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A CASE STUDY OF QUANTIZATION ON CURVED SURFACES: THE MYLAR BALLOON

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ABSTRACT. After a short review of the existing methods for quantization of curved manifolds thę free particle motion (geodesic flow) on the relatively new surface - the so called Mylar balloon is quantized using a combination of methods developed within geometric quantization scheme and constrained quantum mechanics.

1. INTRODUCTION

Modern quantum mechanics starts with the clear idea that the quantization is a map from the space of the classical observables (i.e the smooth functions on the phase space $(M,\omega) \equiv (\mathbb{R}^{2n}, \mathrm{d}p_i \wedge \mathrm{d}q^i)$) to the self-adjoint(symmetric operators) in the Hilbert space $\mathcal{H}$, $\quad Q : \phi \longrightarrow Q(\phi)$, $Q(\phi) : \mathcal{H} \longrightarrow \mathcal{H}$ which has the following properties:

$$\begin{array}{ll} Q1 & Q(\phi+\psi) = Q(\phi) + Q(\psi) \\ Q2 & Q(\lambda\phi) = \lambda Q(\phi), \quad \lambda \in \mathbb{R} \\ Q3 & Q(\{\phi,\psi\}) = \mathrm{i}[Q(\phi), Q(\psi)] \\ Q4 & Q(1) = Id_{\mathcal{H}} \text{ is the identity operator in } \mathcal{H} \\ Q5 & Q(q^i),\ Q(p_i) - \text{are irreducible operators in } \mathcal{H}. \end{array}$$

The various partial realizations of the above so–called Dirac programme [4] are known as algebraic, asymptotic, deformation, geometric, group-theoretical, ... etc quantizations but van Hove [6] proves rigorously that this can not be done at all! However, he proves also that $Q1 - Q4$ part has a solution and that there exists an unique realization in the large for the algebra of polynomials up to second degree in the canonical coordinates q^i, p_i.
Later on Segal [24] had transferred the above theorems to the phase spaces $(T^*Q,\ \mathrm{d}\theta)$ which are cotangent bundles of some configurational manifold and finally Kostant [11] and Souriau [27] present a scheme suitable for an arbitrary symplectic manifold (M,ω).

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Below we will present a short reviews of the Kostant-Souriau quantization scheme and the quantum mechanics of particles constrained on surfaces in $\mathbb{R}^3$ as both methods of quantization will be applied in the subsequent sections for finding the spectrum of the free particle motions on the Mylar balloon.

2. Geometric Quantization

The non-trivial moment in the Kostant-Souriau approach is that the wave functions are not moreover functions but a sections of a line bundle L over M, i.e. $\pi : L \longrightarrow M$, $s : M \longrightarrow L$ and $\pi \circ s = \mathrm{Id}_M$.
Such L exists if the symplectic manifold $(M\,,\,\omega)$ is pre-quantizable [11, 27], i.e. if $[\omega/2\pi]$ is in the image of the map

$$H^2_{\mathrm{Cech}}(M,\mathbb{Z}) \to H^2_{\mathrm{de\ Rham}}(M,\mathbb{R})$$

where $[\quad]$ denotes the de Rham cohomological class.
When M is a compact manifold the above condition is equivalent to

$$\frac{1}{2\pi}\int_\sigma \omega \in \mathbb{Z}, \quad \text{for every two}-\text{cycle} \quad \sigma \in H_2(M,\mathbb{Z}) \tag{1}$$

and the quantum operator associated with f acts in $\mathcal{H} \equiv \Gamma(L)$ – the space of sections of the corresponding line bundle as follows:

$$Q(f)s = -\mathrm{i}\nabla_{X_f}s + fs.$$

Here ∇_{X_f} is the covariant derivative along the Hamiltonian vector field generated by the symplectic form via $i(X_f)\omega = -\mathrm{d}f$. Identifying the sections of L with functions on M (which can be done in general only locally!) the action of $Q(f)$ in $\Gamma(L)$ can be written in the form

$$Q(f)\varphi = (-\mathrm{i}X_f - \theta(X_f) + f)\varphi$$

where θ is some local potential one-form of the symplectic structure $\omega = \mathrm{d}\theta$.
The irreducibility of the representation which is the second stage (quantization) of the programme is achieved by introducing additionally a new structure called polarization which we will not discuss here and refer the interested reader to the books by Simms and Woodhouse [25], Puta [23] and Sniatycki [26] which provide more details.
Much more interesting issue for us at this moment is how such nontrivial manifolds (two-cycles, surfaces) appear in Physics and Chemistry?
The most natural situation for this to happen is the reduction procedure which is known since the time of Newton and Jacobi and which modern formulation is due to Marsden and Weinstein [14]. The setting of the reduction theorem is the following: if a Lie group G acts in a Hamiltonian fashion on the symplectic manifold (M,ω), i.e.

$$\Phi_g : M \longrightarrow M, \qquad \Phi_g^*\omega = \omega$$

and preserves the energy function H of the Hamiltonian system (M,ω,H)

$$\Phi_g^*H = H \circ \Phi_g = H$$

then there exist a natural mapping called momentum

$$J : M \longrightarrow \mathfrak{g}^* - \text{the dual of the Lie algebra } \mathfrak{g} \text{ of } G$$

such that if μ is a fixed regular element in $\mathfrak{g}^*$ then

$$J^{-1}(\mu)/G_\mu \cong M_\mu$$

is an even-dimensional manifold and moreover there exists a two-form ω_μ such that $(M_\mu,\ \omega_\mu)$ is a symplectic manifold. When applied to such reduced manifolds geometric quantization scheme produces the quantization of charge, spin and energy levels of some physical systems [18, 19].

3. Geometry and Quantum Mechanics of Surfaces

The alternative to the Kostant-Souriau quantization of curved manifolds has been introduced in a few year by Jensen and Koppe [9] under the name "constrained" quantum mechanical systems. As a matter of fact, at that time they have considered these systems as pure mathematical ones, "since such systems do no exist" [9]. Fifteen years later, the experimental search for new materials arrived at the exotic C_{60} and C_{70} molecules, and subsequently the quantization of these molecules was faced with an entirely new puzzle of curiosity, which was nothing but the above mentioned "unphysical" problem. Nevertheless, quantum-mechanical study of these molecules remained so far mainly in the framework of quantum chemistry and its simplest approximations. This situation is a little bit strange in view of the fact that the rigorous quantum mechanical description of the behaviour of a particle constrained on a curved manifold is relatively old and well-known problem (cf. Jensen and Koppe [9, 10] and for later developments da Costa [3], Duclos et al [5], Ikegami et al [7], Londergan et al [13], Matsutani [15, 16], Ogawa [20] and Tolar [28]).
As this approach is based on some notions of the classical differential geometry we will sketch them below. Modern exposition of the subject can be found e.g. in the books by Berger and Gostiaux [2] and Oprea [21]. If $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ form the standard orthonormal basis of $\mathbb{R}^3$ and the surface $\mathcal{S}$ is represented as

$$\mathbf{x} = \mathbf{x}[u, v] = x(u, v)\mathbf{e}_1 + y(u, v)\mathbf{e}_2 + z(u, v)\mathbf{e}_3$$

the above books explain in depth that any such surface is specified (up to Euclidean motion) by its first and second fundamental forms

$$I = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\mathrm{d}v + G\,\mathrm{d}v^2 \qquad \text{and} \qquad II = L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\mathrm{d}v + N\,\mathrm{d}v^2 \tag{2}$$

and that their coefficients are given by

$$\begin{aligned} &E = E[u, v] = \mathbf{x}_u\cdot, \mathbf{x}_u\,, \quad F = F[u, v] = \mathbf{x}_u \cdot \mathbf{x}_v\,, \quad G = G[u, v] = \mathbf{x}_v \cdot \mathbf{x}_v\,,\\ &L = L[u, v] = \mathbf{x}_{uu} \cdot \mathbf{n}\,, \quad M = M[u, v] = \mathbf{x}_{uv} \cdot \mathbf{n}\,, \quad N = N[u, v] = \mathbf{x}_{vv} \cdot \mathbf{n}\,, \end{aligned} \tag{3}$$

where $\mathbf{n}$ is the unit vector normal to $\mathcal{S}$

$$\mathbf{n} = \mathbf{n}[u, v] = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}\,. \tag{4}$$

By definition the *normal curvature* κ_n in the direction $(\mathrm{d}u : \mathrm{d}v)$ is

$$\kappa_n = \frac{II}{I} = \frac{L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\mathrm{d}v + N\,\mathrm{d}v^2}{E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\mathrm{d}v + G\,\mathrm{d}v^2} \tag{5}$$

and the directions at which it attains extremal values (maximum and minimum) are called *principal directions*. If the coordinate curves coincide with the principal directions then

$$F = M \equiv 0 \tag{6}$$

and the corresponding curvatures of these directions can be found by the formulae

$$\kappa_1 = \frac{L}{E}, \qquad \kappa_2 = \frac{N}{G}. \tag{7}$$

Besides, it should be noted also that in this situation κ_1 and κ_2 are the principal curvatures along the meridians and parallels of latitude respectively. Classical differential geometry operates also with other important notions which are of immediate interest for us. These are the Gaussian curvature K and the mean curvature $\bar{H}$

$$K = \kappa_1 \cdot \kappa_2, \qquad \bar{H} = \frac{\kappa_1 + \kappa_2}{2} \tag{8}$$

and the surface area element $\mathrm{d}A$

$$\mathrm{d}A = \sqrt{EG - F^2}\,\mathrm{d}u\mathrm{d}v = \sqrt{EG}\,\mathrm{d}u\mathrm{d}v. \tag{9}$$

The systems in which we are interested after Jensen and Koppe [9] are of the following type: a particle of mass m is constrained to move on some surface $\mathcal{S}$. In this setting the naïve approach to quantization of such systems refers to association of the kinetic energy with the Laplacian operator $\Delta_{\mathcal{S}}$ of the natural Riemannian metric induced on this surface. More consistent considerations show that the correct Hamiltonian operator of the free particle motion is of the form

$$\widehat{H}_0 = -\frac{1}{2m}\Delta_{\mathcal{S}} + V_{\mathcal{S}} \tag{10}$$

where the additional surface potential $V_{\mathcal{S}}$ takes into account the actual embedding of $\mathcal{S}$. Using the mean $\bar{H}$ and Gaussian curvature K or the principal curvatures κ_1, κ_2 of the surface $\mathcal{S}$, this additional term can be expressed as follows:

$$V_{\mathcal{S}} = -\frac{1}{2m}(\bar{H}^2 - K) = -\frac{1}{8m}(\kappa_1 - \kappa_2)^2. \tag{11}$$

It should be noted also that the only two-dimensional surface for which this potential vanishes is the sphere, since in this case the two principal curvatures are equal. As the real systems rarely have a purely spherical shape it is desirable to know the spectrum for surfaces of more general shapes. Unfortunately, for such surfaces the above scheme results in a heavy mathematical problem. It is our goal here to investigate the influence of the geometrical characteristics on the quantum-mechanical spectrum of the free motion on the Mylar balloon. For that purpose we will combine the techniques presented in the previous sections to this surface which is related at least to fullerenes since from the mathematical point of view they can be considered to represent a family of closed curved two-dimensional manifolds.

4. The Mylar Balloon

The Mylar* balloon is constructed by taking two circular disks of Mylar, sewing them along their boundaries and then inflating with either air or helium. Somewhat surprisingly, these balloons are not spherical as one naïvely might expect from the well-known fact that the sphere possesses the maximal volume for a given surface area. This experimental fact suggests the following mathematical problem: given a circular Mylar balloon of deflated radius a, what will be the shape of the balloon when it is fully inflated? This question was first raised by W. Paulsen [22] who succeeded in determining the radius, thickness and volume of the inflated balloon.

Paulsen's answers were in terms of the gamma function. Elsewhere [17], we have shown that elliptic functions are equally as effective in answering these questions. Moreover, we achieve a deeper understanding of the geometry of the Mylar balloon because the approach also gives:

1. calculations of the Gauss and mean curvatures of the balloon;
2. a formula for the surface area of the balloon;
3. a characterization of the balloon in terms of curvature.

Furthermore, combining our results with Paulsen's, we have found some interesting relationships between the gamma function and elliptic integrals. However, until the description in terms of elliptic functions, more refined geometric qualities of the Mylar balloon were out of reach. Now we have the opportunity to apply the tools of differential geometry to truly understand a beautiful example of a physical principle constraining shape.

So, let us start with the mathematical model of the balloon. When the Mylar disk is inflated, the radius deforms to a curve $z = z(x)$ which we take to be in the first quadrant of the xz-plane. Of course, the curve proceeds from its highest point on the z-axis to a point of intersection with the x-axis. This is the right hand side of the curve which, when revolved about the z-axis, produces the top half of the balloon. The bottom half is just a reflection of the upper through the xy-plane.

Let r be the radius of the inflated balloon. Because of its physical properties, the Mylar does not stretch significantly so that the arclength of the curve $z(x)$ from $x = 0$ to $x = r$ is equal to the initial radius a. That is, we have

$$\int_0^r \sqrt{1 + z'(x)^2}\, \mathrm{d}x = a\,. \tag{12}$$

The basic shape of the balloon was determined by this constraint and the requirement that the enclosed volume is maximal [17]. There, we have proved

Theorem 1. *The surface of revolution $\mathcal{S}$ which models the Mylar balloon is parametrized by*

$$\begin{aligned} x(u,v) &= r\,\mathrm{cn}\left(u, \frac{1}{\sqrt{2}}\right)\cos v\,, \qquad y(u,v) = r\,\mathrm{cn}\left(u, \frac{1}{\sqrt{2}}\right)\sin v\,, \\ z(u,v) &= r\sqrt{2}\left[E\left(\mathrm{sn}\left(u, \frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right) - \frac{1}{2}F\left(\mathrm{sn}\left(u, \frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)\right] \end{aligned} \tag{13}$$

*According to Webster's New World Dictionary, *Mylar* is a trademark for a polyester made in extremely thin sheets of great tensile strength.

where $u \in [-K(1/\sqrt{2}), K(1/\sqrt{2})]$, $v \in [0, 2\pi]$ *and* $F(w,k)$, $E(w,k)$ *are the incomplete elliptic integrals of first, respectively second kind,* $K(k)$ *is the complete elliptic integral of the first kind,* sn, cn *are the Jacobian elliptic functions and* k *is the modulus of the abovementioned elliptic functions and integrals (more details can be found in* [8]).

One can put this parametrization into a computer algebra system like *Maple* or *Mathematica* and plot. We then see the familiar shape of a Mylar balloon in Fig 1.

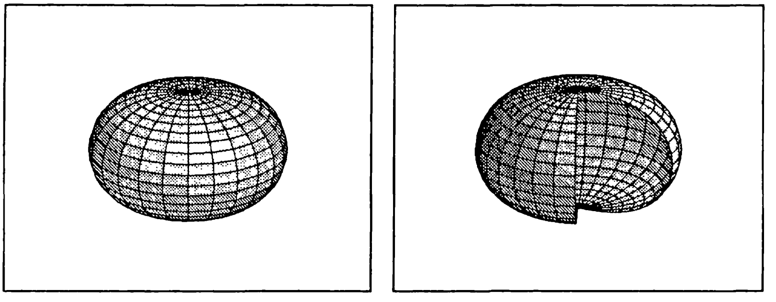

FIGURE 1. Two views of the Mylar balloon

5. Geometry of the Mylar Balloon

Having the explicit parametrizations of the surface of the Mylar balloon (13), we now turn to the study of its geometry. We calculate the coefficients of the first and second fundamental forms to be:

$$
\begin{aligned}
&E = \frac{r^2}{2}, \qquad &&F = 0, \qquad &&G = r^2 \operatorname{cn}^2\left(u, \frac{1}{\sqrt{2}}\right) \\
&L = r \operatorname{cn}\left(u, \frac{1}{\sqrt{2}}\right), \qquad &&M = 0, \qquad &&N = r \operatorname{cn}^3\left(u, \frac{1}{\sqrt{2}}\right).
\end{aligned}
\tag{14}
$$

Our first application of these calculations gives us something which is quite surprising. The formula for the volume of the Mylar balloon involves either the complete elliptic integral of the first kind [17] or the gamma function [22], so we might expect that a formula for surface area would be equally as complicated. Nevertheless, we have

Theorem 2. *The surface area of the Mylar balloon* $\mathcal{S}$ *of inflated radius* r *is given by* $A(\mathcal{S}) = \pi^2 r^2$.

Proof. The surface area element $\mathrm{d}A(\mathcal{S})$ is given by

$$
\mathrm{d}A(\mathcal{S}) = \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v = \sqrt{EG}\, \mathrm{d}u\, \mathrm{d}v = \frac{r^2 \operatorname{cn}\left(u, \frac{1}{\sqrt{2}}\right)}{\sqrt{2}}\, \mathrm{d}u\, \mathrm{d}v .
$$

Now it is quite easy to find the total surface area $A(\mathcal{S})$ of the Mylar balloon $\mathcal{S}$ by computing the following integral (where we denote $K(1/\sqrt{2})$ by K):

$$
\begin{aligned}
A(\mathcal{S}) &= \int_{\mathcal{S}} \mathrm{d}A(\mathcal{S}) \\
&= \frac{r^2}{\sqrt{2}} \int_0^{2\pi} \int_{-K}^{K} \operatorname{cn}\left(u, \frac{1}{\sqrt{2}}\right) \mathrm{d}u \, \mathrm{d}v \\
&= 4\pi \frac{r^2}{\sqrt{2}} \int_0^{K} \frac{\operatorname{cn}\left(u, \frac{1}{\sqrt{2}}\right) \operatorname{dn}\left(u, \frac{1}{\sqrt{2}}\right)}{\operatorname{dn}\left(u, \frac{1}{\sqrt{2}}\right)} \mathrm{d}u \\
&= 4\pi \frac{r^2}{\sqrt{2}} \int_0^1 \frac{\mathrm{d}w}{\sqrt{1 - \frac{1}{2} w^2}} \qquad \text{for} \quad w = \operatorname{sn}(u, 1/\sqrt{2}) \\
&= 4\pi \frac{r^2}{\sqrt{2}} \sqrt{2} \arcsin\left(\frac{w}{\sqrt{2}}\right) \Big|_0^1 \\
&= 4\pi \frac{r^2}{\sqrt{2}} \frac{\pi\sqrt{2}}{4} \\
&= \pi^2 r^2 .
\end{aligned}
$$

□

Now let's focus on qualities of the balloon central to its shape. We can easily obtain the curvatures for the balloon from the coefficients of the first and second fundamental forms (14). The Gauss curvature K and the mean curvature $\bar{H}$ are computed to be:

$$
\begin{aligned}
K &= \frac{LN - M^2}{EG - F^2} = \kappa_1 \cdot \kappa_2 = \frac{r \operatorname{cn}\left(u, \frac{1}{\sqrt{2}}\right) r \operatorname{cn}^3\left(u, \frac{1}{\sqrt{2}}\right)}{r^2/2 \cdot r^2 \operatorname{cn}^2\left(u, \frac{1}{\sqrt{2}}\right)} = \frac{2 \operatorname{cn}^2\left(u, \frac{1}{\sqrt{2}}\right)}{r^2} \\
\bar{H} &= \frac{EN + GL - 2FM}{2(EG - F^2)} = \frac{\kappa_1 + \kappa_2}{2} \\
&= \frac{r^2/2 \cdot r \operatorname{cn}^3\left(u, \frac{1}{\sqrt{2}}\right) + r^2 \operatorname{cn}^2\left(u, \frac{1}{\sqrt{2}}\right) r \operatorname{cn}\left(u, \frac{1}{\sqrt{2}}\right)}{2\left(r^2/2 \cdot r^2 \operatorname{cn}^2\left(u, \frac{1}{\sqrt{2}}\right)\right)} = \frac{3 \operatorname{cn}\left(u, \frac{1}{\sqrt{2}}\right)}{2r} .
\end{aligned}
\tag{15}
$$

These formulas actually allow us to verify our intuition about one particular aspect of the balloon's geometry. When we look at the balloon, we "see" the North and South poles as being "flat", but it is difficult to make this precise. However, we can prove the following geometric result which tells us that the poles are very flat indeed.

Theorem 3. *The North and South pole of the Mylar balloon are planar points (i.e. points whose normal curvatures are zero in all tangent directions).*

Proof. The North pole of the balloon corresponds to $u = K(1/\sqrt{2})$ and we know that $\operatorname{cn}(K(1/\sqrt{2}), 1/\sqrt{2}) = 0$. Therefore, we see from the formulas for K and $\bar{H}$ above that both Gauss curvature K and mean curvature $\bar{H}$ are zero. Hence, we have $\kappa_1 = 0$ and

$\kappa_2 = 0$. Since these are the maximal and minimal normal curvatures, we see that all normal curvatures are zero. The same is true for the South pole by symmetry. □

The Gauss and mean curvatures satisfy $K = (8/9)\bar{H}^2$. From (15), we also see that, for every u, the principal curvatures satisfy

$$\kappa_\mu = \kappa_1 = \frac{2\mathrm{cn}(u, 1/\sqrt{2})}{r} = 2\,\kappa_2 = 2\,\kappa_\pi\,. \tag{16}$$

Either of these relationships identify the Mylar balloon as a very special type of *Weingarten surface* (i.e. a surface whose principal curvatures satisfy a functional relation). *Surprisingly*, this relation between principal curvatures actually *characterizes* the balloon *uniquely* and leads to the following

Theorem 4. *The only surface of revolution $\mathcal{M}$ for which $\kappa_\mu = 2\,\kappa_\pi$ is the Mylar balloon.*

Relying on this theorem we can state that the surface

(17)

$$x(u,v) = \frac{r}{\sqrt{\cosh(2\,u)}}\cos(v)\,, \qquad y(u,v) = \frac{r}{\sqrt{\cosh(2\,u)}}\sin(v)\,,$$

$$z(u,v) = \sqrt{2}\,r\left[E\left(\arcsin(\frac{\sqrt{2}\sinh(u)}{\sqrt{\cosh(2\,u)}}), \frac{1}{\sqrt{2}}\right) - \frac{1}{2}F\left(\arcsin(\frac{\sqrt{2}\sinh(u)}{\sqrt{\cosh(2\,u)}}), \frac{1}{\sqrt{2}}\right)\right]$$

where $u \in (-\infty, \infty)$, $v \in [0, 2\pi]$ and for which

$$I = \frac{r^2}{\cosh(2u)}(\mathrm{d}u^2 + \mathrm{d}v^2) \qquad \text{and} \qquad II = \frac{r}{\cosh(2\,u)^{\frac{3}{2}}}(2\,\mathrm{d}u^2 + \mathrm{d}v^2) \tag{18}$$

is just the Mylar balloon and that (17) provides its conformal representation.

6. Quantization of the Mylar Balloon

The shape of C_{60} as well of certain multiple-shell fullerenes is classified as rather spherical, whereas other fullerenes are quite similar to either oblate or prolate rotational ellipsoids. The Mylar balloon in which we are interested here belongs to the former class.

On any two-dimensional manifold the symplectic form ω coincides up to a multiplicative factor with the respective surface element $\mathrm{d}A$. In conjunction with (1) this means that the integration of $\frac{\mathrm{d}A}{2\pi}$ over S should produce integers. Accordingly, in our case we will have

$$\frac{\text{Area}}{2\pi} = \frac{A(\mathcal{S})}{2\pi} = \frac{\pi r^2}{2} = N \in \mathbb{Z}^+ \tag{19}$$

which means that the radii of the inflated balloon are quantized!

Next, the Laplacian and the Eulerian difference $\bar{H}^2 - K$ which enters into expression for V_S can be easily found as well so that our quantization procedure leads to well posed analytical problem on the chosen coordinate patch.

In conformal (isothermal) coordinates (17) the Laplacian and the potential V_S take respectively the forms

$$\Delta_S = \frac{\cosh(2u)}{r^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \tag{20}$$

and

$$V_S = -\frac{1}{2m}(\bar{H}^2 - K) = -\frac{1}{2m}\frac{\operatorname{sech}(2u)}{4r^2}. \tag{21}$$

After separation of the variables in the quantum-mechanical time-independent Schrödinger equation

$$\widehat{H}_0\Psi = \left[-\frac{1}{2m}\Delta_S + V\right]\Psi = E\Psi \tag{22}$$

by introducing

$$\Psi(u,\, v) = \tilde{U}(u)\, e^{-ikv}, \qquad k \in \mathbb{Z} \tag{23}$$

and $\tanh(2u) = \zeta$ we end up with Sturm-Liouville type problem

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[(1-\zeta^2)\frac{\mathrm{d}U(\zeta)}{\mathrm{d}\zeta}\right] + \left[\frac{1}{16} + \frac{mEr^2}{2\sqrt{1-\zeta^2}} - \frac{k^2}{1-\zeta^2}\right]U(\zeta) = 0\,. \tag{24}$$

Unfortunately, the above differential equation is of a formidable complexity for analytical treatment and this is a serious obstruction for finding in a closed form either the wave functions or the spectrum of the problem in question.

However, confining ourself to the interval where the powers of ζ higher than second can be neglected we end up with the equation

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[(1-\zeta^2)\frac{\mathrm{d}U(\zeta)}{\mathrm{d}\zeta}\right] + \left[\lambda + \varepsilon^2\zeta^2 - \frac{k^2}{1-\zeta^2}\right]U(\zeta) = 0. \tag{25}$$

Here

$$\lambda = \frac{1}{16} + \frac{mEr^2}{2} \quad \text{and} \quad \varepsilon^2 = \frac{mEr^2}{4}\,. \tag{26}$$

One can easily recognize in (25) the defining equation for the oblate angular spheroidal functions $S_{kl}(\varepsilon,\zeta)$, $l \geq k$, corresponding to the eigenvalues

$$\lambda_{kl} = l(l+1) + \sum_{\sigma=1}^{\infty}(-1)^{\sigma} b_{2\sigma}\varepsilon^{2\sigma}\,. \tag{27}$$

which can be evaluated with any desired precision using various type of the existing formulae for the coefficients $b_{2\sigma}$, e.g.,

$$b_2 = \frac{1}{2}\left[1 - \frac{(2k-1)(2k+1)}{(2l-1)(2l+1)}\right],$$

$$b_4 = \frac{(l-k-1)(l-k)(l+k-1)(l+k)}{2(2l-3)(2l-1)^3(2l+1)} - \frac{(l-k+1)(l-k+2)(l+k+1)(l+k+2)}{2(2l+1)(2l+3)^3(2l+5)},$$

and so on. For more details see Abramowitz and Stegun [1] and Larsson et al [12] where the authors have advocated an alternative approach to the evaluation of the above functions.

What is more interesting here is that the above formula for λ_{kl} combined with (26) produces the energy spectrum of the geodesic flow on the Mylar balloon as given below

$$E_{kl} = -\frac{1}{8mr^2} + \frac{2l(l+1)}{mr^2} + \frac{2}{mr^2}\sum_{\sigma=1}^{\infty}(-1)^{\sigma} b_{2\sigma}\varepsilon^{2\sigma}. \tag{28}$$

Let us remember however that the radii of the Mylar balloon in accordance with (19) are discretized and the above formula should be written as

$$E_{Nkl} = -\frac{\pi}{16mN} + \frac{\pi l(l+1)}{mN} + \frac{\pi}{mN}\sum_{\sigma=1}^{\infty}(-1)^{\sigma} b_{2\sigma}\varepsilon^{2\sigma}. \tag{29}$$

Having the spectrum we have to comment the wave functions as well. Actually their properties and other spectral results follow directly from the general Sturm-Liouville theory. E.g., the wave functions $S_{kl}(\varepsilon, \zeta)$, with fixed k form a complete orthogonal system in $\mathcal{L}^2(-1,1)$. Besides, any of these functions has $l-k$ zeros in the interval $(-1,1)$ and the energy levels E_{Nkl} obviously increase when the indices l and N increase.

Finally, let us notice that the wave functions are labelled implicitly by the "principal" quantum number N via the definition of the first argument ε given in (26).

7. Concluding Remarks

The geodesic flow on the the Mylar balloon was quantized using a combination of methods from geometric quantization and constrained quantum mechanics. While geometric quantization scheme has found many concrete applications there were not such up to now of the constrained quantum mechanics. The reason is quite simple - the extra correction term resulting of surface embedding leads to a heavy analytical problem and this prevents the possibility of obtaining analytical results. One has to notice also that for two isometric surfaces (i.e. with the same induced metrics) these correction terms will depend on their second fundamental forms as well. This is in great contrast with the situation in the classical mechanics where the surface motion depends only on the metric properties of the surface. At the same time this hints also to make a search for surfaces for which the Eulerian difference is a simple one as much as possible. Potential candidates are at first place within the class of the so called Weingarten surfaces – i.e. those with a functional dependence among their principal curvatures. The most natural surfaces – spheres and the axisymmetric ellipsoids are just in this class – for the sphere one has $\kappa_1 = \kappa_2$ and in the case of the rotational ellipsoids $\kappa_1 \sim \kappa_2^3$. The new surface – the so-called Mylar balloon [17, 22] with a remarkably simple relationship $\kappa_1 = 2\kappa_2$ has been considered here in some details. Other well studied classes in the classical differential geometry are those of the surfaces with constant curvatures – both mean and Gaussian – present also a challenge and deserve profound treatment as well.

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