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Re-establishing supersymmetry between harmonic oscillators in $D \neq 1$ dimensions

RE-ESTABLISHING SUPERSYMMETRY BETWEEN HARMONIC OSCILLATORS IN $D \neq 1$ DIMENSIONS

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ABSTRACT. We offer a plausible resolution of the paradox (first formulated by Jevicki and Rodriguez in Phys. Lett. B 146, 55 (1984)) that the two shifted harmonic oscillator potentials $V(q) = q^2 + G/q^2 + const$ may, in spite of their exact solvability in a non-empty interval of the couplings G, become supersymmetric partners if and only if G vanishes. We show that and how their $G \neq 0$ SUSY may be re-established via a regularization provided by pseudo-Hermitian quantum mechanics.

1. MOTIVATION

A suitable algebraic background of the theoretical construction of multiplets which would unify some experimentally observable bosons with fermions is provided by the graded Lie algebras, so called superalgebras. In such a setting, an exceptional role is played by the linear harmonic oscillator in one dimension, $D = 1$. Indeed, its Hamiltonian may be factorized and, subsequently, shifted to the left or to the right,

$$H^{(LHO)} = p^2 + q^2 = A \cdot B - 1 = B \cdot A + 1, \qquad A = q + ip, \quad B = q - ip$$

$$H_{(L)} = H^{(LHO)} - 1 = B \cdot A, \qquad H_{(R)} = H^{(LHO)} + 1 = A \cdot B.$$

According to Witten [1] one may then introduce a "super-Hamiltonian" and two "supercharges",

$$\mathcal{H} = \begin{bmatrix} H_{(L)} & 0 \\ 0 & H_{(R)} \end{bmatrix}, \qquad \mathcal{Q} = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \qquad \tilde{\mathcal{Q}} = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$$

and notice that they generate a representation of Lie superalgebra sl(1/1),

$$\{\mathcal{Q}, \tilde{\mathcal{Q}}\} = \mathcal{H}, \qquad \{\mathcal{Q}, \mathcal{Q}\} = \{\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}\} = 0, \qquad [\mathcal{H}, \mathcal{Q}] = [\mathcal{H}, \tilde{\mathcal{Q}}] = 0.$$

In the language of physics, one can speak about the bosonic *and* fermionic vacuum annihilated by both the supercharges,

$$\langle q|0,0\rangle = \begin{bmatrix} \exp(-q^2/2)/\sqrt{\pi} \\ 0 \end{bmatrix}, \qquad \mathcal{Q}|0,0\rangle = \tilde{\mathcal{Q}}|0,0\rangle = 0.$$

The "bosons" themselves may be then introduced as created and/or annihilated by the first-order differential operators $\mathbf{a}^\dagger \sim B$ and/or $\mathbf{a} \sim A$, respectively. The parallel creation, annihilation and occupation-number operators are also easily defined for "fermions",

$$\mathcal{F}^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \mathcal{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{N}_\mathcal{F} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The supercharges become factorized as well, $Q \sim \mathbf{a}\mathcal{F}^\dagger$ and $\tilde{Q} \sim \mathbf{a}^\dagger \mathcal{F}$. In terms of the harmonic-oscillator eigenstates $|n\rangle$ the Fock space will be spanned by the states $|n_b, n_f\rangle$ characterized by the presence of n_b bosons and n_f fermions where n_f is equal to zero or one,

$$|n, 0\rangle = \begin{bmatrix} |n\rangle \\ 0 \end{bmatrix}, \qquad |n, 1\rangle = \begin{bmatrix} 0 \\ |n\rangle \end{bmatrix}.$$

In this way the harmonic oscillator may be understood as a next-to-trivial supersymmetric field theory in a one-dimensional space-time which unifies the bosons with fermions. More details may be found, e.g., in the concise review paper [2].

In what follows, we shall analyze what happens if we replace the one-dimensional $H^{(LHO)}$ by its radial, D-dimensional generalization with any real $D > 2$,

$$H^{(\alpha)} = -\frac{d^2}{dq^2} + \frac{\alpha^2 - 1/4}{q^2} + q^2, \qquad \alpha = (D-2)/2 + \ell, \qquad \ell = 0, 1, \dots.$$

We shall be guided by the Witten's supersymmetric quantum mechanics where the use of the operators $A = \partial_q + W$ and $B = -\partial_q + W$ with arbitrary superpotentials W leads to the same supersymmetric pattern as above, forming the superHamiltonian from the two partner operators

$$H_{(L)} = B \cdot A = \hat{p}^2 + W^2 - W', \qquad H_{(R)} = A \cdot B = \hat{p}^2 + W^2 + W'.$$

In the spirit of our recent letter [3] we shall admit that these operators are pseudo-Hermitian [4].

2. BOUND STATES IN THE PSEUDO-HERMITIAN SETTING

Beyond the elementary harmonic oscillator let us now contemplate a generalized superpotential

$$(1) \qquad\qquad W^{(\gamma)}(r) = r - \frac{\gamma + 1/2}{r} \qquad r = r(x) = x - i\varepsilon.$$

In the other words, we assume that we start from the choice of a real parameter γ and *define* the pair $H_{(L,R)}$ of non-Hermitian operators. Such a recipe gives the partner Hamiltonians in the above D-dimensional harmonic oscillator form where $D_L \neq D_R$ in general,

$$(2) \quad H_{(L)}^{(\gamma)} = H^{(\alpha)} - 2\gamma - 2, \qquad H_{(R)}^{(\gamma)} = H^{(\beta)} - 2\gamma, \qquad \alpha = |\gamma|, \qquad \beta = |\gamma + 1|.$$

In the light of ref. [5] the complexified line of coordinates $r = x - i\varepsilon$ circumvents the singularity in the origin so that the bound state wavefunctions are regular and expressible in terms of the Laguerre polynomials,

$$\psi(r) = \frac{N!}{\Gamma(N + \varrho + 1)} \cdot r^{\varrho+1/2} \exp(-r^2/2) \cdot L_N^{(\varrho)}(r^2).$$

Together with their energies

$$E = E_N^{(\varrho)} = 4N + 2\varrho + 2, \qquad \varrho = -Q \cdot \alpha, \qquad Q = \pm 1, \qquad N = 0, 1, \dots$$

these states are numbered by the integer N and by the so called quasi-parity Q.

3. SUPERSYMMETRY, PSEUDO-HERMITIAN WAY

For reasons explained in ref. [5] we must assume that $\gamma \neq 0, 1, 2, \dots$. Up to that constraint, we may visualize the above construction as one of the regularizations recommended in the recent literature [6]. Here we intend to summarize and discuss the subject in more detail.

In the first step we notice that the quasi-parity Q coincides with the ordinary spatial parity P in the limit $\alpha \to 1/2$. In such a limit the basis states are well known (cf. Appendix A). Once we move to $\alpha \neq 1/2$ we notice that the quasi-even states $\psi(r) \sim r^{1/2-\alpha}$ still lie below their quasi-odd complements $\psi(r) \sim r^{1/2+\alpha}$ at any fixed N.

Whenever we choose $\alpha \geq 1$, the limiting transition $\varepsilon \to 0$ moves the quasi-even solutions out of the Hilbert space completely. Otherwise, these states remain normalizable in a way depicted in Figure 1 where the following ordering is obtained for the N-th bunch of the energy levels,

$$(3) \qquad E_{(L)}^{(-\alpha)} \ [\equiv a(N)] \leq E_{(R)}^{(-\beta)} \ [\equiv b(N)] \leq E_{(L)}^{(+\alpha)} \ [\equiv c(N)] \leq E_{(R)}^{(+\beta)} \ [\equiv d(N)].$$

This ordering is preserved along all their γ-dependent variation. Each of these four curves is just a once broken straight line but, in our picture, our eyes are guided by an infinitesimal shift of the levels in such a way that their shape may be easily followed (one should only recollect that the system remains undefined at all the integers $\gamma \in \mathbb{N}$).

In the Figure the physical, Hermitian limiting transition $\varepsilon \to 0$ has been performed. The general, $\varepsilon \neq 0$ has been discussed elsewhere [7]. We may only note here that in contrast to the latter and manifestly non-Hermitian, $\varepsilon \neq 0$ scheme of ref. [7], all our present states belong to the Hilbert space of the ordinary quantum mechanics. Thus, our new scheme may be interpreted as a result of a pseudo-Hermitian regularization recipe studied, in more detail, elsewhere [8] (cf. also Appendix B for some more details).

The inspection of Figure 1 reveals a certain generalized supersymmetry (SUSY) where the standard requirements of quadratic integrability tolerate the quasi-odd levels at all γ but confine the existence of the levels $a(n)$ to the very short interval of $\gamma \in (-1, 1)$ and the existence of the levels $b(n)$ to the interval of $\gamma \in (-2, 0)$. As a consequence, one has to distinguish between the following five mutually significantly different regimes.

(1) "Far left" with $\gamma \in (-\infty, -2)$ and with the complete degeneracy

$$E_{(L)}^{(+\alpha)} \ [\equiv c(N)] = E_{(R)}^{(+\beta)} \ [\equiv d(N)]$$

where the Witten's index vanishes [3] and where SUSY itself is broken because the ground state energy remains positive. All the spectrum is equidistant.

(2) "Near left" with $\gamma \in (-2, -1)$ and with the mere partial degeneracy which survives from the preceding interval. There emerges the new series of energies $E_{(R)}^{(-\beta)}$ $[\equiv b(N)]$ without any left partners; this possibility represents just a weaker form (and/or a more singular analogue) of the Jevicki-Rodriguez breakdown of SUSY [9] as mentioned above in Abstract.

(3) "Central domain" with $\gamma \in (-1, 0)$. This is the most interesting domain where the properties of the well known linear special case $H^{(LHO)}$ (which has $\gamma = -1/2$) appear generalized to the whole neighboring interval. Up to the exceptional (and newly emerging) ground state $a(0)$ we may spot here the well known pattern of degeneracy,

$$E_{(L)}^{(-\alpha)} \ [\equiv a(N)] < E_{(R)}^{(-\beta)} \ [\equiv b(N)] = E_{(L)}^{(+\alpha)} \ [\equiv c(N)] <$$

$$< E_{(R)}^{(+\beta)} \ [\equiv d(N)] = a(N+1) < \ldots$$

so that SUSY becomes unbroken even for the spectrum which ceased to be equidistant at $\gamma \neq -1/2$.

(4) "Near right" with $\gamma \in (0, 1)$ and with the properties which "mirror" the far left (the series of energies $E_{(R)}^{(-\beta)}$ $[\equiv b(N)]$ ceases to exist, etc).

(5) "Far right" with $\gamma \in (1, \infty)$, degeneracy

$$E_{(L)}^{(+\alpha)} \ [\equiv c(N)] = d(N-1), \quad N > 0$$

and with the characteristic γ-independence of the almost completely degenerate unbroken SUSY spectrum.

We may summarize that the resulting SUSY pattern is fairly unusual. It may be characterized by several above-listed appealing properties but one should re-emphasize, first of all, that near $\gamma = -1/2$ a nice non-equidistant generalization of the textbook $D = 1$ SUSY oscillators is obtained.

FIGURE CAPTIONS

Figure 1. SUSY and the γ-dependence of the spectrum which is generated by the superpotential (1).

Figure 2. Graphical solution of the selfconsistency condition $E(\varrho) = \varrho$ in the schematic example of Appendix B.3. with $g_{N-k} = 1$ and $f_k = 3$.

(A) curve (11) at $a_k = 0.7$ in the Hermitian regime,

(B) curve (11) at $a_k = 0.7$ in the non-Hermitian regime (both energies are real),

(C) curve (11) at $a_k = 1.4$ in the non-Hermitian regime (no real root, both energies are complex),

(D) selfconsistency line $E(\varrho) = \varrho$.

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Appendix A: The standard harmonic oscillator basis on $L_2(\mathbb{R})$

Eigenstates of a Hamiltonian $H(g)$ which commutes with the parity $\mathcal{P}$ may be numbered by an integer n and by the superscript $\pm$ which characterizes the spatial parity of the state,

$$(4) \qquad H(g) \, |n^{(\pm)}(g)\rangle = E_n^{(\pm)}(g) \, |n^{(\pm)}(g)\rangle$$

These eigenstates form a complete basis in the Hilbert space $L_2(\mathbb{R})$,

$$(5) \qquad \sum_{\sigma=\pm} \sum_{m=0}^{\infty} |m^{(\sigma)}(g)\rangle \, \langle m^{(\sigma)}(g)| = I.$$

The usual condition of their orthonormalization reads

$$\langle n^{(\tau)}(g)|\, m^{(\sigma)}(g)\rangle = \int_{-\infty}^{\infty} \langle n^{(\tau)}(g)|\, x\rangle\langle x|\, m^{(\sigma)}(g)\rangle \; dx = \delta_{mn}\delta_{\sigma,\tau}.$$

For the particular and exceptional harmonic oscillator $H(0) \equiv H^{(LHO)}$ these eigenstates are proportional to the well known Hermite polynomials,

$$(6) \qquad \begin{aligned} \langle x|n^{(+)}(0)\rangle &= \mathcal{N}_{2n}\mathcal{H}_{2n}(x) \, \exp(-x^2/2) \equiv \langle x|s_n\rangle, \\ \langle x|n^{(-)}(0)\rangle &= \mathcal{N}_{2n+1}\mathcal{H}_{2n+1}(x) \, \exp(-x^2/2) \equiv \langle x|t_n\rangle, \\ x \in \mathbb{R}, \qquad \mathcal{N}_n &= \left(\sqrt{2^n \, n! \, \sqrt{\pi}}\right)^{-1}, \qquad n = 0, 1, \ldots . \end{aligned}$$

At each particular subscript $n = m$ the pairs of the latter harmonic-oscillator basis states have an opposite parity, $\mathcal{P}\,|s\rangle = +|s\rangle$, $\mathcal{P}\,|t\rangle = -|t\rangle$. They may be transformed into the unitarily equivalent pairs of states

$$\begin{pmatrix} |S\rangle \\ |T\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} |s\rangle \\ |t\rangle \end{pmatrix}$$

with the real $\mathcal{PT}$ parities, $\mathcal{PT}\,|S\rangle = +|S\rangle$, $\mathcal{PT}\,|T\rangle = -|T\rangle$ where the complex conjugation $\mathcal{T}$ defined by the simple rule $\mathcal{T}i\mathcal{T} = -i$ mimics the usual antilinear time reversal.

<center>APPENDIX B: MAIN DIFFERENCES BETWEEN THE HERMITIAN AND NON-HERMITIAN HAMILTONIANS</center>

B. 1: $\mathcal{P}$-symmetric models and the bases on $L_2(\mathbb{R}^+)$. Any eigenstate $|\psi\rangle$ of $H = H^\dagger = \mathcal{P}H\mathcal{P}$ satisfies its Schrödinger equation even after a pre-multiplication by the parity $\mathcal{P}$. Both the old and new eigenstates belong to the same real eigenvalue E which cannot be degenerate due to the Sturm-Liouville oscillation theorems. One of the superpositions $|\psi\rangle \pm \mathcal{P}|\psi\rangle$ must vanish while the other one acquires a definite parity. This is the essence of the mathematical proof that the $\mathcal{P}$ symmetry of wave functions cannot be spontaneously broken, $\mathcal{P}|n^{(\pm)}(g)\rangle = \pm|n^{(\pm)}(g)\rangle$.

The knowledge of the spatial symmetry of the Hermitian Hamiltonian $H(g)$ enables us to simplify many considerations and calculations by choosing and fixing the parity of the solutions in advance,

$$\langle x|\,n^{(\pm)}(g)\rangle = \pm\langle(-x)|\,n^{(\pm)}(g)\rangle.$$

This permits us to live, conveniently, on the semi-axis of $x \in (0,\infty) = \mathbb{R}^+$. In such a setting we rarely imagine that we are *tacitly changing the Hilbert space* from $L_2(\mathbb{R})$ to $L_2(\mathbb{R}^+)$. We feel that this is a technicality which deserves a separate remark.

On the new space (or, if you wish, in the old space equipped by the projector or singular metric Π) the inner product changes its meaning since we have to integrate over the mere semi-axis (symbolically, $\langle\psi|\psi'\rangle \rightarrow \langle\psi|\Pi|\psi'\rangle$). This re-scales the orthogonality relations,

$$\langle n^{(\sigma)}(g)|\,\Pi\,|\,m^{(\sigma)}(g)\rangle = \int_0^\infty \langle n^{(\sigma)}(g)|\,x\rangle\langle x|\,m^{(\sigma)}(g)\rangle\;dx = \frac{1}{2}\delta_{mn}.$$

Alternatively, we may omit the symbols Π and switch to the two re-normalized bases

$$\langle x|\sigma_n\rangle = \mathcal{M}_{2n}\mathcal{H}_{2n}(x)\,\exp(-x^2/2),$$
$$\langle x|\tau_n\rangle = \mathcal{M}_{2n+1}\mathcal{H}_{2n+1}(x)\,\exp(-x^2/2),$$
$$x \in \mathbb{R}^+, \qquad \mathcal{M}_n = \left(\sqrt{2^{n-1}\,n!\,\sqrt{\pi}}\right)^{-1}, \quad n = 0,1,\ldots$$

which are both orthonormal on the half-line. In parallel, condition (5) splits in the two independent completeness relations

$$\sum_{m=0}^\infty |\,\sigma_m\rangle\,\langle\sigma_m| = I, \qquad \sum_{n=0}^\infty |\,\tau_n\rangle\,\langle\tau_n| = I.$$

The overlaps of the states with different superscripts do not vanish and form a unitary matrix which changes the basis in $L_2(\mathbb{R}^+)$. Its elements

$$(7) \qquad \mathcal{U}_{n,m} = 2\int_0^\infty \langle n^{(+)}(0)|\,x\rangle\langle x|\,m^{(-)}(0)\rangle\;dx = 2\,\langle s_n\,|\Pi|\,t_m\rangle = \langle\sigma_n\,|\,\tau_m\rangle$$

may be computed by the direct symbolic integration in MAPLE giving the exact values sampled in Table 1 from which one may extract some closed formulae, e.g.,

$$\langle s_n | \Pi | t_n \rangle = \frac{(2n)! \sqrt{4n+2}}{2^{2n+1}(n!)^2 \sqrt{\pi}} = \frac{\sqrt{n+\frac{1}{2}}}{\pi} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}.$$

This means that the original vectors (6) form an over-complete set and we may make a choice between the two alternative basis sets $\{|\sigma\rangle\}$ and $\{|\tau\rangle\}$. They are both complete on the new Hilbert space $L_2(\mathbb{R}^+)$.

B. 2: $\mathcal{PT}$-symmetric models. The above-mentioned rigidity of the conservation of parity is lost during the transition to the $\mathcal{PT}$ symmetric models $H = H^\ddagger = \mathcal{PT}H\mathcal{PT}$ where any quantity $\exp(i\varphi)$ is an admissible eigenvalue of the operator $\mathcal{PT}$ since its component $\mathcal{T}$ is defined as anti-linear, $\mathcal{T}i = -i$. In more detail, every rule $\mathcal{PT}|\psi\rangle = \exp(i\varphi)|\psi\rangle$ implies that we have

$$\mathcal{PT}\mathcal{PT}|\psi\rangle = \exp(-i\varphi)\,\mathcal{PT}|\psi\rangle = |\psi\rangle$$

as required. The Schur's lemma ceases to be applicable. In the basis of Appendix A with the properties $\mathcal{PT}|S\rangle = |S\rangle$ and $\mathcal{PT}|L\rangle = -|L\rangle$, the general expansion formula

$$H = \sum_{m,n=0}^{\infty} \left(\; |S_m\rangle \mathcal{F}_{m,n}\langle S_n| \; + \; |L_m\rangle \mathcal{G}_{m,n}\langle L_n| \; + \; i\,|S_m\rangle \mathcal{C}_{m,n}\langle L_n| \; + \; i\,|L_m\rangle \mathcal{D}_{m,n}\langle S_n| \; \right)$$

contains four separate complex matrices of coefficients. Once it is subdued to the requirement $H = \mathcal{PT}H\mathcal{PT}$, we get the necessary and sufficient condition demanding that all the above matrix elements of $H = H^\ddagger$ must be real,

(8) $\qquad \mathcal{F}_{m,n} = \mathcal{F}_{m,n}^*, \qquad \mathcal{G}_{m,n} = \mathcal{G}_{m,n}^*, \qquad \mathcal{C}_{m,n} = \mathcal{C}_{m,n}^*, \qquad \mathcal{D}_{m,n} = \mathcal{D}_{m,n}^*.$

As long as the similar trick has led to the superselection rules for the spatially symmetric Hamiltonians, we may conclude that the $\mathcal{PT}$ symmetric analogue of the direct-sum decompositions and superselection rule is just the much weaker constraint (8).

B. 3: Schematic finite-dimensional matrix model with and without $\mathcal{PT}$ symmetry. Let us contemplate the partitioned matrix Schrödinger equation

(9) $\qquad \begin{pmatrix} F - EI & \alpha A \\ A^\dagger & G - EI \end{pmatrix} \cdot \begin{pmatrix} \vec{u} \\ \vec{w} \end{pmatrix} = 0$

where $F = F^\dagger$, $G = G^\dagger$ and either $\alpha = 1$ (Hermitian case) or $\alpha = -1$ ($\mathcal{PT}$ symmetric case). Schrödinger equations with the matrix representation (9) generalize the models with $\mathcal{PT}$ symmetry [10]. Their spectrum may happen to be real and discrete, at least in the limit $A \to 0$, or containing the complex conjugate pairs. Let us now describe their nontrivial, non-perturbative solvable example.

Preliminarily, both the Hermitian submatrices F and G of the Hamiltonian should be diagonalized via a pair of some suitable unitary transformations, $F \to \hat{f}$, $G \to \hat{g}$. Their respective spectra $\{f_n\}$ and $\{g_n\}$ will be assumed real and discrete.

Secondly, we shall ignore all the small elements of the coupling matrix $\mathcal{A}$ in our pre-diagonalized effective Schrödinger equation (9),

(10)
$$\left(\hat{f} - E\, I - \alpha\, \mathcal{A}\, \frac{1}{\hat{g} - E\, I}\, \mathcal{A}^{\dagger} \right) \vec{y} = 0.$$

Here, $\alpha = \pm 1$ "remembers" its respective Hermitian and non-Hermitian origin and all the small elements in $\mathcal{A}$ are irrelevant causing just a perturbative, small deformation of the decoupled spectrum $\{f_n\} + \{g_n\}$.

Thirdly, let us choose the latter coupling matrix in the form dominated by the transposed diagonal,

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & \ldots & 0 & a_0 \\ 0 & \ldots & 0 & a_1 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & a_{N-1} & 0 & \ldots & 0 \\ a_N & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Quite unexpectedly, this choice makes the problem exactly solvable since the secular equation $\det(H_{eff}(\varrho) - E\, I) = 0$ can be immediately factorized,

$$0 = \left(f_0 - E - \alpha \frac{|a_0|^2}{g_N - \varrho} \right) \cdot \left(f_1 - E - \alpha \frac{|a_1|^2}{g_{N-1} - \varrho} \right) \cdots \cdots \left(f_N - E - \alpha \frac{|a_N|^2}{g_0 - \varrho} \right).$$

The explicit evaluation of zeros of the k-th factor is trivial,

(11)
$$E(\varrho) = f_k - \alpha\, a_k\, \frac{1}{g_{N-k} - \varrho}\, a_k^*.$$

The implementation of the selfconsistency $\varrho = E(\varrho)$ gives the sequence of the mere quadratic algebraic equations

$$f_k - E - \alpha\, a_k\, \frac{1}{g_{N-k} - E}\, a_k^* = 0.$$

All their roots are available in closed form,

$$E_{k\pm} = \frac{1}{2} \left(f_k + g_{N-k} \pm \sqrt{(f_k - g_{N-k})^2 + 4\alpha\, |a_k|^2} \right).$$

This confirms our *a priori* expectations since the Hermitian energies with $\alpha = 1$ are always real while, at $\alpha = -1$, we get the real spectrum if and only if

(12)
$$|a_k| < |f_k - g_{N-k}|/2, \quad k = 0, 1, \ldots, N.$$

Vice versa, we get a complex conjugate pair $E_{k\pm}$ whenever we move to the strongly non-Hermitian regime and encounter a large and strong off-diagonality or coupling of modes in $\mathcal{A}$. This is an independent linear-algebraic re-confirmation of the similar observations made during the explicit computations using differential equations.

Table 1. Overlaps $\sqrt{\pi}\,\langle s_n|\Pi| t_m\rangle$ defined by eq. (7). Rows are numbered by $n = 0, 1, \ldots, 6$, columns by $m = 0, 1, \ldots, 4$.

$1/2\sqrt{2}$	$-1/6\sqrt{3}$	$1/20\sqrt{15}$	$-\frac{1}{56}\sqrt{70}$	$1/48\sqrt{35}$
$1/2$	$1/4\sqrt{3}\sqrt{2}$	$-1/24\sqrt{15}\sqrt{2}$	$\frac{1}{80}\sqrt{70}\sqrt{2}$	$-\frac{3}{224}\sqrt{35}\sqrt{2}$
$-1/24\sqrt{2}\sqrt{6}$	$1/8\sqrt{3}\sqrt{6}$	$1/16\sqrt{15}\sqrt{6}$	$-\frac{1}{96}\sqrt{70}\sqrt{6}$	$\frac{3}{320}\sqrt{35}\sqrt{6}$
$1/40\sqrt{2}\sqrt{5}$	$-1/24\sqrt{3}\sqrt{5}$	$1/16\sqrt{15}\sqrt{5}$	$1/32\sqrt{70}\sqrt{5}$	$-\frac{1}{64}\sqrt{35}\sqrt{5}$
$-\frac{1}{224}\sqrt{70}\sqrt{2}$	$\frac{1}{160}\sqrt{3}\sqrt{70}$	$-\frac{1}{192}\sqrt{15}\sqrt{70}$	$\frac{35}{64}$	$\frac{3}{256}\sqrt{35}\sqrt{70}$
$\frac{1}{96}\sqrt{2}\sqrt{7}$	$-\frac{3}{224}\sqrt{3}\sqrt{7}$	$\frac{3}{320}\sqrt{15}\sqrt{7}$	$-\frac{1}{128}\sqrt{70}\sqrt{7}$	$\frac{9}{256}\sqrt{35}\sqrt{7}$
$-\frac{1}{704}\sqrt{2}\sqrt{231}$	$\frac{1}{576}\sqrt{3}\sqrt{231}$	$-\frac{1}{896}\sqrt{15}\sqrt{231}$	$\frac{1}{1280}\sqrt{70}\sqrt{231}$	$-\frac{1}{512}\sqrt{35}\sqrt{231}$

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