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NILPOTENT SPACELIKE JORDAN OSSERMAN PSEUDO-RIEMANNIAN MANIFOLDS

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ABSTRACT. Pseudo-Riemannian manifolds of balanced signature which are both spacelike and timelike Jordan Osserman nilpotent of order 2 and of order 3 have been constructed previously. In this short note, we shall construct pseudo-Riemannian manifolds of signature $(2s, s)$ for any $s \geq 2$ which are spacelike Jordan Osserman nilpotent of order 3 but which are not timelike Jordan Osserman. Our example and techniques are quite new because they are adapted to a completely new situation.

1. INTRODUCTION

Let (M, g) be a pseudo-Riemannian manifold of signature (p, q). Let

$$S^{\pm}(M, g) := \{x \in TM : (x, x) = \pm 1\}$$

be the bundles of unit spacelike and unit timelike vectors, respectively. Let R be the associated Riemann curvature tensor. If $x \in T_PM$, then the *Jacobi operator* $J(x)$ is the self-adjoint linear map of T_PM which is characterized by the identity:

$$g(J(x)y, z) = R(y, x, x, z)\,. \tag{1.a}$$

One says that (M, g) is *spacelike Osserman* or *timelike Osserman* if the eigenvalues of J are constant on $S^+(M, g)$ or on $S^-(M, g)$, respectively. These are equivalent notions if $p \geq 1$ and $q \geq 1$ [12] so such manifolds are simply said to be *Osserman*.

If $p = 0$, and similarly if $q = 0$, then one is in the *Riemannian setting*. If (M, g) is a rank 1 symmetric space or if (M, g) is flat, then the local isometries of (M, g) act transitively on $S^+(M, g)$ so the eigenvalues of J are constant on $S^+(M, g)$. Osserman [19] wondered if the converse held. Work of Chi [7] and of Nikolayevsky [17] has shown this to be the case if the dimension is different from 8 and 16.

If $p = 1$, and similarly if $q = 1$, then one is in the *Lorentzian setting*. Blažić, Bokan and Gilkey [1] and García-Río, Kupeli and Vázquez-Abal [9] have shown that Lorentzian Osserman manifolds have constant sectional curvature.

The situation is quite different in the higher signature setting where $p \geq 2$ and $q \geq 2$. There exist Osserman pseudo-Riemannian manifolds which are not symmetric spaces

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[2, 3, 4, 5, 11]; we refer to [10] for an excellent and quite comprehensive treatment of the subject.

In the higher signature setting, it is natural to impose a more restrictive hypothesis and study the Jordan normal form of the Jacobi operator. We say that (M,g) is *spacelike Jordan Osserman* or is *timelike Jordan Osserman* if the Jordan normal form of $J(x)$ is constant on $S^+(M,g)$ or on $S^-(M,g)$, respectively. Relatively few examples of such manifolds are known.

The eigenvalue 0 is distinguished. One says that (M,g) is *nilpotent Osserman* if $J(x)^{p+q}=0$ or equivalently if 0 is the only eigenvalue of $J(x)$ for any $x\in TM$. The *orders of nilpotency* $n(x)$ and $n(M)$ are then defined by the properties:

$$J(x)^{n(x)}=0\,,\quad J(x)^{n(x)-1}\neq 0\,,\quad \text{and}\quad n(M):=\sup_{x\in TM} n(x)\,.$$

Fiedler and Gilkey [8] gave examples of m dimensional pseudo-Riemannian manifolds for any $m\geq 4$ where $n(M)=m-2$; thus $n(M)$ can be arbitrarily large. However for these examples, $n(x)$ was constant neither on $S^+(M,g)$ or on $S^-(M,g)$ so these manifolds were neither spacelike nor timelike Jordan Osserman.

Results of Gilkey and Ivanova [13] show that if (M,g) is spacelike Jordan Osserman of signature (p,q) where $p<q$, then the Jacobi operator is diagonalizable and hence (M,g) can not be nilpotent. Thus we suppose $p\geq q$ henceforth. Examples of spacelike and timelike Jordan Osserman manifolds of neutral signature (s,s) which are nilpotent of order 2 have been constructed Gilkey, Ivanova, and Zhang [14] for any $s\geq 2$. Examples of spacelike and timelike Jordan Osserman manifolds of signature $(2,2)$ which are nilpotent of order 3 have been constructed by García-Rió, Vázquez-Abal and Vázquez-Lorenzo [11]. This brief note is devoted to the proof of the following result:

Theorem 1.1. *If $s\geq 2$, then there exist pseudo-Riemannian manifolds of signature $(2s,s)$ which are spacelike Jordan Osserman nilpotent of order* 3 *and which are not timelike Jordan Osserman.*

Our examples are quite different in flavor from those described in [11, 14] in several respects. The primary feature is that we are **not** in the *balanced setting* where $p=q$; the extra timelike directions play a central role in our construction. Additionally, the examples of [11, 14] are also timelike Jordan Osserman; this is not the case for our examples.

To prove Theorem 1.1, it is convenient to work first in a purely algebraic context. In Section 2, we shall construct a family of algebraic curvature tensors R on a vector space V of signature $(2s,s)$ which are spacelike Jordan Osserman nilpotent of order 3 and which are not timelike Jordan Osserman. We complete the discussion in Section 3 by realizing this family geometrically. Our construction will show that in fact there are many such examples; although we shall use quadratic polynomials to define the metric in question, this is an inessential feature.

2. Algebraic curvature tensors

Let V be a finite dimensional real vector space which is equipped with a non-degenerate symmetric bilinear form $g(\cdot,\cdot)$ of signature (p,q). Let $R\in\otimes^4V^*$. We say that R is an *algebraic curvature tensor* if R satisfies the symmetries of the Riemann curvature tensor:

$$\begin{aligned}&R(x,y,z,w) = -R(y,x,z,w),\\ \text{(2.a)}\qquad &R(x,y,z,w) = R(z,w,x,y),\\ &R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0\,.\end{aligned}$$

The associated Jacobi operator is then defined using equation (1.a) and the notions spacelike Jordan Osserman and so forth are defined analogously.

Definition 2.1. *Let $s \geq 2$. Let $\{U_1,\dots,U_s,V_1,\dots,V_s,T_1,\dots,T_s\}$ be a basis for $\mathbb{R}^{3s}$. We let indices a,b,c,d range from 1 through s. Let $g_{ab} = g_{ba}$ be an arbitrary symmetric matrix. Define an inner product g of signature $(2s,s)$ on $\mathbb{R}^{3s}$ whose only non-zero entries are among the components:*

$$g(U_a,U_b) = g_{ab}\,,\quad g(U_a,V_b) = g(V_b,U_a) = \delta_{ab}\,,\quad g(T_a,T_b) = -\delta_{ab}\,.$$

Let $\mathfrak{R}$ and $\mathcal{R}$ be algebraic curvature tensors on $\operatorname{Span}\{U_a\} = \mathbb{R}^s$. *Define a 4 tensor $R = R_{\mathfrak{R},\mathcal{R}}$ on $\mathbb{R}^{3s}$ whose only non-zero entries are among the components*

$$\begin{aligned}R(U_a,U_b,U_c,U_d) :&= \mathfrak{R}(U_a,U_b,U_c,U_d),\\ R(U_a,U_b,U_c,T_d) &= R(U_a,U_b,T_c,U_d) = R(U_a,T_b,U_c,U_d)\\ &= R(T_a,U_b,U_c,U_d) := \mathcal{R}(U_a,U_b,U_c,U_d)\,.\end{aligned}$$

Lemma 2.2. *Let $\mathfrak{R}$ and $\mathcal{R}$ be algebraic curvature tensors on $\mathbb{R}^s$. Let $R := R_{\mathfrak{R},\mathcal{R}}$ be the associated 4 tensor on $\mathbb{R}^{3s}$ given in Definition 2.1. Then:*

1. *R is an algebraic curvature tensor on $\mathbb{R}^{3s}$.*
2. *If $\mathcal{R}(U_a,U_b,U_c,U_d) := \delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}$, then R is spacelike Jordan Osserman nilpotent of order 3 and not timelike Jordan Osserman.*

Proof. Let $\mathcal{U} := \{U_1,\dots,U_s\}$, $\mathcal{V} := \{V_1,\dots,V_s\}$, and $\mathcal{T} := \{T_1,\dots,T_s\}$. The sum of algebraic curvature tensors is again an algebraic curvature tensor. Because we have $R = R_{\mathfrak{R},0} + R_{0,\mathcal{R}}$, the problem decouples. Clearly $R_{\mathfrak{R},0}$ is an algebraic curvature tensor since we may assume $x,y,z,w \in \mathcal{U}$ in establishing the relations appearing in display (2.a). Thus we may set $\mathfrak{R} = 0$ and consider only the effect of $\mathcal{R}$ in proving assertion (1). In that case, exactly one of the vectors x,y,z,w must be taken from $\mathcal{T}$ and the remaining 3 vectors must be taken from $\mathcal{U}$. Suppose, for example, $x \in \mathcal{T}$ while $y,z,w \in \mathcal{U}$. Then replacing x by the corresponding element $\bar{x} \in \mathcal{U}$ replaces the tensor R by $\mathcal{R}$ and thus the relations of display (2.a) follow for R due to the corresponding relations for $\mathcal{R}$; Assertion (1) follows.

The tensor $\mathcal{R}$ of assertion (2) is the algebraic curvature tensor of constant sectional curvature $+1$ with respect to the standard scalar product $(U_a,U_b) = \delta_{ab}$. Consequently, it is invariant under the action of the orthogonal group $O(s)$.

Expand a spacelike vector $X \in \mathbb{R}^{3s}$ in the form $X = u_aU_a + v_aV_a + t_aT_a$ where we adopt the Einstein convention and sum over repeated indices. Then

$$g(X,X) = g_{ab}u_au_b + 2\delta_{ab}u_av_b - \delta_{ab}t_at_b\,.$$

If $\vec{u} = 0$, then $g(X,X) \leq 0$. Consequently $\vec{u} \neq 0$.

We now define a new basis of U-vectors by making an orthogonal transformation of the original U basis such that, in the new basis, we have $u_1 = 1$ and $u_\alpha = 0$ for $\alpha > 1$

(we use also rescaling of the vector X if need be). If we now define a new basis of V vectors, and that of T vectors, respectively, using the same orthogonal transformation, then the general form of g and of R remains the same with respect to the new bases. Thus we may suppose without loss of generality that $u_1 = 1$, and that $u_a = 0$ for $a > 1$. For $1 \le a, b, c, d \le s$, define $\mathcal{R}_{abcd} := \mathcal{R}(U_a, U_b, U_c, U_d)$. Then:

$$\begin{array}{lll} (J(X)U_a, U_b) = C_{ab}, & (J(X)U_a, T_b) = \mathcal{R}_{a11b}, & (J(X)U_a, V_b) = 0, \\ (J(X)T_a, U_b) = \mathcal{R}_{a11b}, & (J(X)T_a, T_b) = 0, & (J(X)T_a, V_b) = 0, \\ (J(X)V_a, U_b) = 0, & (J(X)V_a, T_b) = 0, & (J(X)V_a, V_b) = 0, \end{array}$$

where $C_{ab} = C_{ba}$ is an appropriately chosen matrix. We then have:

$$J(X)U_a = C_{ab}V_b - \mathcal{R}_{a11b}T_b, \quad J(X)T_a = \mathcal{R}_{a11b}V_b, \; J(X)V_a = 0.$$

It is now clear that $J(X)^3 = 0$. We have $J(X)X = 0$ and $J(X)V_a = 0$. Since $\mathcal{R}_{a11b} = 0$ if $a = 1$ or $b = 1$, $J(X)T_1 = 0$. Furthermore, $\mathcal{R}_{a11b} = \delta_{ab}$ for $a \ge 2$. Since $u_1 = 1$, $\{X, U_2, \dots, U_s, T_1, \dots, T_s, V_1, \dots, V_s\}$ is a basis for $V = \mathbb{R}^{3s}$. Consequently:

$$\begin{aligned} \text{Range}(J(X)) &= \text{Span}\{J(X)X, J(X)U_2, \dots, J(X)U_s, J(X)T_1, \dots, J(X)T_s, \\ &\qquad J(X)V_1, \dots, J(X)V_s\} \\ &= \text{Span}\{J(X)U_2, \dots, J(X)U_s, J(X)T_2, \dots, J(X)T_s\} \\ &= \text{Span}\{C_{2b}V_b - T_2, \dots, C_{sb}V_b - T_s, V_2, \dots, V_s\}. \end{aligned}$$

The set $\{C_{2b}V_b - T_2, \dots, C_{sb}V_b - T_s, V_2, \dots, V_s\}$ is linearly independent. Furthermore:

$$\begin{aligned} &\text{Range}(J(X)) \cap \ker(J(X)) = \text{Span}\{V_2, \dots, V_s\}, \\ &\text{Range}(J(X)^2) = \text{Span}\{V_2, \dots, V_s\}. \end{aligned}$$

Clearly the tensor R is spacelike Jordan Osserman nilpotent of order 3. Since $J(T_1) = 0$ while $J(U_1 - V_1) = J(U_1) \ne 0$, R is not timelike Jordan Osserman. □

3. Geometric Realizations

We complete the proof of Theorem 1.1 by showing that the structures of Lemma 2.2 are geometrically realizable. The metrics we shall consider are similar those described in different contexts in [6, 15, 18]. We take coordinates (u, v, t) on $\mathbb{R}^{3s}$ where $u = (u_1, ..., u_s)$, $v = (v_1, ..., v_s)$, and $t = (t_1, ..., t_s)$. Let

$$\partial_a^u := \frac{\partial}{\partial u_a}, \quad \partial_a^v := \frac{\partial}{\partial v_a}, \quad \text{and} \quad \partial_a^t := \frac{\partial}{\partial t_a}$$

be the associated coordinate frame for the tangent bundle.

Definition 3.1. *Let $\mathcal{R}$ be an algebraic curvature tensor on the vector space $\mathbb{R}^s$. Set $\psi_{abcd} := -\frac{2}{3}(\mathcal{R}_{acdb} + \mathcal{R}_{adcb})$; $\psi_{abcd} = \psi_{bacd} = \psi_{abdc}$. Define a pseudo-Riemannian metric g on $\mathbb{R}^{3s}$ of signature $(2s, s)$ whose only nonzero entries are among the components*

$$\begin{aligned} &g(u, v, t)(\partial_a^u, \partial_b^u) = \psi_{abcd} u_c t_d, \\ &g(u, v, t)(\partial_a^u, \partial_b^v) = g(u, v, t)(\partial_b^v, \partial_a^u) = \delta_{ab}, \\ &g(u, v, t)(\partial_a^t, \partial_b^t) = -\delta_{ab}. \end{aligned}$$

Theorem 1.1 will follow from Lemma 2.2 and from the following Lemma:

Lemma 3.2. *Let $\mathcal{R}$ be an algebraic curvature tensor on $\mathbb{R}^s$. Let g be the associated pseudo-Riemannian metric of signature $(2s, s)$ on $\mathbb{R}^{3s}$ given in Definition 3.1. Let $P = (u, v, t) \in \mathbb{R}^{3s}$. Let $R(P)$ be the curvature tensor of g at P. Let $\mathfrak{R}$ be the algebraic curvature tensor on $\mathbb{R}^s$ with $\mathfrak{R}(P)_{abcd} := R(P)(\partial_a^u, \partial_b^u, \partial_c^u, \partial_d^u)$. Then $R(P) = R_{\mathfrak{R}(P),\mathcal{R}}$ is given in Definition 2.1.*

Proof. At this point, we change our indexing convention slightly for the remainder of the proof. We shall let indices a, b, c index elements of $\mathcal{U} := \{\partial_1^u, \ldots, \partial_s^u\}$, indices α, β, γ index elements of $\mathcal{V} := \{\partial_1^v, \ldots, \partial_s^v\}$, and indices i, j, k index elements of $\mathcal{T} := \{\partial_1^t, \ldots, \partial_s^t\}$. Indices $r_1, r_2, \ldots$ will index the full coordinate frame

$$\{e_1, \ldots, e_{3s}\} := \{\partial_1^u, \ldots, \partial_s^u, \partial_1^v, \ldots, \partial_s^v, \partial_1^t, \ldots, \partial_s^t\}.$$

By an abuse of notation, we shall set $\Gamma_{abc} = g(\nabla_{\partial_a^u}\partial_b^u, \partial_c^u)$, $\Gamma_{abi} = g(\nabla_{\partial_a^u}\partial_b^u, \partial_i^t)$, etc. We replace an element of $\mathcal{T}$ by the corresponding element of $\mathcal{U}$ to define $\tilde\psi_{abci}$, $\tilde{\mathcal{R}}_{abci}$, $\tilde{\mathcal{R}}_{abic}$, $\tilde{\mathcal{R}}_{aibc}$, and $\tilde{\mathcal{R}}_{iabc}$. The non-zero components of the Christoffel symbols of the metric are:

$$\text{(3.a)} \qquad \begin{aligned} \Gamma_{abc} &= \tfrac{1}{2}(\tilde\psi_{bcai} + \tilde\psi_{acbi} - \tilde\psi_{abci})t_i, \\ \Gamma_{iab} &= \Gamma_{aib} = -\Gamma_{abi} = \tfrac{1}{2}\tilde\psi_{abci}u_c. \end{aligned}$$

We raise indices to see:

$$\text{(3.b)} \qquad \Gamma_{r_1r_2}{}^a = 0, \quad \Gamma_{r_1r_2}{}^i = -\Gamma_{r_1r_2i}, \quad \text{and} \quad \Gamma_{r_1r_2}{}^\alpha = \Gamma_{r_1r_2\alpha}.$$

The curvature tensor is given by:

$$R_{r_1r_2r_3r_4} = e_{r_1}\Gamma_{r_2r_3r_4} - e_{r_2}\Gamma_{r_1r_3r_4} + \Gamma_{r_1r_5r_4}\Gamma_{r_2r_3}{}^{r_5} - \Gamma_{r_2r_5r_4}\Gamma_{r_1r_3}{}^{r_5}.$$

If r_5 indexes an element of $\mathcal{V}$, then $\Gamma_{\star r_5 \star} = 0$ by equation (3.a) while if r_5 indexes an element of $\mathcal{U}$, then $\Gamma_{\star\star}{}^{r_5} = 0$ by equation (3.b). Thus r_5 must index an element of $\mathcal{T}$ and consequently, we may express:

$$\text{(3.c)} \qquad R_{r_1r_2r_3r_4} = e_{r_1}\Gamma_{r_2r_3r_4} - e_{r_2}\Gamma_{r_1r_3r_4} + \Gamma_{r_1ir_4}\Gamma_{r_2r_3}{}^{i} - \Gamma_{r_2ir_4}\Gamma_{r_1r_3}{}^{i}.$$

Thus by equation (3.a), quadratic terms in Γ can only appear in equation (3.c) if r_1, r_2, r_3, and r_4 all index elements of $\mathcal{U}$. The only other non-zero curvatures occur when exactly one of r_ν indexes an element of $\mathcal{T}$ and the remaining r_ν index elements of $\mathcal{U}$. We may therefore compute the proof by computing:

$$\begin{aligned} R(\partial_a^u, \partial_b^u, \partial_c^u, \partial_i^t) &= \partial_a^u\Gamma_{bci} - \partial_b^u\Gamma_{aci} = \tfrac{1}{2}(\tilde\psi_{acbi} - \tilde\psi_{bcai}) \\ &= -\tfrac{1}{3}(\tilde{\mathcal{R}}_{abic} + \tilde{\mathcal{R}}_{aibc} - \tilde{\mathcal{R}}_{baic} - \tilde{\mathcal{R}}_{biac}) \\ &= -\tfrac{1}{3}(2\tilde{\mathcal{R}}_{abic} - \tilde{\mathcal{R}}_{iabc} - \tilde{\mathcal{R}}_{ibca}) = -\tfrac{1}{3}(2\tilde{\mathcal{R}}_{abic} + \tilde{\mathcal{R}}_{icab}) = \tilde{\mathcal{R}}_{abci}. \end{aligned}$$

□

Remark 3.3. It is worth giving a very specific example. Define an inner product g on $\mathbb{R}^6$ whose non-zero components are, up to the usual $\mathbb{Z}_2$ symmetries, included among

$$\begin{aligned} &g(\partial_1^u, \partial_1^u) = -2u_2t_2, \quad g(\partial_2^u, \partial_2^u) = -2u_1t_1, \quad g(\partial_1^u, \partial_2^u) = u_1u_2, \\ &g(\partial_1^u, \partial_1^v) = g(\partial_2^u, \partial_2^v) = -g(\partial_1^t, \partial_1^t) = -g(\partial_2^t, \partial_2^t) = 1. \end{aligned}$$

This manifold has signature $(4, 2)$. It is spacelike Jordan Osserman nilpotent of order 3. It is not timelike Jordan Osserman. Furthermore, it is curvature homogeneous up to order 0 as defined by Kowalski, Tricerri, and Vanhecke [16].

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