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In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 21st Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2002. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 69. pp. [209]–217.

Persistent URL: http://dml.cz/dmlcz/701698

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 69 (2002), pp. 209-217

AN INFINITE DIMENSIONAL VERSION OF THE THIRD LIE THEOREM

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ABSTRACT. The notion of a weak Lie subgroup of a regular Lie group is introduced by exploiting the concept of the evolution operator. This enables to consider the problem of integrability of infinite dimensional Lie algebras. Given a regular Lie group G with its Lie algebra $\mathfrak{g}$ conditions ensuring that a Lie subalgebra $\mathfrak{h}$ is the Lie algebra of a weak Lie subgroup H are formulated. Several examples are presented.

1. INTRODUCTION

Since a celebrated paper [5] by van Est and Korthagen on the non-enlargibility of some infinite dimensional Lie algebras it is well known that the third Lie theorem is, in general, no longer true in the infinite dimension. On the other hand, important counterparts of the local version of the theorem are known, cf. [11], [3]. Also the third Lie theorem has been extended to some Lie algebras of vector fields, see e.g. [2], [7]. Recently, in author's papers [13], [14] it has been shown that some strict Lie algebras (i.e. Lie algebras of sections of a Lie algebroid) can be integrated to groups of bisections of a corresponding Lie groupoid.

We shall use the definition of an infinite dimensional Lie group in the sense of the convenient setting of the infinite dimensional Lie theory [8] due to A.Kriegl and P.Michor. A clue point in this setting is the idea of testing smoothness along smooth curves. A mapping between two possibly infinite dimensional manifolds is smooth if, by definition, it sends smooth curves to smooth curves. This is based on Boman's theorem which states that a mapping $f : \mathbb{R}^n \to \mathbb{R}$ is smooth whenever $f \circ c$ is smooth for any smooth curve $c : \mathbb{R} \to \mathbb{R}$.

Our generalization of the third Lie theorem appeals to the concept of a weak Lie subgroup and has very immediate proof. However, it encompasses many nontrivial

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2000 *Mathematics Subject Classification.* Primary 22E65, 17B65.

Key words and phrases. Infinite dimensional Lie algebra, regular Lie group, weak Lie subgroup, integrability, third Lie theorem.

Supported in part by the AGH grant n. 11.420.04.

The paper is in final form and no version of it will be submitted elsewhere.

situations as it is illustrated by examples in section 4. An essential assumption imposed on a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in our theorem is based on a distinction between smooth curves in G consisting of elements of H and "H-isotopies", where H is a subgroup of G with the Lie algebra $\mathfrak{g}$ corresponding to $\mathfrak{h}$.

A philosophy that lies behind our approach is that the existence of the evolution mapping

$$\mathrm{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to C^\infty((\mathbb{R}, 0), (G, e)),$$

creates a Lie group structure on G. Of course, the smooth structure on such a Lie group G is not the usual one (i.e. it is not defined by charts), however basic concepts of Lie theory are still true. A more complete exposition of this approach concerning regular Lie groups will be presented in a forthcoming paper.

We would like also to indicate that there are some "weak" settings of the infinite dimensional Lie theory and that they usually do not correspond strictly to each other. One example is the notion of diffeological groups due to J. M. Souriau [15]. A smooth structure is there defined by establishing sets of local smooth mappings from $\mathbb{R}^n$ to G, $n = 1, 2, \dots$, and by imposing some conditions on them. As another example of a "weak" setting can serve the concept of generalized Lie groups of H. Omori [10]. The definition is based on a continuous mapping $\exp : G \to \mathfrak{g}$ between a topological group G and a topological Lie algebra $\mathfrak{g}$ with some conditions which mimics essential properties of the exponential map. A common feature of such theories is that any closed subgroup of a Lie group is a Lie subgroup. This corresponds to the lack of a meaningful counterpart of the third Lie theorem on the ground of them.

2. Regular Lie groups and their weak Lie subgroups

Recall that a convenient Lie group G is called *regular* [8] if for $\mathfrak{g} = T_e G$ there exists a bijective evolution map

$$\mathrm{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to C^\infty((\mathbb{R}, 0), (G, e)),$$

and its evaluation map at 1

$$C^\infty(\mathbb{R}, \mathfrak{g}) \ni X \mapsto \mathrm{evol}^r_G(X)(1) \in G$$

is smooth. The right logarithmic derivative δ^r_G is then the inverse of evol^r_G. The notion of regularity has been introduced by J. Milnor [9].

We let $\exp : \mathfrak{g} \to G$ be given by $\exp(X) = \mathrm{evol}^r_G(X)(1)$. In particular, $\exp tX = \mathrm{evol}^r(X)(t)$.

Clearly $\exp(t+s)X = \exp tX . \exp sX$.

The following situation often arise. Given a regular group G as above, there is a closed subgroup $H \subset G$ and a Lie algebra $\mathfrak{h}$ such that smooth curves with values in $\mathfrak{h}$ are sent bijectively by evol^r_G to isotopies with values in H. Unfortunately, such a bijection does not yield a Lie group structure on H. However, a common intuition is that such a situation is not bad, and sometimes by abuse one states that H is a Lie subgroup with the Lie algebra $\mathfrak{h}$. Our goal here is to formalize this intuition, i.e. to define a (weak) Lie subgroup without using charts or the concept of submanifold.

Definition. A subgroup $H \subset G$ is said to be a *weak Lie subgroup* if there exists a closed subspace $\mathfrak{h} \subset \mathfrak{g}$ such that the mapping

$$\mathrm{evol}^r_G|_{C^\infty(\mathbb{R},\mathfrak{h})} : C^\infty(\mathbb{R},\mathfrak{h}) \to C^\infty_e(\mathbb{R},G) \cap \mathrm{Map}_e(\mathbb{R},H)$$

is a bijection. Here $\mathrm{Map}_e(\mathbb{R},H)$ stands for the totality of mappings of $\mathbb{R}$ onto H sending 0 to e, and $C^\infty_e(\mathbb{R},G) := C^\infty((\mathbb{R},0),(G,e))$.

Then we set $\mathrm{evol}^r_H := \mathrm{evol}^r_G|_{C^\infty(\mathbb{R},\mathfrak{h})}$ and $C^\infty_e(\mathbb{R},H) := C^\infty(\mathbb{R},G) \cap \mathrm{Map}_e(\mathbb{R},H)$.

A well known examples of weak Lie subgroups of $\mathrm{Diff}^\infty(M)(M)$ for which it is not known whether they are Lie subgroups constitute the volume preserving diffeomorphism group on a possibly noncompact manifold with volume element (cf.[1]), or the leaf preserving diffeomorphism group on a manifold with foliation. In the latter case the affirmative answer has been given for regular foliation in [12]. However, the problem of introducing a Lie group structure on the leaf preserving diffeomorphism group of arbitrary foliation seems to be difficult and central in the infinite dimensional Lie theory. For further examples, see section 4.

Let us consider basic properties of weak Lie subgroups. First note that evol^r_H sends $\mathfrak{h} \subset C^\infty(\mathbb{R},\mathfrak{h})$ onto the space of all one-parameter subgroups of H. The set $\mathrm{evol}^r_H(\mathfrak{h})$ of all smooth one-parameter subgroups will be designated by T_eH. We will identify $\mathfrak{h}$ with T_eH. We have $\exp : \mathfrak{h} \to H$ given by $\exp(X) = \mathrm{evol}^r_H(X)(1)$. Consequently

$$X \in \mathfrak{h} \quad \Longleftrightarrow \quad \exp tX \in C^\infty_e(\mathbb{R},H)$$

i.e. $\mathfrak{h} = \mathfrak{g} \cap C^\infty_e(\mathbb{R},H)$.

Given $H \subset G$ a weak Lie subgroup and a morphism $\phi : H \to H$ it is easily seen that the tangent map $T_e\phi : \mathfrak{h} \to \mathfrak{h}$ is well defined. We have also the commutative diagram

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{T\phi} & \mathfrak{h} \\ {\scriptstyle \exp_H}\downarrow & & \downarrow{\scriptstyle \exp_H} \\ H & \xrightarrow{\phi} & H \end{array}$$

Let $\mathrm{conj}_g : H \ni h \mapsto ghg^{-1} \in H$ be the conjugation by $g \in H$. Define $\mathrm{Ad}(g) = T_e\,\mathrm{conj}_g$. Then we have

$$g(\exp tX)g^{-1} = \exp(t\,\mathrm{Ad}(g)X).$$

Clearly for every $g,h \in G$ one has $\mathrm{Ad}(gh) = \mathrm{Ad}(g)\,\mathrm{Ad}(h)$.

As usual, we let $\mathrm{ad} := T_e\,\mathrm{Ad}$. Then

$$\mathrm{ad}_X(Y) = [X,Y] = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\,\mathrm{Ad}(\exp tX)Y.$$

It is not hard to check that the bracket $[,]$ is the Lie algebra bracket on $\mathfrak{h}$.

Some further properties are possible. So it is apparent that a large portion of Lie theory is inherited from a Lie group to its weak Lie subgroup.

3. A version of the third Lie theorem

Let G be a regular Lie group with its Lie algebra $\mathfrak{g}$. We begin with the following fact.

Lemma. *For all $f, g \in C^\infty_e(\mathbb{R}, G)$ the following formulae hold:*
(i) $\delta^r_G(fg) = \delta^r_G f + \mathrm{Ad}(f).\delta^r_G g$;
(ii) $\delta^r_G(f^{-1}) = -\mathrm{Ad}(f^{-1}).\delta^r_G f$.

Proof. We show only (ii) as the proof of (i) is in [8], p.405.

First let us observe that the tangent map to the inversion $\nu : G \to G$ can be expressed by

$$T_a\nu.X_a = -T_e(\mu_a)^{-1}.T_a(\mu^{a^{-1}}).X_a = -T_e(\mu_{a^{-1}}).T_a(\mu^{a^{-1}}).X_a,$$

where μ is the multiplication in G, and μ_a (resp. μ^a) is the left (resp. right) translation by a. Next remind that the right logarithmic derivative $\delta^r_G : \mathbb{R} \to \mathfrak{g}$ is given by

$$\delta^r_G f(t) = T_{f(t))}(\mu^{f(t)^{-1}}).T_t f.(\frac{\partial}{\partial t}|_t).$$

Therefore

$$\begin{aligned}
\delta^r_G f^{-1}(t) &= T_{f^{-1}(t)}(\mu^{f(t)}).T_t f^{-1}.(\frac{\partial}{\partial t}|_t) \\
&= T_{f^{-1}(t)}(\mu^{f(t)}).T_{f(t)}\nu.T_t f.(\frac{\partial}{\partial t}|_t) \\
&= -T_{f^{-1}(t)}(\mu^{f(t)}).T_e(\mu_{f(t)^{-1}}).T_{f(t)}(\mu^{f(t)^{-1}}).T_t f.(\frac{\partial}{\partial t}|_t) \\
&= -T_e(\mathrm{conj}_{f^{-1}(t)}).\delta^r_G f(t) \\
&= -\mathrm{Ad}(f^{-1}(t)).\delta^r_G f(t),
\end{aligned}$$

as required. □

Next we formulate two conditions related to the (weak) integrability of a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

(A) If $g \in C^\infty_e(\mathbb{R}, G)$ and for any $t \in \mathbb{R}$ there exists $h \in C^\infty_e(\mathbb{R}, G)$ with $\delta^r_G(h) \in C^\infty(\mathbb{R}, \mathfrak{h})$ and $h_t = g_t$ then $\delta^r_G(g) \in C^\infty(\mathbb{R}, \mathfrak{h})$. In other words, any G-isotopy with values in H is an H-isotopy, where the group H is defined as in the theorem below.

(B) For any $g \in C^\infty_e(\mathbb{R}, G)$ with $\delta^r_G(g) \in C^\infty(\mathbb{R}, \mathfrak{h})$ one has $\mathrm{Ad}(g)\mathfrak{h} \subset \mathfrak{h}$.

Theorem. *Let $\mathfrak{h}$ be a Lie subalgebra satisfying the conditions (A) and (B). Define*

$$H = \{h \in G : h = g_1, \quad \textit{where} \quad \delta^r_G(g) \in C^\infty(\mathbb{R}, \mathfrak{h})\}.$$

Then H is a weak Lie subgroup of G with the Lie algebra $\mathfrak{h} = T_e H$.

Proof. First we show that H is a subgroup of G. Indeed, if $h^i \in H$, $i = 1,2$, then there are $g^i \in C_e^\infty(\mathbb{R}, G)$ with $h^i = g_1^i$ such that $\delta_G^r(g^i) \in C^\infty(\mathbb{R}, \mathfrak{h})$. Therefore $(g^1 g^2)_1 = h^1 h^2$ and $\delta_G^r(g^1 g^2) \in C^\infty(\mathbb{R}, \mathfrak{h})$. The latter holds by Lemma (i) and condition (A). Next, if $g \in C_e^\infty(\mathbb{R}, G)$ with $\delta_G^r(g) \in C^\infty(\mathbb{R}, \mathfrak{h})$ then $(t \mapsto g_t^{-1}) \in C_e^\infty(\mathbb{R}, G)$ as G is a (convenient) Lie group, and $\delta_G^r(g^{-1}) \in C^\infty(\mathbb{R}, \mathfrak{h})$ in view of Lemma (ii) and (A). Thus H is a subgroup of G.

It remaines to check that

$$\mathrm{evol}_G^r|_{C^\infty(\mathbb{R},\mathfrak{h})} : C^\infty(\mathbb{R}, \mathfrak{h}) \to C_e^\infty(\mathbb{R}, G) \cap \mathrm{Map}_e(\mathbb{R}, H)$$

is a well-defined bijection. We use the formula

$$\delta_G^r(g^\lambda) = \lambda(\delta_G^r(g), \quad \text{where} \quad g^\lambda(t) = g(\lambda t),\ \lambda, t \in \mathbb{R}\,.$$

It follows that if $h = g_t$ for some t, where $g \in C_e^\infty(\mathbb{R}, G)$ with $\delta_G^r(g) \in C^\infty(\mathbb{R}, \mathfrak{h})$, then $h \in H$. Consequently, $\mathrm{evol}_G^r|_{C^\infty(\mathbb{R},\mathfrak{h})}$ is well defined.

The injectivity is trivial, while the surjectivity is equivalent to condition (B). □

4. Examples

1. Given any smooth manifold M, in the framework of [8] $\mathrm{Diff}^\infty(M)$ is a Lie group modeled on the complete locally convex vector space $\mathfrak{X}_c(M) = \varinjlim_K \mathfrak{X}_K(M)$. Its tangent space at id is $\mathfrak{X}_c(M)$ but the Lie bracket is the negative of the usual Lie bracket of vector fields. The adjoint action of $g \in \mathrm{Diff}_c^\infty(M)$ on $\mathfrak{X}_c(M)$ is given by $(g^{-1})^*$.

For any $X \in C^\infty(\mathbb{R}, \mathfrak{X}_c(M))$ there is $g : \mathbb{R} \to \mathrm{Diff}^\infty(M)$ such that $\dot{g} = X$ where

$$\dot{g} : \mathbb{R} \to \mathfrak{X}_c(M) \qquad \dot{g}(t)(x) := \tfrac{\partial}{\partial s}|_t (g_s(g_t^{-1}(x)))\,.$$

$\dot{g}$ is the time-dependent vector field defining uniquely g with $g(0) = id$. Thus there exists a bijective evolution map

$$\mathrm{evol}_{\mathrm{Diff}^\infty(M)}^r : C^\infty(\mathbb{R}, \mathfrak{X}_c(M)) \to C^\infty((\mathbb{R}, 0), (\mathrm{Diff}^\infty(M), \mathrm{id}))$$

and so $\mathrm{Diff}^\infty(M)$ is regular.

Specifically it admits the exponential mapping, namely

$$\exp : \mathfrak{X}_c(M) \to \mathrm{Diff}_c^\infty(M)_\circ \qquad X \mapsto \mathrm{Fl}_1^X\,,$$

the flow of the vector field at time 1. It is well known that exp is not locally surjective around 0. Observe that in the convenient setting the identity component of $\mathrm{Diff}^\infty(M)$ consists only of compactly supported elements.

Now let $\sigma \in \Omega^k(M)$ be any closed k-form on M. Let us consider automorphisms of the geometric structure (M, σ). Namely we set ($\mathcal{L}$ is the Lie derivative)

$$L(M, \sigma) = \{X \in \mathfrak{X}(M) :\ \mathcal{L}_X \sigma = 0\}$$
$$G(M, \sigma) = \{f \in \mathrm{Diff}^\infty(M) :\ f^*\sigma = \sigma\}\,.$$

Then $L(M, \sigma)$ is a Lie subalgebra of $\mathfrak{X}(M)$, and $G(M, \sigma)$ is its "Lie group". The latter statement is justified by the following observation.

Proposition 1. *$G(M,\sigma)$ is a weak Lie subgroup of* $\mathrm{Diff}^\infty(M)$ *with the Lie algebra* $L(M,\sigma)$.

Proof. Suppose that $f \in C_e^\infty(\mathbb{R}, \mathrm{Diff}^\infty(M))$ is a smooth curve of diffeomorphisms and $X = \delta^r_{\mathrm{Diff}^\infty(M)}(f)$. Then $f(t) \in G(M,\sigma)$ for each t if and only if $X_t \in L(M,\sigma)$ for each t. In other words, for any smooth path $X : \mathbb{R} \to L(M,\sigma)$ its evolution curve takes its values in $G(M,\sigma)$. Indeed, we have for any t

$$\frac{d}{dt} f(t)^*\sigma = f(t)^*(\mathcal{L}_{X_t}\sigma) = f(t)^*(\iota(X_t)d\sigma + d(\iota(X_t)\sigma)) = f(t)^* d(\iota(X_t)\sigma).$$

It follows that the definition of a weak Lie subalgebra is fulfilled. □

The most important examples of such structures are symplectic structures, volume elements, locally conformal symplectic structures (cf.[6]), cosymplectic structures. Note that the symplectomorphism group is also a usual Lie subgroup ([1]). From [7] we know that also the group of conformal automorphisms of a locally conformal symplectic structures is a Lie subgroup.

Recall that a cosymplectic structure on $2k+1$-dimensional manifold M is given by a pair (θ,ω), where θ is a closed 1-form, and ω is a closed 2-form, and the nondegeneracy condition $\omega^k \wedge \theta \neq 0$ is fulfilled. It follows from Proposition 1 that

$$G(M,\theta,\omega) = \{X \in \mathfrak{X}(M) : \mathcal{L}_X\theta = 0,\ \mathcal{L}_X\omega = 0\}$$

is a weak Lie subgroup with the Lie algebra $L(M,\theta,\omega)$. Analogous statement can be made for other automorphism groups studied of (M,θ,ω)..

2. Let ω be a symplectic structure on M, and let $\sharp^\omega : \Omega^1(M) \to \mathfrak{X}(M)$ be the corresponding musical isomorphism.

A smooth path of diffeomorphisms satisfying Proposition 1 is called a symplectic isotopy. A symplectic isotopy $f(t)$ is Hamiltonian (or exact) if the corresponding $X_t \in L^*(M,\omega)$ for each t. Here $L^*(M,\omega)$ stands for the Lie algebra Hamiltonian vector fields, i.e. $X \in L^*(M,\omega)$ iff $\iota(X)\omega$ is exact, ι being the insertion. A diffeomorphism f of M is called Hamiltonian if there exists a Hamiltonian isotopy $f(t)$ such that $f(0) = \mathrm{id}$ and $f(1) = f$. The totality of all Hamiltonian diffeomorphisms is denoted by $G^*(M,\omega)$.

Proposition 2. *$G^*(M,\omega)$ is a normal subgroup of $G(M,\omega)$.*

Proof. First we show that $G^*(M,\omega)$ verifies the group axioms. Let $f(t)$, $g(t)$ be Hamiltonian isotopies, that is $\delta^r_{\mathrm{Diff}^\infty(M)} f(t) = (du_t)^{\sharp^\omega}$, $\delta^r_{\mathrm{Diff}^\infty(M)} g(t) = (dv_t)^{\sharp^\omega}$ for some smooth families of C^∞-functions u_t and v_t. Then $f(t) \circ g(t)$ is still a Hamiltonian isotopy. Indeed by Lemma

$$\delta^r_{\mathrm{Diff}^\infty(M)}(f(t) \circ g(t) = (d(u_t + v_t \circ f(t)^{-1}))^{\sharp^\omega}.$$

We have that $f(t)^{-1}$ is Hamiltonian as well since by Lemma

$$\delta^r_{\mathrm{Diff}^\infty(M)}(f^{-1}(t)) = (d(-u_t \circ f(t))^{\sharp^\omega}.$$

Therefore $G^*(M,\omega)$ is a group. Next, if $f(t)$ is a Hamiltonian isotopy as above and g is a symplectomorphism then

$$\delta^r_{\mathrm{Diff}^\infty(M)}(g^{-1}\circ f(t)\circ g) = (d(u_t\circ g))^{\sharp^\omega}.$$

This means that $G^*(M,\omega)$ is a normal subgroup of $G(M,\omega)$. □

Observe that $G^*(M,\omega)$ is a weak Lie subgroup. It is a usual Lie group, if the group of periods of ω (i.e. the image by the flux homomorphism of the first homotopy group of $G(M,\omega)$) is discrete. Analogous statements still hold for locally conformal symplectic structures, cf.[6], in view of the existence the flux homomorphism and other invariants.

3. Let $\mathcal{F}$ be a generalized foliation on M, cf.[16], and let $G(M,\mathcal{F})$ be the group of leaf preserving diffeomorphisms of $\mathcal{F}$. It is straightforward that $G(M,\mathcal{F})$ is a weak Lie subgroup. It has been shown in [12] that it is a Lie group if $\mathcal{F}$ is regular, however there is no hope to extend the proof for the singular case.

4. Recall that a Poisson structure on M can be defined by a bivector Λ which satisfies the integrability condition $[\Lambda,\Lambda]=0$ in terms of the Schouten bracket $[,]$. This yields a Lie algebra bracket on $C^\infty(M)$ given by

$$\{u,v\} = \Lambda(\mathrm{d}\,u,\mathrm{d}\,v\}$$

for any $u,v\in C^\infty(M)$. We have the musical homomorphism $\sharp^\Lambda:\Omega^1(M)\to\mathfrak{X}(M)$ which is an isomorphism iff Λ defines a symplectic structure.

A smooth mapping f of (M,Λ) into itself is called a Poisson morphism if

$$\{u\circ f, v\circ f\} = \{u,v\}\circ f \quad \text{for any } u,v\in C^\infty(M).$$

Let $G(M,\Lambda)$ be the group of all Poisson diffeomorphisms of (M,Λ) which are tangent to the leaves of the symplectic foliation $\mathcal{F}(\Lambda)$.

Recall that a vector field X is an infinitesimal automorphism of (M,Λ) if $[\Lambda,X]=0$, that is if $\mathcal{L}_X\Lambda=0$. By $L(M,\Lambda)$ we denote the Lie algebra of all infinitesimal automorphisms with compact support which are tangent to $\mathcal{F}(\Lambda)$. Next, let $L^*(M,\Lambda)$ be the ideal of $L(M,\Lambda)$ of all Hamiltonian vector fields, i.e. $X\in L^*(M,\Lambda)$ iff there exists compactly supported $u\in C^\infty(M)$ such that

$$X=[\Lambda,u] \quad \text{or, equivalently,} \quad X=(du)^{\sharp^\Lambda}.$$

We have the inclusion $[L(M,\Lambda),L(M,\Lambda)]\subset L^*(M,\Lambda)$ as a consequence of the equality $[X_1,X_2]=[\Lambda,u]$, where u is defined by $u(x)=\iota(X_1(x)\wedge X_2(x))\omega^\Lambda_{L_x}$.

As in the symplectic case one defines $G^*(M,\Lambda)$, the group of Hamiltonian automorphisms.

It is easily checked that counterparts of Propositions 1 and 2 hold for arbitrary Poisson manifolds. This is due to the fact that on each leaf L of the symplectic foliation lives a symplectic form ω^Λ_L. Consequently, the groups $G(M,\Lambda)$ and $G^*(M,\Lambda)$ are weak Lie subgroups of $\mathrm{Diff}^\infty(M)$.

Let us remind that Λ is said to be regular iff the symplectic foliation $\mathcal{F}(\Lambda)$ is so. It is proven in [12] that in the regular case the above automorphism groups admit a (usual) Lie algebra structure (under some extra assumption for $G^*(M,\Lambda)$). But the method of [12], which consists in a use of the space of foliated forms as the model space, cannot be generalized to the arbitrary case.

5. A reasonably broad class of groups in geometry constitute strict groups (this idea goes back to C. Ehresmann, cf. [4]). Recently, by using the convenient setting in global analysis it has been shown that important strict groups carry a regular Lie group structure, cf. [13], [14].

Recall that G is a strict group if it is a subgroup of the group of all bisections $\mathrm{Bis}(\Gamma)$ of a Lie groupoid Γ. A groupoid structure on a set Γ is given by two surjections (the source and target) $\alpha, \beta : \Gamma \to M \subset \Gamma$, by a multiplication $m : \Gamma_2 \to \Gamma$, where $\Gamma_2 = \{(x,y) \in \Gamma \times \Gamma : \alpha(x) = \beta(y)\}$, and by an inversion $i : \Gamma \to \Gamma$, satisfying compatibility axioms, see eg. [11], [4]. Then M is the space of units of Γ.

By a bisection of a Lie groupoid Γ we mean a submanifold B of Γ such that $\alpha|B$ and $\beta|B$ are diffeomorphisms onto M. Let $\mathrm{Bis}(\Gamma)$ be the set of all bisections. It is a group endowed with the product law induced by m. Note that if $\Gamma = M \times M$ is the coarse groupoid then $\mathrm{Bis}(\Gamma) = \mathrm{Diff}^\infty(M)$, so that all diffeomorphism groups are strict.

Let us remind that a left-invariant vector fields X on Γ is characterized by $T\beta(X) = 0$ and $Tl_x(X) = X$ for any left translation

$$l_x : \beta^{-1}(\alpha(x)) \ni y \mapsto x.y \in \beta^{-1}(\beta(x)).$$

Let $\mathfrak{X}_L(\Gamma)$ be the Lie algebra of all left-invariant on Γ. Then $\mathfrak{X}_L(\Gamma) \simeq \ker T\beta|_M$. Also $T\alpha|_{\mathfrak{X}_L(\Gamma)} : \mathfrak{X}_L(\Gamma) \to \mathfrak{X}(M)$ is well defined.

It is well known (J. Pradines [11]) that to any Lie groupoid $\Gamma \rightrightarrows M$ is assigned the associated Lie algebroid $\mathcal{A}(\Gamma)$. Namely, $\mathcal{A}(\Gamma) = (\mathcal{N}_\Gamma M, [[,]], T\alpha)$, where $[[,]]$ is a Lie algebra bracket on $\mathrm{Sect}(\ker T\beta|_M)$ introduced by means of the above identification and $\mathcal{N}_\Gamma M \simeq \ker T\beta$ with $\mathcal{N}_\Gamma M$ being the normal bundle to M.

Let $X \in C^\infty(\mathbb{R}, \mathrm{Sect}_c(\mathcal{N}_\Gamma M))$. There is a unique $\tilde{X} \in C^\infty(\mathbb{R}, \mathfrak{X}_L(\Gamma))$ which extends X. Let $\phi = \mathrm{evol}^r_{\mathrm{Diff}^\infty(M)}(\tilde{X})$. By definition $\beta \circ \phi(t)$ is a bijection for all t. With some effort it can be shown that $\alpha \circ \phi(t)$ is a bijection for all t as well, cf.[14]. Thus we get that $C_t = f(t)(M) \in \mathrm{Bis}(\Gamma)$, and we get the evolution operator

$$\mathrm{evol}^r_{\mathrm{Bis}(\Gamma)} : C^\infty(\mathbb{R}, \mathrm{Sect}_c(\mathcal{N}_\Gamma M)) \to C^\infty_{e=M}(\mathbb{R}, \mathrm{Bis}(\Gamma))$$

given by

$$X \mapsto \mathrm{evol}^r_{\mathrm{Diff}^\infty(M)}(\tilde{X})(M).$$

Specifically, $\mathrm{Bis}(\Gamma)$ admits the exponential mapping

$$\exp : \mathrm{Sect}_c(\mathcal{N}_\Gamma M) \to \mathrm{Bis}(\Gamma) \quad X \mapsto \mathrm{Fl}_1^{\tilde{X}}(M).$$

All this gives a flavour how to show that $\mathrm{Bis}(\Gamma)$ is a (convenient) regular Lie group. The complete proof is in [14]. The fact that $\mathrm{Bis}(\Gamma)$ is a Lie group enables to introduce a weak Lie subgroup structure on several strict groups. Some of them are studied in [13], [14].

Final remark. The presented examples are rather standard. However we would like to emphasize that it is usually difficult to built up a usual Lie group structure on a subgroup of regular Lie group. The reason is that such a construction deeply involves the geometry determined by the subgroup which is usually not known enough. This strengthens our belief of the usufulness of the concept of weak Lie subgroup.

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