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THE DECOMPOSITION OF TENSOR SPACES WITH ALMOST COMPLEX STRUCTURE

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ABSTRACT. Decomposition of tensor spaces with almost complex structures is a standard task in representation theory and thus in differential geometry. Our aim is to deduce explicit formulae by an elementary and straightforward approach. This decomposition is computed for tensors of the type (1,3) with symmetries of certain curvature tensors, providing an illustration of the general method on this well known example.

1. INTRODUCTION

Let E be a real n-dimensional vector space and E^p_q the tensor space of tensors of the type (p,q). A fixed basis of E determines a unique basis of E^p_q. The components of any tensor A with respect to this basis will be denoted $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$. Now let F^i_j be an arbitrary tensor of the type $(1,1)$ such that $F^\alpha_\alpha = 0$. A tensor $A \in E^p_q$ is called F-traceless if the following conditions hold

$$\forall k = 1, \cdots, p; \quad \forall r = 1, \cdots, q: \quad F^\alpha_\beta A^{\cdots i_{k-1} \beta i_{k+1} \cdots}_{\cdots j_{k-1} \alpha j_{k+1} \cdots} = 0, \quad A^{\cdots i_{k-1} \alpha i_{k+1} \cdots}_{\cdots j_{r-1} \alpha j_{r+1} \cdots} = 0.$$

The following theorem was proved in [5] and it shows F-decomposition for e-structures. e-structures are structures where the condition $F^i_\alpha F^\alpha_j = e\delta^i_j$, $e = \pm 1$ is fulfilled.

Theorem 1. *Let A be a tensor of the type (p,q). If $n > 2(p+q)$ then there exists a unique decomposition of A in the form*

$$(1) \qquad A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} = B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} + \sum_{t=1}^{\min\{p,q\}} \sum_{\oplus} \overset{\star}{Q}{}^{i_{\rho_1} i_{\rho_1} \cdots i_{\rho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots j_{\sigma_t}} \overset{\diamond}{B}$$

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where

$$\overset{\star}{Q}{}^{i_{\rho_1} i_{\rho_1} \cdots i_{\rho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots j_{\sigma_t}} \equiv \overset{\{\tau_1\}}{F}{}^{i_{\rho_1}}_{j_{\sigma_1}} \overset{\{\tau_2\}}{F}{}^{i_{\rho_2}}_{j_{\sigma_2}} \cdots \overset{\{\tau_t\}}{F}{}^{i_{\rho_t}}_{j_{\sigma_t}} ; \quad \overset{\{0\}}{F}{}^{i}_{j} = \delta^i_j ; \ \overset{\{1\}}{F}{}^{i}_{j} = F^i_j$$

$$\bigoplus = \begin{cases} \rho_1, \rho_2, \cdots, \rho_t \in \{1, 2, \cdots, p\} & (\rho_1 < \rho_2 < \cdots < \rho_t) , \\ \sigma_1, \sigma_2, \cdots, \sigma_t \in \{1, 2, \cdots, q\} & (\sigma_i \textit{ are mutually different}) , \\ \tau_1, \tau_2, \cdots, \tau_t \in \{0, 1\} \end{cases}$$

$$\star = \{\tau_1, \tau_2, \cdots, \tau_t\}, \qquad \diamond = \left\{ \begin{matrix} \rho_1, \rho_2, \cdots, \rho_t \\ \sigma_1, \sigma_2, \cdots, \sigma_t \\ \tau_1, \tau_2, \cdots, \tau_t \end{matrix} \right\} .$$

2. Decomposition of tensors of the type $(1,3)$.

In this section we will compute F- decompositon of tensors of the type $(1,3)$ for e-structures with $e = -1$. This structure is called almost complex structure. It was proved in [4] that we get 48 algebraic equations in 48 unknowns in generally and this system is not easy solvable. Therefore in [4] the contents of the theorem 1 was extended.

We compute the decomposition of tensors of the type $(1,3)$ which have following properties.

$$\text{(2)} \qquad \begin{aligned} &\text{(a) } A^h_{ijk} + A^h_{ikj} = 0 ; \quad &&\text{(b) } A^h_{ijk} + A^h_{jki} + A^h_{kij} = 0 ; \\ &\text{(c) } A^\alpha_{\alpha jk} = 0 ; &&\text{(d) } A^h_{i\overline{jk}} = A^h_{ijk} . \end{aligned}$$

We will use next notation.

$$\text{(3)} \qquad A^{\cdots}_{\cdots\bar{i}\cdots} \equiv F^\alpha_i A^{\cdots}_{\cdots\alpha\cdots} , \quad A^{\cdots\bar{i}\cdots}_{\cdots} \equiv F^i_\alpha A^{\cdots\alpha\cdots}_{\cdots}$$

It follows from this notation $A^{\cdots}_{\cdots\bar{\bar{i}}\cdots} = -A^{\cdots}_{\cdots i \cdots}$. If we denote

$$\text{(4)} \qquad A_{ij} \equiv A^\alpha_{ij\alpha}$$

we can also deduce following properties

$$\text{(5)} \qquad A_{ij} = A_{ji} .$$

For example the Riemannian tensors of Kählerian space, K-space, CR-space have these properties ([6], [7]). We have the next theorem.

Theorem 2. *Let A be a tensor of the type $(1,3)$ with properties (2), (5). Let F be almost complex structure. If $n > 4$ then there exists unique F-decomposition of the tensor A in the form*

$$\text{(6)} \qquad A^h_{ijk} = B^h_{ijk} + \delta^h_i C_{jk} + \delta^h_j D_{ik} + \delta^h_k E_{ij} + F^h_i G_{jk} + F^h_j H_{ik} + F^h_k I_{ij} ,$$

where tensors $B^h_{ijk}, C_{jk}, D_{jk}, E_{jk}, G_{jk}, H_{jk}, I_{jk}$ have following form

$$
\begin{aligned}
C_{jk} &= 0\,;\\
D_{jk} &= -\tfrac{1}{n+2}A_{jk}\,;\\
E_{jk} &= \tfrac{1}{n+2}A_{jk}\,;\\
G_{jk} &= -\tfrac{2}{n+2}A_{j\bar{k}}\,;\\
H_{jk} &= -\tfrac{1}{n+2}A_{j\bar{k}}\,;\\
I_{jk} &= \tfrac{1}{n+2}A_{j\bar{k}}\,;\\
B^h_{ijk} &= A^h_{ijk} + \tfrac{1}{n+2}\left(\delta^h_j A_{ik} - \delta^h_k A_{ij} + 2F^h_i A_{j\bar{k}} + F^h_j A_{i\bar{k}} - F^h_k A_{i\bar{j}}\right)
\end{aligned}
\tag{7}
$$

and the tensor B is F-traceless.

The aim of the following text is to prove the main Theorem 2. We will suppose, that the tensor A can be expressed in a form (6) where B^h_{ijk} is F-traceless tensor and $C_{jk}, D_{ij}, E_{ij}, G_{jk}, H_{ik}, I_{ij}$ are certain tensors. We will suppose that the tensor B^h_{ijk} has algebraic properties analogous to algebraic properties of the tensor A^h_{ijk}, i.e.

$$
B^h_{ijk} + B^h_{ikj} = 0\,; \quad B^h_{ijk} + B^h_{jki} + B^h_{kij} = 0\,; \quad B^h_{\bar{i}\bar{j}\bar{k}} = B^h_{ijk}\,. \tag{8}
$$

Let us alternate the expression (6) in j, k. Using

$$
A^h_{ijk} + A^h_{ikj} = 0\,; \qquad B^h_{ijk} + B^h_{ikj} = 0
$$

we can write

$$
\begin{aligned}
&\delta^h_i \left(C_{jk} + C_{kj}\right) + F^h_i \left(G_{jk} + G_{kj}\right) + \delta^h_j \left(D_{ik} + E_{ik}\right) + \delta^h_k \left(D_{ij} + E_{ij}\right)\\
&\quad + F^h_j \left(H_{ik} + I_{ik}\right) + F^h_k \left(H_{ij} + I_{ij}\right) = 0
\end{aligned}
\tag{9}
$$

Suppose that $C_{jk} + C_{kj} \neq 0$. Then there exists a tensor ϵ^j such that

$$
\epsilon^j \epsilon^k \left(C_{jk} + C_{kj}\right) = \pm 1\,.
$$

Contracting (9) by $\epsilon^j \epsilon^k$, we obtain

$$
\pm\delta^h_i + a\delta^h_{\bar{i}} + \epsilon^h \overset{1}{Q}_i + \epsilon^{\bar{h}} \overset{2}{Q}_i = 0\,, \tag{10}
$$

where $\overset{1}{Q}_i = 2\epsilon^\alpha \left(D_{i\alpha} + E_{i\alpha}\right)$ and $\overset{2}{Q}_i = 2\epsilon^\alpha \left(H_{i\alpha} + I_{i\alpha}\right)$. After contraction (10) by $\bar{i}$ we have

$$
\pm\delta^h_{\bar{i}} - a\delta^h_i + \epsilon^h \overset{1}{Q}_{\bar{i}} + \epsilon^{\bar{h}} \overset{2}{Q}_{\bar{i}} = 0\,. \tag{11}
$$

Let's substitute $\delta^h_{\bar{i}}$ from (11) in (10) then we get the following condition

$$
\pm\delta^h_j \left(1 + a^2\right) + \epsilon^h \left(\mp a \overset{1}{Q}_{\bar{j}} + \overset{1}{Q}_j\right) + \epsilon^{\bar{h}} \left(\mp a \overset{2}{Q}_{\bar{j}} + \overset{2}{Q}_j\right) = 0 \tag{12}
$$

Since $\mathrm{Rank}||\delta^h_j|| \leq 2$ it contradicts the assumption $n > 2$. We have following lemma.

Lemma 1. *The condition*

$$C_{jk} + C_{kj} = 0 \tag{13}$$

holds for coefficients C_{jk}.

We can use the previous arguments for the coefficients G_{jk} and we get

Lemma 2. *The condition*

$$G_{jk} + G_{kj} = 0 \tag{14}$$

holds for coefficients C_{jk}.

The equation (9) now has a form

$$\delta_j^h (D_{ik} + E_{ik}) + \delta_k^h (D_{ij} + E_{ij}) + F_j^h (H_{ik} + I_{ik}) + F_k^h (H_{ij} + I_{ij}) = 0 . \tag{15}$$

Suppose that $D_{ik} + E_{ik} \neq 0$. Similarly to the previous cases we get the existence of a bivector $\epsilon^i \eta^k$ such that

$$\epsilon^i \eta^k (D_{ik} + E_{ik}) = 1 .$$

When we contract (15) by $\epsilon^i \eta^k$ we obtain the equation

$$\delta_j^h + a\delta_{\bar{j}}^h + \eta^h \overset{1}{Q}_j + \eta^{\bar{h}} \overset{2}{Q}_j = 0 , \tag{16}$$

where $\overset{1}{Q}_i = \epsilon^\alpha (D_{\alpha j} + E_{\alpha j})$ and $\overset{2}{Q}_i = \epsilon^\alpha (H_{\alpha j} + I_{\alpha j})$. Contracting (16) we express $\delta_{\bar{j}}^h$, then we replace it in equation (16):

$$\delta_j^h (1 + a^2) + \eta^h \left(\overset{1}{Q}_j - a \overset{1}{Q}_{\bar{j}} \right) + \eta^{\bar{h}} \left(\overset{2}{Q}_j - a \overset{2}{Q}_{\bar{j}} \right) = 0 . \tag{17}$$

The equation (17) has no solution for $n > 2$.

Lemma 3. *The condition*

$$D_{jk} + E_{kj} = 0 . \tag{18}$$

holds for coefficients D_{jk}, E_{jk}.

In a similar way we obtain

Lemma 4. *The condition*

$$H_{jk} + I_{kj} = 0 . \tag{19}$$

holds for coefficients H_{jk}, I_{jk}.

When we apply lemmas to the Theorem 2 we have

Lemma 5. *When the condition* (2(a)) *is fulfilled then for* $n > 2$ *the tensor* A *may be expressed in a form*

$$A_{ijk}^h = B_{ijk}^h + \delta_i^h C_{jk} + \delta_j^h D_{ik} - \delta_k^h D_{ij} + F_i^h G_{jk} + F_j^h H_{ik} - F_k^h H_{ij} , \tag{20}$$

where

$$C_{jk} + C_{kj} = 0 , \quad G_{jk} + G_{kj} = 0 .$$

Using properties $A^h_{ijk}+A^h_{jki}+A^h_{kij}=0$; $B^h_{ijk}+B^h_{jki}+B^h_{kij}=0$ we get the equation

$$\delta^h_i\Omega_{jk}+\delta^h_j\Omega_{ki}+\delta^h_k\Omega_{ij}+F^h_i\overline{\Omega}_{jk}+F^h_j\overline{\Omega}_{ki}+F^h_k\overline{\Omega}_{ij}=0\,, \tag{21}$$

where

$$\Omega_{jk}=C_{jk}-D_{jk}+D_{kj}\,;\quad \overline{\Omega}_{jk}=G_{jk}-H_{jk}+H_{kj}\,.$$

But $\Omega_{jk}=0$; $\overline{\Omega}_{jk}=0$ for $n>4$, i.e.

$$C_{jk}=D_{jk}-D_{kj}\,;\qquad G_{jk}=H_{jk}-H_{kj}\,. \tag{22}$$

Let us replace C_{jk} and G_{jk} in (20) by (22). Then we get

Lemma 6. *If conditions* (2 (a), (b)) *are fulfilled then for $n>4$ the tensor A may be expressed in a form*

$$\begin{aligned}A^h_{ijk}&=B^h_{ijk}+\delta^h_i\left(D_{jk}-D_{kj}\right)+\delta^h_jD_{ik}-\delta^h_kD_{ij}\\&\quad+F^h_i\left(H_{jk}-H_{kj}\right)+F^h_jH_{ik}-F^h_kH_{ij}\,.\end{aligned} \tag{23}$$

The condition $A^h_{\overline{ijk}}=A^h_{ijk}$ gives

$$\begin{aligned}&\delta^h_i\left(D_{jk}-D_{kj}-D_{\overline{jk}}+D_{\overline{kj}}\right)\\&+\delta^h_j\left(D_{ik}+H_{i\bar{k}}\right)-\delta^h_k\left(D_{ij}+H_{i\bar{j}}\right)\\&+F^h_i\left(H_{jk}-H_{kj}-H_{\overline{jk}}+H_{\overline{kj}}\right)\\&+F^h_j\left(H_{ik}-D_{i\bar{k}}\right)-F^h_k\left(H_{ij}-D_{i\bar{j}}\right)=0\,.\end{aligned} \tag{24}$$

Equation (24) implies

$$H_{ik}=D_{i\bar{k}}\,;\quad D_{jk}-D_{kj}=D_{\overline{jk}}-D_{\overline{kj}}\,. \tag{25}$$

Using conditions $A^\alpha_{\alpha jk}=B^\alpha_{\alpha jk}=0$ and conditions (25) in the equation (23) we have after contraction by δ

$$(n+1)\left(D_{jk}-D_{kj}\right)+D_{\overline{jk}}-D_{\overline{kj}}=0\,. \tag{26}$$

It follows from (26)

$$D_{jk}=D_{kj}\,. \tag{27}$$

Substitute (27) to (22). We obtain

$$C_{jk}=0\,;\qquad G_{jk}=2D_{j\bar{k}}\,. \tag{28}$$

We can rewrite the equation (23) in a form

$$A^h_{ijk}=B^h_{ijk}+\delta^h_jD_{ik}-\delta^h_kD_{ij}+2F^h_iD_{j\bar{k}}+F^h_jD_{i\bar{k}}-F^h_kD_{i\bar{j}}\,. \tag{29}$$

Contract (29) by δ^k_h then

$$A_{ij}=-\left(n+2\right)D_{ij} \tag{30}$$

and therefore

$$D_{ij}=-\frac{1}{n+2}A_{ij}\,. \tag{31}$$

Substituting (18) to (19), (25), (28), (31) we get coefficients C_{jk}, D_{jk}, E_{jk}, H_{jk}, I_{jk} in the form mentioned in the Theorem 2. Now the tensor B^h_{ijk} has a form

$$B^h_{ijk} = A^h_{ijk} + \frac{1}{n+2}\left(\delta^h_j A_{ik} - \delta^h_k A_{ij} + 2F^h_i A_{j\bar{k}} + F^h_j A_{i\bar{k}} - F^h_k A_{i\bar{j}}\right) . \tag{32}$$

All computed tensors are F-traceless and the proof is complete.

When A^h_{ijk} is the Riemannian tensor then B^h_{ijk} (32) is well known tensor of the holomorphically-projective curvature.

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