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In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 24th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2005. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 75. pp. [241]–250.

Persistent URL: <http://dml.cz/dmlcz/701750>

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LOWER BOUNDS FOR THE EIGENVALUES OF THE BASIC DIRAC OPERATOR

SEOUNG DAL JUNG

ABSTRACT. This talk is a survey on the eigenvalue estimates of the basic Dirac operator on the Riemannian manifold with the transverse spin foliation, which is based on the works of the author [9, 10, 11].

1. INTRODUCTION

In 1963, A. Lichnerowicz [18] proved that on a Riemannian spin manifold the square of the Dirac operator D is given by

$$(1.1) \quad D^2 = \Delta + \frac{\sigma}{4},$$

where Δ is the positive spinor Laplacian and σ the scalar curvature. In 1980, Th. Friedrich [5] gave a lower bound for the square for the eigenvalues of the Dirac operator D . In fact, by using a suitable Riemannian spin connection, he proved the inequality

$$(1.2) \quad \lambda^2 \geq \frac{n}{4(n-1)} \inf_M \sigma$$

on manifolds (M^n, g) with positive scalar curvature $\sigma > 0$. He also proved, in the limiting case, that the manifold is an Einstein. The inequality (1.2) has been improved in several directions by many authors [2, 3, 7, 8, 14, 15, 16].

In this talk, we estimate the lower bound of the eigenvalues for the basic Dirac operator D_b on the foliated Riemannian manifold, which are defined by J. Brüning and F. W. Kamber [4, 6]. They obtained the Lichnerowicz type formula on the transverse spin foliation with the basic-harmonic mean curvature form κ ;

$$(1.3) \quad D_b^2 = \nabla_{\text{tr}}^* \nabla_{\text{tr}} + \frac{1}{4} K_\sigma,$$

2000 *Mathematics Subject Classification*: 53C12, 53C27, 57R30.

Key words and phrases: transverse spin foliation, basic Yamabe operator, transversal Dirac operator, basic Dirac operator, transversally Einsteinian, Kähler spin foliation.

This paper was supported by grant No. R01-2003-000-10004-0 from Korea Science and Engineering Foundations (KOSEF).

The paper is in final form and no version of it will be submitted elsewhere.

where $K_\sigma = \sigma^\nabla + |\kappa|^2$, σ^∇ the transversal scalar curvature of \mathcal{F} and κ the mean curvature form of \mathcal{F} . By using the similar method to ordinary case, we obtain the following theorem which is corresponding to (1.2).

Theorem 1.1 ([9]). *Let (M, g_M, \mathcal{F}) be a compact Riemannian manifold with the transverse spin foliation \mathcal{F} of codimension $q > 1$ and bundle-like metric g_M such that κ is basic-harmonic. Assume $K_\sigma \geq 0$. Then the eigenvalue λ of the basic Dirac operator D_b satisfies*

$$(1.4) \quad \lambda^2 \geq \frac{1}{4} \frac{q}{q-1} K_\sigma^0,$$

where $K_\sigma^0 = \inf_M K_\sigma$.

By transversally conformal change of the metric g_M , we have the following sharp estimation, which is corresponding to the result of Hijazi [7] in ordinary manifold.

Theorem 1.2 ([11]). *Let (M, g_M, \mathcal{F}) be a compact Riemannian manifold with a transverse spin foliation \mathcal{F} of codimension $q \geq 3$ and bundle-like metric g_M such that κ is basic-harmonic. If the transversal scalar curvature satisfies $\sigma^\nabla \geq 0$, then we have*

$$(1.5) \quad \lambda^2 \geq \frac{q}{4(q-1)} (\mu_1 + \inf_M |\kappa|^2).$$

On the Kähler spin foliation, if we use the basic Kähler form Ω acting on the basic spinor field, we have the following theorem (see [14] for ordinary case).

Theorem 1.3 ([10]). *Let (M, g_M, \mathcal{F}) be a compact Riemannian manifold with a Kähler spin foliation \mathcal{F} of codimension $q = 2n$ and a bundle-like metric g_M such that κ is basic-harmonic and transversally holomorphic. If $K_\sigma \geq 0$, then the eigenvalue λ of D_b satisfies*

$$(1.6) \quad \lambda^2 \geq \frac{q+2}{4q} K_\sigma^0,$$

where $K_\sigma^0 = \inf_M K_\sigma$.

In the limiting case, the foliation is minimal, transversally Einsteinian with positive constant transversal scalar curvature σ^∇ . In particular, the limiting foliation in (1.6) is minimal, transversally Einsteinian with odd complex codimension. This implies that when complex codimension of \mathcal{F} is even, there exists a shaper estimate than (1.6).

2. PRELIMINARIES AND KNOWN FACTS

Let (M, g_M, \mathcal{F}) be a $(p+q)$ -dimensional Riemannian manifold with a foliation \mathcal{F} of codimension q and a bundle-like metric g_M with respect to \mathcal{F} . We recall the exact sequence

$$0 \rightarrow L \rightarrow TM \xrightarrow{\pi} Q \rightarrow 0$$

determined by the tangent bundle L and the normal bundle $Q = TM/L$ of \mathcal{F} . The assumption of g_M to be a bundle-like metric means that the induced metric g_Q on the normal bundle $Q \cong L^\perp$ satisfies the holonomy invariance condition $\overset{\circ}{\nabla} g_Q = 0$, where $\overset{\circ}{\nabla}$ is the Bott connection in Q ([12]).

For a distinguished chart $\mathcal{U} \subset M$ the leaves of \mathcal{F} in \mathcal{U} are given as the fibers of a Riemannian submersion $f : \mathcal{U} \rightarrow \mathcal{V} \subset N$ onto an open subset \mathcal{V} of a model Riemannian manifold N .

For overlapping charts $U_\alpha \cap U_\beta$, the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on N are isometries.

Further, we denote by ∇ the canonical connection of the normal bundle $Q = TM/L$ of \mathcal{F} . It is defined by

$$(2.1) \quad \begin{cases} \nabla_X s = \pi([X, Y_s]) & \text{for } X \in \Gamma L, \\ \nabla_X s = \pi(\nabla_X^M Y_s) & \text{for } X \in \Gamma L^\perp, \end{cases}$$

where $s \in \Gamma Q$, and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $L^\perp \cong Q$. The connection ∇ is metric and torsion free. It corresponds to the Riemannian connection of the model space N^q , [12]. The curvature R^∇ of ∇ is defined by

$$R_{XY}^\nabla = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for } X, Y \in TM.$$

Since $i(X)R^\nabla = 0$ for any $X \in \Gamma L$ ([12, 13, 20]), we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \rightarrow \Gamma Q$ and the (transversal) scalar curvature σ^∇ of \mathcal{F} by

$$\rho^\nabla(s) = \sum_\alpha R_{sE_\alpha}^\nabla E_\alpha, \quad \sigma^\nabla = \sum_\alpha g_Q(\rho^\nabla(E_\alpha), E_\alpha),$$

where $\{E_\alpha\}_{\alpha=1, \dots, q}$ is an orthonormal basis for Q . The foliation \mathcal{F} is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, that is,

$$(2.2) \quad \rho^\nabla = \frac{1}{q} \sigma^\nabla \cdot \text{id}$$

with constant transversal scalar curvature σ^∇ .

The *mean curvature vector field* of \mathcal{F} is then defined by

$$(2.3) \quad \tau = \sum_i \pi(\nabla_{E_i}^M E_i),$$

where $\{E_i\}_{i=1, \dots, p}$ is an orthonormal basis of L . The dual form κ , the *mean curvature form* for L , is then given by

$$(2.4) \quad \kappa(X) = g_Q(\tau, X) \quad \text{for } X \in \Gamma Q.$$

The foliation \mathcal{F} is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

Let $\Omega_B^r(\mathcal{F})$ be the space of all *basic r -forms*, i.e.,

$$\Omega_B^r(\mathcal{F}) = \{\phi \in \Omega^r(M) \mid i(X)\phi = 0, \theta(X)\phi = 0, \text{ for } X \in \Gamma L\}.$$

Since the exterior derivative preserves the basic forms (that is, $\theta(X)d\phi = 0$ and $i(X)d\phi = 0$ for $\phi \in \Omega_B^r(\mathcal{F})$), the restriction $d_B = d|_{\Omega_B^r(\mathcal{F})}$ is well defined. Let δ_B the adjoint operator of d_B . Then it is well-known ([1, 9]) that

$$(2.5) \quad d_B = \sum_\alpha \theta_\alpha \wedge \nabla_{E_\alpha}, \quad \delta_B = - \sum_\alpha i(E_\alpha) \nabla_{E_\alpha} + i(\kappa_B),$$

where κ_B is the basic component of κ , $\{E_\alpha\}$ is a local orthonormal basic frame in Q and $\{\theta_\alpha\}$ its g_Q -dual 1-form.

The *basic Laplacian* acting on $\Omega_B^*(\mathcal{F})$ is defined by

$$(2.6) \quad \Delta_B = d_B \delta_B + \delta_B d_B.$$

If \mathcal{F} is the foliation by points of M , the basic Laplacian is the ordinary Laplacian.

3. THE BASIC DIRAC OPERATOR

Let (M, g_M, \mathcal{F}) be a Riemannian manifold with a transversally oriented Riemannian foliation \mathcal{F} of codimension q and a bundle-like metric g_M with respect to \mathcal{F} . Let $SO(q) \rightarrow P \rightarrow M$ be the principal bundle of (oriented) transverse orthonormal framings. Then a *transverse spin structure* is a principal $\text{Spin}(q)$ -bundle \tilde{P} together with two sheeted covering $\xi : \tilde{P} \rightarrow P$ such that $\xi(p \cdot g) = \xi(p)\xi_0(g)$ for all $p \in \tilde{P}$, $g \in \text{Spin}(q)$, where $\xi_0 : \text{Spin}(q) \rightarrow SO(q)$ is a covering. In this case, the foliation \mathcal{F} is called a *transverse spin foliation*. We then define the vector bundle S associated with \tilde{P} by

$$(3.1) \quad S(\mathcal{F}) = \tilde{P} \times_{\text{Spin}(q)} S_q,$$

where S_q is the irreducible spinor space associated to Q . The Hermitian metric on $S(\mathcal{F})$ is induced from g_Q , and the Riemannian connection ∇ on P defined by (2.1) can be lifted to one on \tilde{P} , in particular, to one on $S(\mathcal{F})$, which will be denoted by the same letter. $S(\mathcal{F})$ is called the *foliated spinor bundle*. It is well known that the curvature transform R^S ([17]) is given as

$$(3.2) \quad R_{XY}^S \Phi = \frac{1}{4} \sum_{a,b} g_Q(R_{XY}^\nabla E_a, E_b) E_a \cdot E_b \cdot \Phi \quad \text{for } X, Y \in TM.$$

On the foliated spinor bundle $S(\mathcal{F})$, we have

$$(3.3) \quad \sum_a E_a \cdot R_{X E_a}^S \Phi = -\frac{1}{2} \rho^\nabla(\pi(X)) \cdot \Phi,$$

$$(3.4) \quad \sum_{a < b} E_a \cdot E_b \cdot R_{E_a E_b}^S \Phi = \frac{1}{4} \sigma^\nabla \Phi$$

for $X \in TM$, [9, 11]. Taking $\hat{\pi}$ to denote the projection

$$\hat{\pi} : C^\infty(T^*M \otimes S(\mathcal{F})) \rightarrow C^\infty(Q^* \otimes S(\mathcal{F})) \cong C^\infty(Q \otimes S(\mathcal{F}))$$

we define the *transversal Dirac operator* D'_{tr} ([4, 6]) by

$$D'_{\text{tr}} = \cdot \circ \hat{\pi} \circ \nabla.$$

If $\{E_a\}_{a=1, \dots, q}$ is taken to be a local orthonormal basic frame in Q , then

$$D'_{\text{tr}} = \sum_a E_a \cdot \nabla_{E_a}.$$

In [4, 6] it was shown that the formal adjoint D'^*_{tr} is given by $D'^*_{\text{tr}} = D'_{\text{tr}} - \kappa$ and that therefore

$$(3.5) \quad D_{\text{tr}} = D'_{\text{tr}} - \frac{1}{2} \kappa.$$

is a symmetric, transversally elliptic differential operator, with symbol $\sigma_{D_{tr}}$ satisfying $\sigma_{D_{tr}}(x, \xi) = \xi$ for $\xi \in Q_x^*$ and $\sigma_{D_{tr}}(x, \xi) = 0$ for $\xi \in L_x^*$. We define the subspace $\Gamma_B S(\mathcal{F})$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$(3.6) \quad \Gamma_B S(\mathcal{F}) = \{ \Phi \in \Gamma S(\mathcal{F}) \mid \nabla_X \Phi = 0 \text{ for } X \in \Gamma L \}.$$

From (3.5), we see that D_{tr} leaves $\Gamma_B S(\mathcal{F})$ invariant if and only if the foliation \mathcal{F} is isoparametric, i.e., $\kappa \in \Omega_B^1(\mathcal{F})$. Let $D_b = D_{tr}|_{\Gamma_B S(\mathcal{F})} : \Gamma_B S(\mathcal{F}) \rightarrow \Gamma_B S(\mathcal{F})$. This operator D_b is called the *basic Dirac operator* on (smooth) basic sections $\Gamma_B S(\mathcal{F})$. We now define $\nabla_{tr}^* \nabla_{tr} : \Gamma S(\mathcal{F}) \rightarrow \Gamma S(\mathcal{F})$ as

$$(3.7) \quad \nabla_{tr}^* \nabla_{tr} \Phi = - \sum_a \nabla_{E_a, E_a}^2 \Phi + \nabla_\kappa \Phi,$$

where $\nabla_{V,W}^2 = \nabla_V \nabla_W - \nabla_{\nabla_V W}$ for any $V, W \in TM$.

Proposition 3.1 ([9]). *Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be a compact Riemannian manifold with the transverse spin foliation \mathcal{F} and a bundle-like metric g_M with respect to \mathcal{F} . Then*

$$\langle\langle \nabla_{tr}^* \nabla_{tr} \Phi, \Psi \rangle\rangle = \langle\langle \nabla_{tr} \Phi, \nabla_{tr} \Psi \rangle\rangle$$

for all $\Phi, \Psi \in \Gamma E$, where $\langle\langle \Phi, \Psi \rangle\rangle = \int_M \langle \Phi, \Psi \rangle$ is the inner product on $S(\mathcal{F})$.

Proposition 3.2 ([9]). *Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be the same as in Proposition 3.1. Assume that κ is basic-harmonic. Then the basic Dirac operator D_b satisfies*

$$(3.8) \quad D_b^2 = \nabla_{tr}^* \nabla_{tr} + \frac{1}{4} K_\sigma,$$

where $K_\sigma = \sigma^\nabla + |\kappa|^2$.

4. AN ESTIMATION OF THE EIGENVALUES ON RIEMANNIAN SPIN FOLIATION

Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be a compact Riemannian manifold with the transverse spin foliation \mathcal{F} of codimension q , a bundle-like metric g_M with respect to \mathcal{F} and a foliated spinor bundle $S(\mathcal{F})$. Now, we introduce a new connection $\overset{f}{\nabla}$ on $S(\mathcal{F})$ as

$$(4.1) \quad \overset{f}{\nabla}_X \Phi = \nabla_X \Phi + f\pi(X) \cdot \Phi \quad \text{for } X \in TM,$$

where f is a real valued basic function on M . Trivially, this connection $\overset{f}{\nabla}$ is a metric connection on Q . By similar calculation to proposition 3.1, we have

$$(4.2) \quad \langle\langle \overset{f}{\nabla}_{tr}^* \overset{f}{\nabla}_{tr} \Phi, \Psi \rangle\rangle = \langle\langle \overset{f}{\nabla}_{tr} \Phi, \overset{f}{\nabla}_{tr} \Psi \rangle\rangle$$

for all $\Phi, \Psi \in \Gamma S(\mathcal{F})$. Let $D_b \Phi = \lambda \Phi$. From (3.8), (4.1) and (4.2) we have

$$(4.3) \quad \|\overset{f}{\nabla}_{tr} \Phi\|^2 = \int_M \left(\left(\frac{q-1}{q} \lambda^2 - \frac{1}{4} K_\sigma \right) |\Phi|^2 \right),$$

where $K_\sigma = \sigma^\nabla + |\kappa|^2$. From (4.3), we have the following theorem.

Theorem 4.1 ([9]). *Let (M, g_M, \mathcal{F}) be a compact Riemannian manifold with a transverse spin foliation \mathcal{F} of codimension $q > 1$ and bundle-like metric g_M such that κ is*

basic-harmonic. Assume $K_\sigma \geq 0$. Then the eigenvalue λ of the basic Dirac operator D_b satisfies

$$(4.4) \quad \lambda^2 \geq \frac{1}{4} \frac{q}{q-1} \inf_M K_\sigma,$$

where $K_\sigma = \sigma^\nabla + |\kappa|^2$.

Remark. If \mathcal{F} is a point foliation, then the transversal (basic) Dirac operator is just a Dirac operator on an ordinary manifold. Therefore Theorem 4.1 is a generalization of the result on an ordinary manifold (cf.[5]).

Theorem 4.2 ([9]). *Let (M, g_M, \mathcal{F}) be a compact Riemannian manifold with a transverse spin foliation \mathcal{F} of codimension $q > 1$ and a bundle-like metric g_M such that κ is basic-harmonic. Assume $K_\sigma > 0$. If there exists an eigenspinor field Ψ_1 of the basic Dirac operator D_b for the eigenvalue $\lambda_1^2 = \frac{q}{4(q-1)} K_\sigma^0$, then \mathcal{F} is a minimal, transversally Einsteinian with constant transversal scalar curvature.*

Remark. Theorem 4.2 implies that if the foliation \mathcal{F} is not minimal, then $\lambda^2 > \frac{q}{4(q-1)} K_\sigma^0$. So when \mathcal{F} is not minimal, there exists a sharper estimate than (4.4).

5. AN ESTIMATION OF THE EIGENVALUES BY THE CONFORMAL CHANGE

Now, we consider, for any real basic function u on M , the transversally conformal metric $\bar{g}_Q = e^{2u} g_Q$. Let $\bar{P}_{so}(\mathcal{F})$ be the principal bundle of \bar{g}_Q -orthogonal frames. Locally, the section \bar{s} of $\bar{P}_{so}(\mathcal{F})$ corresponding a section $s = (E_1, \dots, E_q)$ of $P_{so}(\mathcal{F})$ is $\bar{s} = (\bar{E}_1, \dots, \bar{E}_q)$, where $\bar{E}_a = e^{-u} E_a$ ($a = 1, \dots, q$). This isometry will be denoted by I_u . Thanks to the isomorphism I_u one can define a transverse spin structure $\bar{P}_{spin}(\mathcal{F})$ on \mathcal{F} in such a way that the diagram

$$\begin{array}{ccc} P_{spin}(\mathcal{F}) & \xrightarrow{I_u} & \bar{P}_{spin}(\mathcal{F}) \\ \downarrow & & \downarrow \\ P_{so}(\mathcal{F}) & \xrightarrow{I_u} & \bar{P}_{so}(\mathcal{F}) \end{array}$$

commutes. Let $\bar{S}(\mathcal{F})$ be the foliated spinor bundle associated with $\bar{P}_{spin}(\mathcal{F})$. For any section Ψ of $S(\mathcal{F})$, we write $\bar{\Psi} \equiv I_u \Psi$. If $\langle \cdot, \cdot \rangle_{g_Q}$ and $\langle \cdot, \cdot \rangle_{\bar{g}_Q}$ denote respectively the natural Hermitian metrics on $S(\mathcal{F})$ and $\bar{S}(\mathcal{F})$, then for any $\Phi, \Psi \in \Gamma S(\mathcal{F})$

$$(5.1) \quad \langle \Phi, \Psi \rangle_{g_Q} = \langle \bar{\Phi}, \bar{\Psi} \rangle_{\bar{g}_Q},$$

and the Clifford multiplication in $\bar{S}(\mathcal{F})$ is given by

$$(5.2) \quad \bar{X} \cdot \bar{\Psi} = \overline{X \cdot \Psi} \quad \text{for } X \in \Gamma Q.$$

Let $\bar{\nabla}$ be the metric and torsion free connection corresponding to \bar{g}_Q . Then we have for $X, Y \in \Gamma TM$,

$$(5.3) \quad \bar{\nabla}_X \pi(Y) = \nabla_X \pi(Y) + X(u)\pi(Y) + Y(u)\pi(X) - g_Q(\pi(X), \pi(Y)) \text{grad}_{\bar{\nabla}}(u),$$

where $\text{grad}_{\bar{\nabla}}(u) = \sum_a E_a(u) E_a$ is a transversal gradient of u and $X(u)$ is the Lie derivative of the function u in the direction of X . The formula (5.3) follows from that $\bar{\nabla}$ is the metric and torsion free connection with respect to \bar{g}_Q . The connection ∇ and

$\bar{\nabla}$ acting respectively on the sections of $S(\mathcal{F})$ and $\bar{S}(\mathcal{F})$, are related, for any vector field X and any spinor field $\bar{\Psi}$ by

$$(5.4) \quad \bar{\nabla}_X \bar{\Psi} = \overline{\nabla_X \Psi} - \frac{1}{2} \overline{\pi(X) \cdot \text{grad}_{\nabla}(u) \cdot \Psi} - \frac{1}{2} g_Q(\text{grad}_{\nabla}(u), \pi(X)) \bar{\Psi}.$$

Now, we introduce a new connection $\overset{f}{\nabla}$ on $\bar{S}(\mathcal{F})$ as

$$(5.5) \quad \overset{f}{\nabla}_X \bar{\Psi} = \bar{\nabla}_X \bar{\Psi} + f \pi(X) \cdot \bar{\Psi} \quad \text{for } X \in TM,$$

where f is a real-valued basic function on M . Trivially, this connection $\overset{f}{\nabla}$ is a metric connection.

Lemma 5.1. *On the foliated spinor bundle $\bar{S}(\mathcal{F})$, we have*

$$\langle\langle \overset{f}{\nabla}_{\text{tr}}^* \overset{f}{\nabla}_{\text{tr}} \bar{\Psi}, \bar{\Phi} \rangle\rangle_{\bar{g}_Q} = \langle\langle \overset{f}{\nabla}_{\text{tr}} \bar{\Psi}, \overset{f}{\nabla}_{\text{tr}} \bar{\Phi} \rangle\rangle_{\bar{g}_Q}$$

for all $\bar{\Psi}, \bar{\Phi} \in \Gamma S(\mathcal{F})$, where $\langle\langle \overset{f}{\nabla}_{\text{tr}} \bar{\Psi}, \overset{f}{\nabla}_{\text{tr}} \bar{\Phi} \rangle\rangle_{\bar{g}_Q} = \sum_a \langle \overset{f}{\nabla}_{\bar{E}_a} \bar{\Psi}, \overset{f}{\nabla}_{\bar{E}_a} \bar{\Phi} \rangle_{\bar{g}_Q}$.

On the other hand, from (3.7) and (5.5) we have

$$(5.6) \quad \overset{f}{\nabla}_{\text{tr}}^* \overset{f}{\nabla}_{\text{tr}} \bar{\Psi} = \bar{\nabla}_{\text{tr}}^* \bar{\nabla}_{\text{tr}} \bar{\Psi} - 2f \bar{D}_{\text{tr}} \bar{\Psi} + qf^2 \bar{\Psi} - e^{-u} \overline{\text{grad}_{\nabla}(f) \cdot \Psi}.$$

Let $D_b \Phi = \lambda \Phi (\Phi \neq 0)$. If we put $f = \frac{\lambda}{q} e^{-u}$, then we have

$$(5.7) \quad \int |\overset{f}{\nabla}_{\text{tr}} \bar{\Psi}|_{\bar{g}_Q}^2 = \frac{q-1}{q} \int e^{-2u} (\lambda^2 - \frac{q}{4(q-1)} e^{2u} K_{\sigma}^{\bar{\nabla}}) |\bar{\Psi}|_{\bar{g}_Q}^2,$$

where $K_{\sigma}^{\bar{\nabla}} = h^{-1} Y_b h + |\kappa|^2$, Y_b is a basic Yamabe operator of \mathcal{F} , which is defined by

$$(5.8) \quad Y_b = 4 \frac{q-1}{q-2} \Delta_B + \sigma^{\nabla}.$$

From (5.7), we have the following theorem ([11]).

Theorem 5.2. *Let (M, g_M, \mathcal{F}) be a compact Riemannian manifold with a transverse spin foliation \mathcal{F} of codimension $q \geq 3$ and bundle-like metric g_M such that $\kappa \in \Omega_B^1(\mathcal{F})$ and $\delta\kappa = 0$. If the transversal scalar curvature is non-negative, then we have*

$$(5.9) \quad \lambda^2 \geq \frac{q}{4(q-1)} (\mu_1 + \inf |\kappa|^2),$$

where μ_1 is the smallest eigenvalue of the basic Yamabe operator.

Remark. Since $\mu_1 \geq \inf \sigma^{\nabla}$, the inequality (5.9) is a sharper estimate than (4.4).

6. AN ESTIMATION OF THE EIGENVALUES ON KÄHLER SPIN FOLIATION

Let \mathcal{F} be a Kähler foliation. Namely, by a *Kähler foliation* \mathcal{F} ([19]) we mean a foliation satisfying the following conditions; (i) \mathcal{F} is Riemannian, with a bundle-like metric g_M on M inducing the holonomy invariant metric g_Q on $Q \equiv L^\perp$, (ii) there is a holonomy invariant almost complex structure $J : Q \rightarrow Q$, where $\dim Q = q (= 2n)$ (real dimension), with respect to which g_Q is Hermitian, i.e.,

$$(6.1) \quad g_Q(JX, JY) = g_Q(X, Y)$$

for $X, Y \in \Gamma Q$, and (iii) if ∇ is almost complex, i.e., $\nabla J = 0$. Note that

$$(6.2) \quad \Omega(X, Y) = g_Q(X, JY)$$

defines a basic 2-form Ω , which is closed as a consequence of $\nabla g_Q = 0$ and $\nabla J = 0$. Then we can express the basic 2-form Ω by

$$(6.3) \quad \Omega = \sum_{k=1}^n \theta^{2k-1} \wedge \theta^{2k},$$

where $\{\theta^a\}$ is a g_Q -dual 1-form on M . For a Kähler foliation, we have the following identities ([19]):

$$(6.4) \quad R_{XY}^\nabla J = JR_{XY}^\nabla, \quad R_{JXJY}^\nabla = R_{XY}^\nabla, \quad R_{XY}^\nabla Z + R_{YZ}^\nabla X + R_{ZX}^\nabla Y = 0,$$

where X, Y and Z are elements of ΓQ .

Let \mathcal{F} be a Kähler spin foliation on a compact oriented Riemannian manifold M . From (6.3), we know that

$$(6.5) \quad \Omega = -\frac{1}{2} \sum_a E_a \cdot JE_a = \frac{1}{2} \sum_a JE_a \cdot E_a,$$

where $\{E_a\}$ is a local orthonormal basic frame in Q .

Note that the foliated spinor bundle $S(\mathcal{F})$ of a Kähler spin foliation \mathcal{F} splits into the orthogonal direct sum

$$(6.6) \quad S(\mathcal{F}) = S_0 \oplus S_1 \oplus \cdots \oplus S_n,$$

where the fiber $(S_r)_x$ of the subbundle S_r is just defined as the eigenspace corresponding to the eigenvalue $i(n - 2r)$ ($r = 0, \dots, n$) of $\Omega_x : S_x(\mathcal{F}) \rightarrow S_x(\mathcal{F})$. If $p_r : S(\mathcal{F}) \rightarrow S_r$ is the projection, then we have

$$(6.7) \quad \Omega = \sum_{r=0}^n i\mu_r p_r, \quad \mu_r = n - 2r.$$

The decomposition (6.6) is compatible with ∇ , i.e., if Ψ is a section of S_r , then $\nabla_X \Psi$ is also a section of S_r for any vector field X .

Let \tilde{D}_{tr} be the operator which is locally defined by

$$(6.8) \quad \tilde{D}_{\text{tr}} \Phi = \sum_a JE_a \cdot \nabla_{E_a} \Phi - \frac{1}{2} J\kappa \cdot \Phi \quad \text{for } \Phi \in \Gamma S(\mathcal{F}).$$

Using Green’s theorem on the foliated Riemannian manifold ([21]), we know for any $\Phi, \Psi \in \Gamma S(\mathcal{F})$

$$(6.9) \quad \int_M \langle \tilde{D}_{\text{tr}}\Phi, \Psi \rangle = \int_M \langle \Phi, \tilde{D}_{\text{tr}}\Psi \rangle,$$

i.e., \tilde{D}_{tr} is self-adjoint transversally elliptic operator.

Proposition 6.1 ([10]). *Let (M, g_M, \mathcal{F}) be a Riemannian manifold with a Kähler spin foliation \mathcal{F} and a bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$. Suppose the mean curvature of \mathcal{F} is a transversally holomorphic. Then we have*

$$D_{\text{tr}}^2 = \tilde{D}_{\text{tr}}^2, \quad D_{\text{tr}}\tilde{D}_{\text{tr}} + \tilde{D}_{\text{tr}}D_{\text{tr}} = 0.$$

On the foliated spinor bundle $S(\mathcal{F})$, we introduce a new connection of the form

$$(6.10) \quad \overset{fg}{\nabla}_X\phi = \nabla_X\phi + f\pi(X) \cdot \phi + igJ\pi(X) \cdot \iota^2\phi \text{ for } X \in TM,$$

where f, g are real valued basic functions on M and $\iota : S(\mathcal{F}) \rightarrow S(\mathcal{F})$ is a bundle map (see [10]). By similar method to section 5, if we put $f = \frac{\lambda}{q+2}$ and $g = \frac{(-1)^\epsilon \lambda}{q+2}$, then we have takes the form

$$(6.11) \quad \|\overset{fg}{\nabla}_{\text{tr}}\phi\|^2 = \int_M \left(\frac{q}{q+2}\lambda^2 - \frac{1}{4}K_\sigma \right) |\phi|^2,$$

where $K_\sigma = \sigma^\nabla + |\kappa|^2$. From (6.11), we have the following theorem ([10]).

Theorem 6.2. *Let (M, g_M, \mathcal{F}) be a compact Riemannian manifold with a Kähler spin foliation \mathcal{F} of codimension $q = 2n$ and a bundle-like metric g_M such that κ is basic-harmonic and transversally holomorphic. If $K_\sigma \geq 0$, then the eigenvalue λ of D_b satisfies*

$$(6.12) \quad \lambda^2 \geq \frac{q+2}{4q} \inf_M K_\sigma,$$

where $K_\sigma = \sigma^\nabla + |\kappa|^2$.

Remark. The estimation of the eigenvalue of the transversal Dirac operator on a Kähler spin foliation is a shaper estimate than the one in Theorem 4.1.

Theorem 6.3 ([10]). *Let (M, g_M, \mathcal{F}) be the same as in Theorem 6.2. If there exists an eigenspinor field $\phi (\neq 0)$ of the basic Dirac operator D_b for the eigenvalue $\lambda^2 = \frac{q+2}{4q} K_\sigma^0$, then \mathcal{F} is a minimal, transversally Einsteinian of odd complex codimension n with nonnegative constant transversal scalar curvature σ^∇ .*

Question. In Theorem 6.3, the limiting foliation is odd complex codimension. This implies that if the codimension of \mathcal{F} is even, then there exists a sharper estimate than (6.12) in Theorem 6.2. What is the estimate?

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