$, and so one can decompose them on irreducible M-modules via (26):

$$\begin{aligned} &(\pi_K\,|_M\,(m)f\,|_{K\setminus\{eM\}})(k) = f\,|_{K\setminus\{eM\}}\,(m^{-1}k)\,,\\ &f\,|_{K\setminus\{eM\}} \in C^\infty(K \setminus \{eM\}, \boldsymbol{V}(\lambda))^M\,,\ k \in K \setminus \{eM\}\,,\ m \in M\,,\\ &C^\infty(K \setminus \{eM\}, \boldsymbol{V}(\lambda))^M \subset C^\infty(K, \boldsymbol{V}(\lambda))^M\,. \end{aligned} \tag{27}$$

The Theorem (4.1) gives the bijection

$$\Gamma(\pi^{-1}(\mathbb{R}^n), \boldsymbol{V}(\lambda)) \xrightarrow{\simeq} C^\infty(K \setminus \{eM\}, \boldsymbol{V}(\lambda))^M\,, \tag{28}$$

where now π denotes the analytical diffeomorphism $\pi : S^n \setminus N \to \mathbb{R}^n$.

In the following Lemma we identify two different spaces of representation-valued invariant functions on appropriate Lie groups.

Lemma 4.3. *Let as in the scheme above G be a connected noncompact Lie group, K its maximal compact subgroup, P the maximal parabolic subgroup of G and M the Langlands-Levi factor in the decomposition of P. Let $(w, \boldsymbol{V}(\lambda))$ be an irreducible representation of P, with λ being a dominant weight of M.*
Then there is a natural bijection

$$C^\infty(G, \boldsymbol{V}_w(\lambda))^P \xrightarrow{\simeq} C^\infty(K, \boldsymbol{V}(\lambda))^M\,, \tag{29}$$

given by restriction of support of any element of $C^\infty(G, V_w(\lambda))^P$ to the maximal compact subgroup K.

Proof. For the proof, see [18]. □

An immediate consequence of this Lemma is the following Corollary.

Corollary 4.4. *The isomorphism of spaces of sections via the Lemma (4.3),*

$$C^\infty(G, V_w(\lambda))^P \xrightarrow{\simeq} C^\infty(K, V(\lambda))^M, \tag{30}$$

implies the bijection of K-representation spaces $C^\infty(G, V_w(\lambda))^P$ and $C^\infty(K, V(\lambda))^M$ as the isomorphism of admissible representations of finite K-types.

Proof. The spaces $C^\infty(G, V_w(\lambda))^P$ resp. $C^\infty(K, V(\lambda))^M$ are suitable invariant subspaces of finite (G, K)-modules resp. K-modules, and hence they are equivalent as the spaces of K-types. □

5. Higher spin twistor operators on spinor valued polynomials

Let us introduce on standard $\mathbb{R}^n$ standard ON-base $e_1, \ldots, e_n$, and let us denote the spinor bundle on $\mathbb{R}^n$ by $\mathbb{S}$. In the even case $n = 2l$, $\mathbb{S}$ is reducible *Spin*-module and decomposes on irreducible parts $\mathbb{S} \simeq \mathbb{S}^+ \oplus \mathbb{S}^-$, in the odd case $\mathbb{S}$ is irreducible. For any n, the spinor bundle $\mathbb{S}$ on $\mathbb{R}^n$ exists, is unique and topologically trivial $Spin(n)$-bundle.

Definition 5.1. *Let us denote the ring of polynomials on $\mathbb{R}^n$ by $\mathcal{P}(\mathbb{R}^n)$. $\mathcal{P}(\mathbb{R}^n)$ is canonically graded by the homogeneity of the polynomial. An element $p \in \mathcal{P}(\mathbb{R}^n)$ is represented by*

$$p = \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_n : i_1 + \cdots + i_n = k} a_i \, x_1^{i_1} \ldots x_n^{i_n},$$

where the coefficients a_i are usually valued in $\mathbb{R}$ or $\mathbb{C}$. Spinor-valued polynomial is defined to be an element of

$$\Gamma(\mathcal{P}(\mathbb{R}^n) \otimes \mathbb{S}^\pm) \simeq \mathcal{P}(\mathbb{R}^n, \mathbb{S}^\pm), \tag{31}$$

where the unified notation $\mathbb{S}^+ \simeq \mathbb{S}^- \simeq \mathbb{S}$ is used in odd cases. We shall denote a spinor-valued polynomial on $\mathbb{R}^n$ by

$$\sum_{I,J} a_{IJ} p_I \otimes s_J \in \mathcal{P}(\mathbb{R}^n) \otimes \mathbb{S}^\pm,$$

for p_I a base of the space of polynomials and s_J the base of the spinor space.
By a spinor-valued polynomial on $\mathbb{R}^n$ of homogeneity k we shall mean an element

$$\sum_{I|_{|I|=k}, J} a_{IJ} p_I^k \otimes s_J \in \mathcal{P}_k(\mathbb{R}^n) \otimes \mathbb{S}^\pm.$$

First of all, let us remind the space of polynomials on $\mathbb{R}^n$ alone. The group $Spin(n)$ naturally operates on the space of polynomials.

Definition 5.2. *Let $\mathcal{P}(\mathbb{R}^n, \mathbb{S}^{\pm})$ be the space of spinor valued polynomials on $\mathbb{R}^n$, and let ρ be the fundamental spinor representation of $Spin(n)$. Then the action of $Spin(n)$ on $\mathcal{P}(\mathbb{R}^n, \mathbb{S}^{\pm})$ is defined by*

$$(\pi_g^{\mathbb{S}} \mathcal{P}_k)(x) := \rho(g)\mathcal{P}_k(g^{-1}xg)\,,\ g \in Spin(n)\,. \tag{32}$$

There is canonical action of the polynomial part $\mathcal{P}_k(\mathbb{R}^n)$ on the spinor space $\mathbb{S}^{\pm}$. It is homomorphism of $Spin(n)$-modules, and for $k = 1$, this action descends to Clifford multiplication,

$$c : \mathbb{R}^n \otimes \mathbb{S}^{\pm} \to \mathbb{S}^{\mp}\,. \tag{33}$$

Using the representation $\pi_g^{\mathbb{S}}$ defined in (32), the decomposition of spinor valued polynomials on irreducible $Spin(n)$-modules is well known, and it can be described by following picture:

- odd case $n = 2l + 1$, $\mathbb{S}$

$k=0$	$k=1$	$k=2$	$k=3$	
$(\frac{1}{2},\dots,\frac{1}{2})$	$(\frac{1}{2},\dots,\frac{1}{2})$	$(\frac{1}{2},\dots,\frac{1}{2})$	$(\frac{1}{2},\dots,\frac{1}{2})$	$\dots$
	$\oplus$	$\oplus$	$\oplus$	
	$(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2})$	$(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2})$	$(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2})$	$\dots$
		$\oplus$	$\oplus$	
		$(\frac{5}{2},\frac{1}{2},\dots,\frac{1}{2})$	$(\frac{5}{2},\frac{1}{2},\dots,\frac{1}{2})$	$\dots$
			$\oplus$	
			$(\frac{7}{2},\frac{1}{2},\dots,\frac{1}{2})$	$\dots$

- even case $n = 2l$, $\mathbb{S}^{\pm}$

$k=0$	$k=1$	$k=2$	$k=3$	
$\bullet$ $(\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})$	$\bullet$ $(\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})$	$\bullet$ $(\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})$	$\bullet$ $(\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})$	$\dots$
	$\oplus$	$\oplus$	$\oplus$	
	$\bullet$ $(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})$	$\bullet$ $(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})$	$\bullet$ $(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})$	$\dots$
		$\oplus$	$\oplus$	
		$\bullet$ $(\frac{5}{2},\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})$	$\bullet$ $(\frac{5}{2},\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})$	$\dots$
			$\oplus$	
			$\bullet$ $(\frac{7}{2},\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})$	$\dots$

A given vertical slope corresponds to the decomposition of polynomials of fixed homogeneity on irreducible pieces. Using unified notation for spinor bundle $\mathbb{S}^{\pm}$, we shall denote the irreducible $Spin(n)$-submodule $(\frac{2p+1}{2},\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})$ of $\mathcal{P}_k(\mathbb{R}^n)\otimes\mathbb{S}^{\pm}$ by $E_{\pm}^{k,p}$. The decomposition of k-th homogeneity component is

$$k=2m+1,\ \mathcal{P}_{2m+1}(\mathbb{R}^n)\otimes\mathbb{S}^{\pm}\simeq E_{\pm}^{k,k}\oplus\cdots\oplus E_{\mp}^{k,0}\,,$$
$$k=2m,\ \mathcal{P}_{2m}(\mathbb{R}^n)\otimes\mathbb{S}^{\pm}\simeq E_{\pm}^{k,k}\oplus\cdots\oplus E_{\pm}^{k,0}\,. \tag{34}$$

Definition 5.3. *The space $E^{k,k}=\mathcal{P}_k(\mathbb{R}^n)\boxtimes\mathbb{S}^{\pm}\subset\mathcal{P}_k(\mathbb{R}^n)\otimes\mathbb{S}^{\pm}$, which is the kernel of Dirac operator D (see (36)) restricted to the k-th homogeneity component, is called the space of* **primitive** *spinor valued polynomials on $\mathbb{R}^n$:*

$$\mathrm{Ker}D\simeq E^{k,k}\,. \tag{35}$$

There are a next few invariant differential operators acting on $\mathcal{P}(\mathbb{R}^n)\otimes\mathbb{S}^{\pm}$, which deserve a special attention. Note, that every general definition of an operator acting on the space $\mathcal{P}(\mathbb{R}^n)\otimes\mathbb{S}^{\pm}$ is accompanied by its special case, used for example on the picture in Lemma (5.5).

Definition 5.4. *The restriction of action of* **Dirac operator** *$D_{\pm}^{\frac{1}{2}}$ on irreducible $Spin(n)$ module $E_{\pm}^{k,i}$ is defined by*

$$D_{\pm}^{\frac{1}{2}}\ :\ E_{\pm}^{k,i}\longrightarrow E_{\mp}^{k-1,i}\,. \tag{36}$$

Its basic realization for $k=1,i=0$ is

$$D_{\pm}^{\frac{1}{2}}\ :\ \mathbb{S}^{\pm}\to S^{\mp}$$
$$D_{\pm}^{\frac{1}{2}}\ :\ E_{\pm}^{1,0}\longrightarrow E_{\mp}^{0,0}\,. \tag{37}$$

The restriction of action of k-th order **higher spin twistor operator** *T_k on irreducible $Spin(n)$-module $E_{\mp}^{l,i}$ is defined by*

$$T_{\mp,k}\ :\ E_{\mp}^{l,i}\longrightarrow E_{\mp}^{l+k,i+k}\,, \tag{38}$$

and its special case of basic twistor operator $T_{\mp}, 1$ *restricted on* $E_{\mp}^{j,0}$ *is*

$$T_{\mp,1} \;:\; (\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})_{Spin(n)} \to (\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})_{Spin(n)},$$

$$T_{\mp,1} \;:\; E_{\mp}^{j,0} \to E_{\mp}^{j+1,1}, \tag{39}$$

resp.

$$T'_{\mp,1} \;:\; (\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})_{Spin(n)} \to (\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})_{Spin(n)},$$

$$T'_{\mp,1} \;:\; E_{\mp}^{j,0} \to E_{\mp}^{j+1,1}. \tag{40}$$

Finally, the restriction of the action of **Laplace operator** *on irreducible* $Spin(n)$*-module* $E_{\mp}^{k,i}$ *is defined by*

$$\triangle_{\mp} \;:\; E_{\mp}^{k,i} \to E_{\mp}^{k-2,i}, \tag{41}$$

with special case

$$\triangle_{\mp} \;:\; (\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})_{Spin(n)} \to (\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})_{Spin(n)}.$$

There is a fundamental relationship among invariant differential operators.

Lemma 5.5. *Let the base manifold be the standard Euclidean space* $\mathbb{R}^n$. *The higher spin twistor operators* $T_{\pm,k}, T'_{\pm,k}$ *and the Laplace operators* $\triangle_{\pm}$ *on* $\mathbb{R}^n$ *fulfill the basic identity*

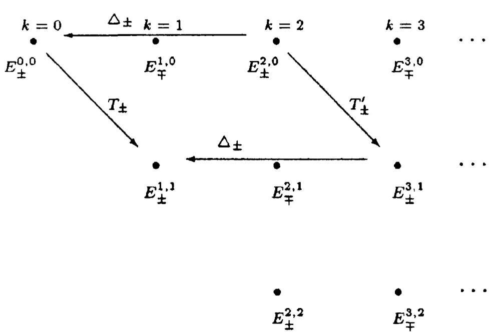
$$\triangle_{\pm} T'_{\pm,k} = T_{\pm,k} \triangle_{\pm}\,. \tag{42}$$

This means, that inside the cone of irreducible $Spin(n)$*-modules with horizontal operators* $\triangle_{\pm}$ *and 'downright' higher spin twistor operators* $T_{\pm,k}$, $T'_{\pm,k}$, *there is a whole set of commutation relations produced by restriction of (42) to a suitable diagrams ('diamonds') inside the cone.*

On the picture, we have chosen the basic twistor operators $T_{\pm}, T'_{\pm}$ *of first order, with the 'diamond' situated in the most upper left part of the cone of irreducible* $Spin(n)$*-modules.*

The pictures for higher spin twistor operators of general order $k \geq 1$ *and the 'diamond' situated in a general position inside the cone are quite analogous.*

The last mentioned example has the following diagrammatic demonstration:



In other words, the diagram is commutative.

Proof. Let $(x_1, \ldots, x_n)$ be the standard coordinates on $\mathbb{R}^n$ with standard Euclidean metric. The operators $T_{\pm,k}, T'_{\pm,k}$ and $\Delta_\pm$ on standard $\mathbb{R}^n$, written in global coordinates, are of the form

$$\begin{aligned} T'_\pm &= \sum_{|k|} B_{\pm,|k|} \frac{\partial}{\partial x_{|k|}}, \\ T_\pm &= \sum_{|k|} A_{\pm,|k|} \frac{\partial}{\partial x_{|k|}}, \\ \Delta_\pm &= \sum_{|2|} C_{\pm,|2|} \frac{\partial}{\partial x_{|2|}}, \\ \frac{\partial}{\partial x_{|k|}} &\equiv \frac{\partial^k}{\partial^{k_1} x_1 \ldots \partial^{k_n} x_n}, \sum_i k_i = k, \end{aligned} \tag{43}$$

where k denotes the order of operator (and also the symbol of multi index summation), and the matrices $A_\pm^{|k|}$ resp. $B_\pm^{|k|}$ are endomorphisms of spinor bundle $\mathbb{S}^\pm$. The conformal group on $\mathbb{R}^n$ is generated by special orthogonal group $SO(n)$, the set of n translations $T_{\mathbb{R}^n}$ and dilatation (scaling) D. Most suitable for our purposes is its realization given for example in [23] - the conformal group of S^n is realized by Vahlen matrices (Clifford algebra valued 2×2-matrices). In fact, the generators $CO(\mathbb{R}^n) \subset CO(S^n)$, conserving by their action $\mathbb{R}^n$ ($\mathbb{R}^n \hookrightarrow S^n$), are just the standard ones $(SO(n), T_{\mathbb{R}^n}, D)$.
Because of trivial connection on $\mathbb{R}^n$, the conformal invariance of all operators means that

$$[T_{\mathbb{R}^n}, T'_{\pm,k}] = 0\,, \; [T_{\mathbb{R}^n}, T_{\pm,k}] = 0\,, \; [T_{\mathbb{R}^n}, \Delta_{\pm,k}] = 0\,,$$

which implies ($T_{\mathbb{R}^n,i} \equiv \partial_i$)

$$\begin{aligned} A_{\pm,|k|}(x+a) &= A_{\pm,|k|}(x)\,, \\ B_{\pm,|k|}(x+a) &= B_{\pm,|k|}(x)\,, \\ C_{\pm,|k|}(x) &= Id\,, \forall a \in \mathbb{R}^n\,, \end{aligned} \tag{44}$$

i.e. $A_{\pm,|k|}, B_{\pm,|k|}, C_{\pm,|k|}$ are the constant endomorphisms of spinor bundle $\mathbb{S}^\pm$ on $\mathbb{R}^n$, and $C_{\pm,|k|}(x)$ are the identity endomorphisms of $\mathbb{S}^\pm$ (note that $\Delta_\pm$ act between isomorphic $Spin(n)$-modules).
Because the endomorphisms represented by $A_{\pm,|k|}$ resp. $B_{\pm,|k|}$ are x-independent and commute with the identity endomorphism, they also commute with $\Delta_\pm$ and we are done. □

A direct consequence of commutativity of the previous diagrams is a useful corollary.

Corollary 5.6. *Let us consider a suitable commutative diagram of higher spin twistor operator $T_{\pm,k}$, whose corners (in anti-clockwise direction) are the $Spin(n)$-modules*

$$E_\pm^{i,j}, E_\pm^{i-2,j}, E_\pm^{i+k-2,j+k}, E_\pm^{i+k,j+k}\,. \tag{45}$$

Let $s \in E_\pm^{i,j}$ be in the kernel of $T'_{\pm,k}$. Then it holds true the implication

$$T'_{\pm,k} s = 0 \Longrightarrow T_{\pm,k} \Delta_\pm s = 0\,. \tag{46}$$

Proof. The proof follows from commutation relations (42). □

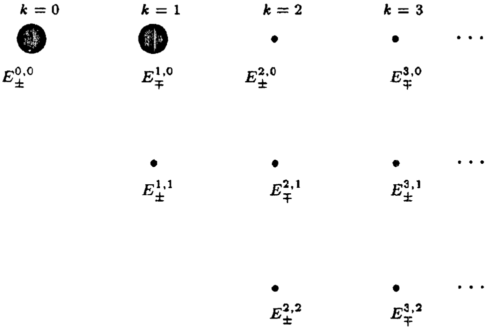
Lemma 5.7. *The kernel of the first order twistor operator $T_{\pm,1}$, acting on $\mathbb{S}^{\pm}\otimes\mathbb{R}^n$, are the first two upper left irreducible $Spin(n)$-modules in the cone of $Spin(n)$-modules, isomorphic to*

$$\text{Ker}T_{\pm,1} \simeq (\frac{1}{2}_1,\dots,\frac{1}{2}_l)_M \oplus (\frac{1}{2}_1,\dots,\frac{1}{2}_{l-1},-\frac{1}{2}_l)_M \simeq E^{0,0}_{\pm}\oplus E^{1,0}_{\mp}. \tag{47}$$

Proof. We determined M-module content of the kernels of higher spin twistor operators, see the equation (47). Using the results of fourth section, it remains to prove that these modules are realized inside invariant decomposition of $\mathbb{R}^n\otimes\mathbb{S}^{\pm}$ as the first two most left modules in the upper horizontal chain of $Spin(n)$-modules.

From the work [4] and [5] it follows, that if f is a $\mathcal{C}^{\infty}$-solution of twistor equation, then f solves a system of elliptic PDE's of 2-nd order, and the theory of elliptic PDE's implies, that the solution is a real analytic function. Then every homogeneous component in this sum solves the twistor equation. Let us now study the polynomial solutions of higher spin twistor operators.

The situation is pictured on the following figure:



Let us number the positions of $Spin(n)$-modules in the first horizontal (upper) row from the left by A_i, i.e.

$$A_0 = E^{0,0}_{\pm},\ A_1 = E^{1,0}_{\mp},\dots,A_i = E^{i,0}_{(-1)^i\pm},\dots. \tag{48}$$

Let the irreducible $Spin(n)$-modules $(\frac{1}{2},\dots,\pm\frac{1}{2})$ be embedded in the space of spinor valued polynomials on $\mathbb{R}^n$. This means, that there are finite positive integers $k<\infty$, $l<\infty$, and complex numbers

$$\begin{gathered}\alpha_1,\alpha_3,\dots,\alpha_{2k+1},\ \forall i\ \alpha_{2i+1}\in\mathbb{C},\\ \beta_0,\beta_2,\dots,\beta_{2l},\ \forall i\ \beta_2 i\in\mathbb{C},\end{gathered} \tag{49}$$

such that

$$\begin{gathered}(\frac{1}{2},\dots,\frac{1}{2}) \equiv f_{odd} = \alpha_1 f_1+\alpha_3 f_3+\dots+\alpha_{2k+1}f_{2k+1}\subset A_1\oplus A_3\oplus\dots\oplus A_{2k+1},\\ (\frac{1}{2},\dots,-\frac{1}{2}) \equiv f_{even} = \beta_0 f_0+\beta_2 f_2+\dots+\beta_{2l}f_{2l}\subset A_0\oplus A_2\oplus\dots\oplus A_{2l},\end{gathered} \tag{50}$$

where $f_i\in A_i$, $\forall i$. The Laplace operator is invariant differential operator, it acts in a given row by decreasing homogeneity degree by 2. Invariance of $\triangle$ means, that there is an implication

$$f \in \mathtt{Ker}T_{\pm,1} \Longrightarrow \triangle f \in \mathtt{Ker}T_{\pm,1}\,, \tag{51}$$

where the restriction of $T_{\pm,1}$ to appropriate irreducible $Spin(n)$-modules is understood in the last implication. Let us rewrite the kernels of $T_{\pm,1}$ via (50),

$$\begin{aligned} f_{odd} &= \alpha_1 f_1 + \alpha_3 x^2 f_1 + \cdots + \alpha_{2k+1} x^{2k} f_1\,, \\ f_{even} &= \beta_0 f_0 + \beta_2 x^2 f_0 + \cdots + \beta_{2l} x^{2l} f_0\,, \end{aligned} \tag{52}$$

where $x = \sum_i x_i \otimes e_i$ is isomorphism

$$E_{\pm}^{k,i} \to E_{\mp}^{k+1,i}\ ,\ \forall i, k \in \mathbb{N}_0\,. \tag{53}$$

Then

$$\begin{aligned} \triangle^k f_{odd} &= \alpha_{2k+1} f_1\,, \\ \triangle^l f_{even} &= \alpha_{2l} f_0\,, \end{aligned} \tag{54}$$

and (52) implies (f_{odd} and f_{even} are the solutions $T_{\pm,1}$), that f_1 and f_0 are also solutions of basic twistor operator. Now the $Spin(n)$-invariance of Laplace operator means, that the space of solutions of basic twistor operator is carried exactly by irreducible modules A_1 and A_0, hence the previous picture with big black dots denoting the kernel of basic twistor operator is correct. □

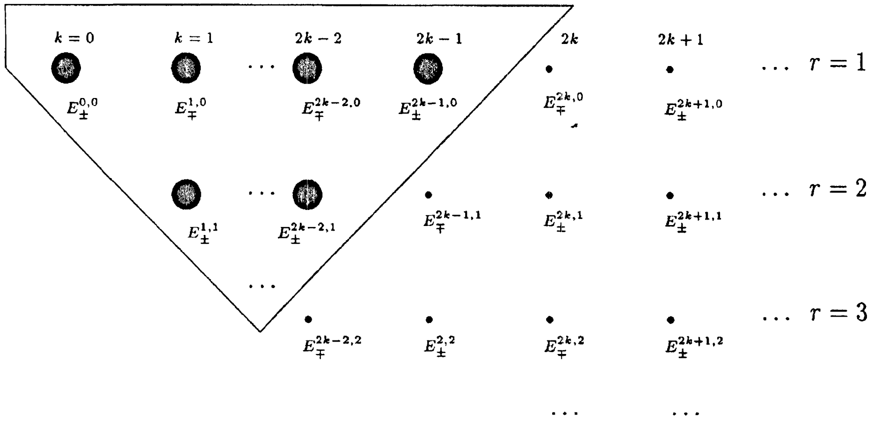
In the case of higher spin twistor operator $T_{\pm,k}$, the kernel has the following description.

Lemma 5.8. *Let r denotes the numbering of rows in the cone of $Spin(n)$-modules of spinor valued polynomials on $\mathbb{R}^n$, such that the upper horizontal one has the number $r = 1$. The kernel of the k-th order twistor operator $T_{\pm,k}$ on $\mathbb{R}^n$ consists in the even case $n = 2l$ of*

$$\mathtt{Ker}T_{\pm,k} \simeq \oplus_{j\in\{-1,1\}} \oplus_{p=0}^{k-1} (p+1)\,(\frac{2k-2p-1}{2}_1, \frac{1}{2}_2, \ldots, \frac{1}{2}_{l-1}, \frac{j}{2}_l)_M\,. \tag{55}$$

It corresponds in the cone of $Spin(n)$-modules of spinor valued polynomials on $\mathbb{R}^n$ in the r-th row to the first $2(k-r+1)$ irreducible $Spin(n)$-modules, $r \in \{1, \ldots, k\}$.

Proof. The proof uses basically the same arguments as the ones in the proof (5.7), so we shall omit it. We only add a pictorial description of the kernel of $T_{\pm,k}$:



□

We can deduce from the previous Lemma analytical description of twistor spinors and their higher spin analogs lying in the kernel of $T_{\pm,k}$.

Theorem 5.9. *Let us denote the projection on irreducible Spin(n)-submodule* $(\frac{2p+1}{2},\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})$ *of* $\mathcal{P}_k(\mathbb{R}^n)\otimes\mathbb{S}^\pm$ *by* $E^{k,p}_{S,\pm}$. *Let* $E^{i,i}_{S,\pm}$ *denotes the projection on the irreducible Spin(n)-submodule corresponding to primitive spinor valued polynomials on* $\mathbb{R}^n$*:*

$$D(E^{i,i}_{S,\pm}(\mathcal{P}_i(\mathbb{R}^n)\otimes\mathbb{S}^\pm))=0\,. \tag{56}$$

Then the general form of higher spin twistor spinor on $(\mathbb{R}^n)$ *with standard Euclidean metric is*

$$\mathrm{Ker}T_{\pm,k}\simeq\sum_{i=0}^{k-1}\Big(\sum_{j=0}^{2k-2i-1}x^j\Big)E^{i,i}_{S,\pm}(\mathcal{P}_i(\mathbb{R}^n)\otimes\mathbb{S}^\pm)\,. \tag{57}$$

In particular, the case $k=1$ *of basic twistor operator* $T_{\pm,1}$ *corresponds to the general form of twistor spinors on* $\mathbb{R}^n$*:*

$$\mathrm{Ker}T_{\pm,1}\simeq\Big(\sum_{j=0}^{1}x^j\Big)E^{0,0}_{S,\pm}(\mathcal{P}_0(\mathbb{R}^n)\otimes\mathbb{S}^\pm)=s_0+xs_1\,, \tag{58}$$

where $s_0,s_1\in S_\pm$ *are constant spinors (with respect to standard flat connection) on* $\mathbb{R}^n$.

Proof. The proof follows immediately from the previous Lemma (5.8) and the fact that, having the primitive spinor valued polynomial $p\otimes s$ on $\mathbb{R}^n$,

$$D(p\otimes s)=0\,, \tag{59}$$

all the other higher spin twistor spinors can be obtained as multiplication of appropriate primitive spinor valued polynomial $p \otimes s$ by appropriate power of invariant differential operator x.
For $k = 1$, the space $E^{0,0}$ of primitive spinor valued polynomials on $\mathbb{R}^n$ consists of constant sections of spinor bundle on $\mathbb{R}^n$, hence the general form of twistor spinor on $\mathbb{R}^n$ is $\mathrm{Ker} T_{\pm,1} \simeq s_0 + x s_1$, with s_0, s_1 constant spinor fields on $\mathbb{R}^n$. □

6. The choice of conformal representants of higher spin twistor spinors

Let us denote by $\boldsymbol{V}(\lambda, w)$ a $\mathcal{P} = CO(n) \rtimes \mathbb{R}^n$-module with highest weight (λ, w), and let $G = Spin(n+1, 1, \mathbb{R})$ denotes the group of conformal transformation of S^n. We shall be interested in the change of kernels of higher spin twistor operators on $\mathbb{R}^n \hookrightarrow G/P \simeq S^n$ by conformal transformations of sections of homogeneous associated bundles $G \times_P \boldsymbol{V}(\lambda, w)$.

It is well known, see [23],[10] and references therein, that the group of conformal transformations of $\mathbb{R}^n$ can be realized by group of 2×2 matrices with entries in Clifford algebra $\mathcal{C}_n$ (Vahlen matrices) satisfying certain conditions. We shall be interested in the tangent mapping to this group action.

Lemma 6.1. *Let $g \in G$ be an element of the group of Vahlen matrices, acting on $(\mathbb{R}^n, g_{\mathbb{R}^n})$ by conformal transformations,*

$$\phi : x \to (ax+b)(cx+d)^{-1}, \, x \in \mathbb{R}^n, \, a, b, c, d \in \mathcal{C}_n \cup 0.$$

Then the conformal factor of the metric g is

$$g_{\mathbb{R}^n} \to \frac{1}{|(cx+d)|^2} g_{\mathbb{R}^n}, \, c, d \in \mathcal{C}_n \cup 0. \tag{60}$$

Proof. The proof is standard exercise in Clifford algebras, so we shall omit it. □

For our purposes, it is sufficient to consider two special cases, treated in (4), (5),(6):

- the realization of hyperbolic space H^n as an open domain in $\mathbb{R}^n$ with metric of constant negative curvature conformally related to Euclidean one,

$$g_{\mathbb{R}^n} \to g_{H^n} = \frac{4}{(1-\|x\|^2)^2} g_{\mathbb{R}^n}. \tag{61}$$

- the realization of $\mathbb{R}^n \hookrightarrow S^n$ with metric of constant positive curvature conformally equivalent with Euclidean metric on $\mathbb{R}^n$,

$$g_{\mathbb{R}^n} \to g_{S^n} = \frac{4}{(1+\|x\|^2)^2} g_{\mathbb{R}^n}, \tag{62}$$

  uniquely extendible to S^n (i.e. to a one missing point) with all the stated properties conserved.

Considering the spaces of sections with suitable equivariance properties with respect to conformal transformations, any section with respect to Euclidean metric on $\mathbb{R}^n$ bijectively corresponds to a section with respect to a conformally related metric, the multiple being a suitable power of conformal factor. Conformal transformations of interest will be the ones in (61) and (62).

We must pay also attention to the normalization of conformal weights. In [2], there is used the conformal weight of the metric to be 1,

$$g \to \tilde{g} = \sigma g\,, \tag{63}$$

so that the basic twistor operator transforms with conformal weight $\frac{1}{4}$. In our normalization, the metric has conformal weight 2, the basic twistor operator has conformal weight $\frac{1}{2}$, and the higher spin twistor operator of k-th order has conformal weight $(k - \frac{1}{2})$.

We shall summarize these facts in an important Corollary of the Lemma6.1.

Corollary 6.2. *Let s be a section of a vector bundle over $(\mathbb{R}^n, g_{\mathbb{R}^n})$ with conformal weight k, such that the metric tensor $g_{\mathbb{R}^n}$ has conformal weight 2. Then the conformal transformation*

$$g \to \tilde{g} = \sigma^2 g\,, \tag{64}$$

corresponds to the action of conformal group on section s

$$s \to \tilde{s} = \sigma^{-k} s\,. \tag{65}$$

The higher spin twistor operators $T_{\pm,k}$ of k-th order act on sections of conformal weights $(k - \frac{1}{2})$, in such a way that the explicit cases (61) resp. (62) correspond to

- *in the hyperbolic case $g_{\mathbb{R}^n} \to g_{H^n}$:*

$$s \to \tilde{s} = s\left(\frac{4}{1 - \| x \|^2}\right)^{k-\frac{1}{2}}; \tag{66}$$

- *in the spherical case $g_{\mathbb{R}^n} \to g_{S^n}$:*

$$s \to \tilde{s} = s\left(\frac{4}{1 + \| x \|^2}\right)^{k-\frac{1}{2}}. \tag{67}$$

Especially, the case of $(k = 1)$ basic twistor operator $T_{\pm,1}$ corresponds to

- *in the hyperbolic case $g_{\mathbb{R}^n} \to g_{H^n}$:*

$$s \to \tilde{s} = s\left(\frac{4}{1 - \| x \|^2}\right)^{\frac{1}{2}}; \tag{68}$$

- *in the spherical case $g_{\mathbb{R}^n} \to g_{S^n}$:*

$$s \to \tilde{s} = s\left(\frac{4}{1 + \| x \|^2}\right)^{\frac{1}{2}}. \tag{69}$$

Proof. The proof consists of counting of degree of conformal factor. Using conformal transformations (61) resp. (62), a section s of conformal weight w transforms

$$s \to \tilde{s} = \frac{1}{| (cx + d) |^w} s \simeq \left(\frac{4}{1 \mp \| x \|^2}\right)^w s\,. \tag{70}$$

□

The following Theorem determines the kernels of higher spin twistor operators in a uniform manner.

Theorem 6.3. *Let $n = 2l$. Let us denote by $E^{j,j}$ the space of spinor valued polynomials of degree j which are solutions of the Dirac equation (so called spherical monogenics of degree j). The kernels of higher spin twistor operators T_k, $k \geq 1$, $k \in \mathbb{N}$, are given by*

- *in the flat case* $g_{\mathbb{R}^n}$:

$$\text{Ker}T_k \simeq \{f(x)|f(x) = \sum_{j=0}^{k-1} \sum_{l=0}^{2k-2j-1} x^l s_l^j \, ; s_l^j = s_l^j(x) \in E^{j,j}\} \, ; \tag{71}$$

- *in the hyperbolic case* g_{H^n}:

$$\text{Ker}T_k \simeq \frac{f(x)}{(1- \| x \|^2)^{k-\frac{1}{2}}} \, , \tag{72}$$

where $f(x)$ *has the same form as in the flat case;*
- *in the spherical case* g_{S^n}:

$$\text{Ker}T_k \simeq \frac{f(x)}{(1+ \| x \|^2)^{k-\frac{1}{2}}} \, . \tag{73}$$

, where $f(x)$ *has the same form as in the flat case.*

Note, that $\{s_0, \ldots, s_{2k-1}\}$ *is the set of* $2k$ *constant sections of unique spinor bundle. In the case* $k = 1$*, we reproduce the twistor spinors, as discussed in* [2].

Proof. The proof follows from Theorem5.9 together with Corrolary 6.2. □

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