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∂_ψ- DIFFERENCE CALCULUS BERNOULLI-TAYLOR FORMULA

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ABSTRACT. In this note we derive the general ∂_ψ-difference Bernoulli-Taylor formula with the rest term of the Cauchy type.

1. BERNOULLI-TAYLOR FORMULA

In [2] O.V. Viskov presents another form of the Bernoulli-Taylor formula with the rest term of the Cauchy type. For that he uses Graves-Heisenberg-Weil (GHW) algebra generators $\hat{p}$ and $\hat{q}$ such that:

$$[\hat{p},\hat{q}] = \hat{p}\,\hat{q} - \hat{q}\hat{p} = 1 \tag{1}$$

where 1 is identity operator. Using (1) and the induction one may prove the following identity:

$$\hat{p}\,\hat{q}^n = \hat{p}^n\hat{q} + n\hat{q}^{n-1}\ (n = 1, 2, \dots). \tag{2}$$

Now consider the obvious identity:

$$\sum_{k=0}^{n}(\alpha_k - \alpha_{k+1}) = \alpha_0 - \alpha_{n+1}. \tag{3}$$

Under the substitution

$$\alpha_0 = 0,\ \alpha_k = (-1)^k\frac{\hat{q}^{k-1}\hat{p}^k}{(k-1)!},\ k = 1, 2, \dots \tag{4}$$

and using (2) one can get from (3):

$$\hat{p}\sum_{k=0}^{n}\frac{(-\hat{q})^k\hat{p}^k}{k!} = \frac{(-\hat{q})^n\hat{p}^{n+1}}{n!} \tag{5}$$

what is Bernoulli identity (see Viskov [2]).

Example 1.1. Let $\hat{p}$ and $\hat{q}$ be as below:

$$\hat{p} = D \equiv \frac{d}{dx},\quad \hat{q} = \hat{x} - y,\ \ y \in \mathbf{F}(\mathbf{R}, \mathbf{C}).$$

where

$$\hat{x}f(x) = xf(x)$$

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The paper is in final form and no version of it will be submitted elsewhere.

for sufficiently smooth function $f : \mathbf{F} \to \mathbf{F}$.
After substitution into Bernoulli identity and application to function f as above we get:

$$D\sum_{k=0}^{n} \frac{(y-x)(D^k f)(x)}{k!} = \frac{(y-x)^n}{n!}(D^{k+1} f)(x).$$

Now after integration $\int_y^x dt$ we get:

$$f(y) = \sum_{k=0}^{n} \frac{(y-x)^k}{k!} f^{(k)}(x) + \int_x^y \frac{(y-t)^n}{n!} \dot{f}^{(n+1)}(t)\, dt$$

what is well known Bernoulli-Taylor formula with the rest term of the Cauchy type.

2. ∂_ψ-UMBRAL CALCULUS

Now we shall present some definitions and theorems of ∂_ψ-umbral calculus. One can find more of them in [3], [6], [7].

We shall denote by $\mathbf{P}$ the algebra of polynomials over the field $\mathbf{F}$ of characteristic zero. Let us consider a one parameter family $\mathcal{F}$ of sequences. Then Ψ is called admissible if $\Psi \in \mathcal{F}$. Where:

$$\mathcal{F} = \{\Psi : \mathbf{R} \supset [a,b];\ q \in [a,b] : \Psi(q) : \mathbf{Z} \to \mathbf{F};\ \Psi_0(q) = 1,$$
$$\Psi_n(q) \neq 0,\ \Psi_{-n}(q) = 0,\ n \in \mathbf{N}\}.$$

Now let us to introduce the Ψ-notation:

$$n_\psi = \Psi_{n-1}(q)\Psi_n^{-1}(q);$$

$$n_\psi! = n_\psi(n-1)_\psi \cdots 2_\psi 1_\psi = \Psi_n^{-1}(q),$$

$$n_\psi^{\underline{k}} = n_\psi(n-1)_\psi \cdots (n-k+1)_\psi,$$

$$\binom{n}{k}_\psi = \frac{n_\psi^{\underline{k}}}{k_\psi!},$$

$$\exp_\psi\{y\} = \sum_{k=0}^{\infty} \frac{y^k}{k_\psi!}$$

Definition 2.1. *Let $\partial_\psi : \mathbf{P} \to \mathbf{P}$ and $\partial_\psi x^n = n_\psi x^{n-1}$; ∂_ψ-linearly extended is called the Ψ-derivative.*

Definition 2.2. *The $\hat{x}_\psi$-operator (∂_ψ-multiplication operator) is the linear map such that*

$$\hat{x}_\psi x^n = \frac{n+1}{(n+1)_\psi} x^{n+1} \quad n \geq 0.$$

Note that $[\partial_\psi, \hat{x}_\psi] = 1$.

Let us to introduce Ψ-multiplication $*_\psi$ of functions as specified below:

Definition 2.3.

$$x *_\psi x^n = \hat{x}_\psi(x^n) = \frac{(n+1)}{(n+1)_\psi} x^{n+1} \quad n \geq 0, \tag{6}$$

$$x^n *_\psi x = \hat{x}^n_\psi(x) = \frac{(n+1)!}{(n+1)_\psi!} x^{n+1} \quad n \geq 0. \tag{7}$$

Therefore

$$x *_\psi \alpha 1 = \alpha 1 *_\psi x = \alpha *_\psi x \tag{8}$$

and

$$f(x) *_\psi x^n = F(\hat{x}_\psi) x^n.$$

Note 2.1. For $k \neq n$, $x^n *_\psi x^k \neq x^k *_\psi x^n$ as well as $x^n *_\psi x^k \neq x^{n+k}$ – in general i.e. for arbitrary admissible Ψ.

Definition 2.4. *Let us to define $*_\psi$-powers of x according to:*

$$x^{n*_\psi} = x *_\psi x^{(n-1)*_\psi} = \hat{x}_\psi(x^{(n-1)*_\psi}) = x *_\psi x *_\psi \cdots *_\psi x = \frac{n!}{n_\psi!} x^n; \quad n \geq 0. \tag{9}$$

Note 2.2. Note that

$$x^{n*_\psi} *_\psi x^{k*_\psi} = \frac{n!}{n_\psi!} x^{(n+k)*_\psi} \neq \frac{k!}{k_\psi!} x^{(n+1)*_\psi} = x^{k*_\psi} *_\psi x^{n*_\psi}$$

for $k \neq n$ and $x^{0*_\psi} = 1$.

This noncommutative Ψ-multiplication $*_\psi$ is devised so as to ensure the following observations:

Observation 2.1. Let f, g be formal series. Then:

(a) $\partial_\psi x^{n*_\psi} = n x^{(n-1)*_\psi}$; $n \geq 0$;
(b) $\exp_\psi[\alpha x] = \exp \alpha \hat{x}_\psi 1$;
(c) $\exp[\alpha x] *_\psi \exp_\psi \beta \hat{x}_\psi 1 = \exp_\psi [\alpha + \beta] \hat{x}_\psi 1$;
(d) $\partial_\psi (x^k *_\psi x^{n*_\psi}) = (Dx^k) *_\psi x^{n*_\psi} + x^k *_\psi (\partial_\psi x^{n*_\psi})$;
(e) $\partial_\psi (f *_\psi g) = (Df) *_\psi g + f *_\psi (\partial_\psi g)$; ($\partial_\psi$-Leibnitz rule);
(f) $f(\hat{x}_\psi) g(\hat{x}_\psi) 1 = f(x) *_\psi \tilde{g}$; $\tilde{g}(x) = g(\hat{x}_\psi) 1$.

3. Ψ-INTEGRATION

Now let us to define Ψ-integration which is a right inverse operation to Ψ-derivative, i.e.

$$\partial_\psi \circ \int d_\psi t = id.$$

Note that $\partial_\psi = \hat{n}_\psi \partial_0$ where $\hat{n}_\psi x^n = (n+1)_\psi x^n$; $n \geq 1$ and $\partial_0 x^n = x^{n-1}$; in general $(\partial_0 f)(x) = \frac{1}{x}(f(x) - f(0))$.

Definition 3.1. *We define Ψ-integral as a linear operator such that*

$$\int x^n \, d_\psi x = [\hat{x} \frac{1}{\hat{n}_\psi}] x^n = \hat{x}(\frac{1}{(n+1)_\psi} x^n) = \frac{1}{(n+1)_\psi} x^{n+1}; \ n \geq 0 \tag{10}$$

for $\hat{x}$ as in Example 1.1.

Note 3.1. Also note that :

$$\partial_\psi \circ \int_\alpha^x f(t)\, d_\psi t = f(x) \tag{11}$$

and

$$\int_\alpha^x (\partial_\psi f)(t)\, d_\psi t = f(x) - f(\alpha) \tag{12}$$

for every formal series f.

Observation 3.1. The following formula for integration "per partes" holds:

$$\int_\alpha^\beta (f *_\psi \partial_\psi g)(x)\, d_\psi x = [(f *_\psi g)(x)]_\alpha^\beta - \int_\alpha^\beta ((Df) *_\psi g)(x)\, d_\psi x.$$

4. ∂_ψ-Bernoulli-Taylor formula

Let us to return to Bernoulli identity

$$\hat{p} \sum_{k=0}^{n} \frac{(-\hat{q})^k \hat{p}^k}{k!} = \frac{(-\hat{q})^n \hat{p}^{n+1}}{n!}. \tag{13}$$

Now let $\hat{p}$ and $\hat{q}$ be as below:

$$\hat{p} = \partial_\psi, \quad \hat{q} = \hat{z} - \psi = \hat{x}_\psi - y, \quad y \in \mathbf{F}. \tag{14}$$

Note that $\partial_\psi, \hat{z}_\psi = id$. After submission into (13) we get:

$$\partial_\psi \sum_{k=0}^{n} \frac{(y - \hat{x}_\psi)^k (\partial_\psi^k f)(x)}{k!} = \frac{(y - \hat{x}_\psi)^n (\partial_\psi^{n+1} f)(x)}{n!}. \tag{15}$$

Using (6)–(9) one can get equivalent identity:

$$\partial_\psi \sum_{k=0}^{n} \frac{(y - x)^{k *_\psi} *_\psi (\partial_\psi^k f)(x)}{k!} = \frac{(y - x)^{n *_\psi} *_\psi (\partial_\psi^{n+1} f)(x)}{n!}. \tag{16}$$

After integration $\int_\alpha^x d_\psi x$ using (11),(12) it gives ∂_ψ-Bernoulli-Taylor formula of the form:

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} (x - \alpha)^{k *_\psi} *_\psi (\partial_\psi^k f)(\alpha) + R_{n+1}(x) \tag{17}$$

with the rest term of the Cauchy type of the form:

$$R_{n+1}(x) = \frac{1}{n!} \int_\alpha^x (x - t)^{n *_\psi} *_\psi (\partial_\psi^{n+1} f)(t)\, d_\psi t. \tag{18}$$

Remark 4.1. In [1] is presented special case of (17),(18) which is ∂_q-Bernoulli-Taylor formula of the form:

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} (x - \alpha)^{k *_q} *_q (\partial_q^k f)(\alpha) + R_{n+1}(x) \tag{19}$$

with

$$R_{n+1}(x) = \frac{1}{n!} \int_\alpha^x (x - t)^{n *_q} *_q (\partial_q^{n+1} f)(t)\, d_q t, \tag{20}$$

where

$$(\partial_q f)(t) = \frac{f(t) - f(qt)}{(1-q)t} \tag{21}$$

and one can get it by the choice $n_\psi = n_q = 1 + q + \cdots + q^{n-1}$.

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